Hyperstability of a Cauchy-Jensen type functional equation

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Abstract. In this paper, we establish some hyperstability results concerning the Cauchy - Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x)$$

in Banach spaces.

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1. Introduction

In 1940, Ulam [25] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?
Let X and Y be Banach spaces with norms $\|.\|$ and $\|.\|$, respectively. In 1941, Hyers [11] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon,$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x)-T(x)\| \leq \epsilon,$$

for all $x \in X$. In 1978, Rassias [23] introduced the following inequality, that we call Cauchy-Rassias inequality. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \theta(\|x\|^p+\|y\|^p),$$

for all $x, y \in X$. Rassias [23] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x)-T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p,$$

for all $x \in X$. The above inequality has produced a lot of influence on the development of what we now call the *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians (see [7], [13]-[16] and [22]).

Recently, interesting results concerning Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x) \tag{1.1}$$

have been obtained in [1],[9],[12],[19] and [20].

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [3] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [17]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. The term hyperstability has been used for the first time in [17], however it seems that the first hyperstability result was published in [3] and concerned the ring homomorphisms. Numerous papers on this subject have been published and we refer to [2], [4], [5] and [10].

Throughout this paper, we present the hyperstability results for the Cauchy-Jensen functional equation (1.1) in Banach spaces.

The method of the proofs used in the main results is based on a fixed point result that can be derived from [6] (Theorem 1). To present it we need the following three hypothesis:

(**H1**) X is a nonempty set, Y is a Banach space, $f_1, ..., f_k : X \longrightarrow X$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi\left(f_i(x)\right)-\mu\left(f_i(x)\right)\|, \qquad \xi, \mu \in Y^X, \quad x \in X.$$

(**H3**) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta\left(f_i(x)\right), \qquad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

The following theorem is the basic tool in this paper. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 1.1. *Let hypotheses **(H1)**-**(H3)** be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x)-\varphi(x)\| \leq \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) < \infty, \qquad x \in X.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x)-\psi(x)\| \le \varepsilon^*(x), \qquad x\in X.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x\in X.$$

2. Hyperstability results

The following theorems are the main results in this paper and concern the hyperstability of equation (1.1).

Theorem 2.1. *Let X be a normed space, Y be a Banach space, $c\ge 0$, $p,q\in\mathbb{R}$, $p+q<0$ and let $f: X\longrightarrow Y$ satisfy*

$$\left\|f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-f(x)\right\| \le c\|x\|^p\cdot\|y\|^q \tag{2.1}$$

for all $x,y\in X\setminus\{0\}$. Then f is Cauchy-Jensen on $X\setminus\{0\}$.

Proof. Since $p+q<0$, one of p,q must be negative. Assume that $q<0$ and replace y by mx, where $m\in\mathbb{N}$, in (2.1). We get that

$$\left\|f\left(\frac{1+m}{2}x\right)+f\left(\frac{1-m}{2}x\right)-f(x)\right\| \le cm^q\|x\|^{p+q} \tag{2.2}$$

for all $x\in X\setminus\{0\}$. Write

$$\mathcal{T}_m\xi(x) := \xi\left(\frac{1+m}{2}x\right)+\xi\left(\frac{1-m}{2}x\right), \quad x\in X\setminus\{0\}, \xi\in Y^{X\setminus\{0\}},$$

$$\varepsilon_m(x) := cm^q\|x\|^{p+q}, \quad x\in X\setminus\{0\},$$

then (2.2) takes the following form

$$\|\mathcal{T}_m f(x)-f(x)\| \le \varepsilon_m(x), \quad x\in X\setminus\{0\}.$$

Define

$$\Lambda_m\eta(x) := \eta\left(\frac{1+m}{2}x\right)+\eta\left(\frac{1-m}{2}x\right), \quad x\in X\setminus\{0\}, \eta\in\mathbb{R}_+^{X\setminus\{0\}}.$$

Then it is easily seen that Λ_m has the form described in (**H3**) with $k=2$ and $f_1(x)=\frac{1+m}{2}x$, $f_2(x)=\frac{1-m}{2}x$, $L_1(x)=L_2(x)=1$ for $x\in X\setminus\{0\}$.
Moreover, for every $\xi,\mu\in Y^{X\setminus\{0\}}$ and $x\in X\setminus\{0\}$, we get that

$$\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\| = \left\|\xi\left(\frac{1+m}{2}x\right)+\xi\left(\frac{1-m}{2}x\right)-\mu\left(\frac{1+m}{2}x\right)-\mu\left(\frac{1-m}{2}x\right)\right\|$$

$$\le \left\|(\xi-\mu)\left(\frac{1+m}{2}x\right)\right\|+\left\|(\xi-\mu)\left(\frac{1-m}{2}x\right)\right\| = \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|.$$

So, (**H2**) is valid. Next, we can find $m_0 \in \mathbb{N}$ such that

$$\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q} < 1 \quad \text{for all} \quad m \geq m_0.$$

Therefore, we obtain that

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= cm^q \|x\|^{p+q} \sum_{n=0}^{\infty} \left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^n \\
&= \frac{cm^q\|x\|^{p+q}}{1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q}}, \qquad x \in X \setminus \{0\}, m \geq m_0.
\end{aligned}$$

Hence, according to Theorem 1.1, for each $m \geq m_0$ there exists a unique solution $A_m : X \setminus \{0\} \to Y$ of the equation

$$A_m(x) = A_m\left(\frac{1+m}{2}x\right) + A_m\left(\frac{1-m}{2}x\right), \qquad x \in X \setminus \{0\}$$

such that

$$\|f(x) - A_m(x)\| \leq \frac{cm^q\|x\|^{p+q}}{1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q}}, \qquad x \in X \setminus \{0\}, m \geq m_0.$$

Moreover,

$$A_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x), \qquad x \in X \setminus \{0\}.$$

To prove that A_m satisfies the Cauchy-Jensen equation on $X \setminus \{0\}$, we show that

$$\left\|\mathcal{T}_m^n f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^n f(x)\right\| \leq c\left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^n \|x\|^p\|y\|^q \tag{2.3}$$

for every $x, y \in X \setminus \{0\}$ and every $n \in \mathbb{N}_0$.

If $n = 0$, then (2.3) is simply (2.1). So, take $r \in \mathbb{N}_0$ and suppose that (2.3) holds for $n = r$. Then

$$\left\|\mathcal{T}_m^{r+1} f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^{r+1} f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^{r+1} f(x)\right\| = \left\|\mathcal{T}_m^r f\left(\frac{1+m}{2}\frac{x+y}{2}\right) + \mathcal{T}_m^r f\left(\frac{1-m}{2}\frac{x+y}{2}\right)\right.$$

$$\left. + \mathcal{T}_m^r f\left(\frac{1+m}{2}\frac{x-y}{2}\right) + \mathcal{T}_m^r f\left(\frac{1-m}{2}\frac{x-y}{2}\right) - \mathcal{T}_m^r f\left(\frac{1+m}{2}x\right) - \mathcal{T}_m^r f\left(\frac{1-m}{2}x\right)\right\|$$

$$\leq \left\| \mathcal{T}_m^n f\left(\frac{1+m}{2}\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{1+m}{2}\frac{x-y}{2}\right) - \mathcal{T}_m^n f(\frac{1+m}{2}x)\right\|$$
$$+ \left\| \mathcal{T}_m^n f\left(\frac{1-m}{2}\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{1-m}{2}\frac{x-y}{2}\right) - \mathcal{T}_m^n f(\frac{1-m}{2}x)\right\|$$
$$\leq c\left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^r \left(\left\|\frac{1+m}{2}x\right\|^p \cdot \left\|\frac{1+m}{2}y\right\|^q + \left\|\frac{1-m}{2}x\right\|^p \cdot \left\|\frac{1-m}{2}y\right\|^q\right)$$
$$= c\left(\left|\frac{1+m}{2}\right|^{p+q} + \left|\frac{1-m}{2}\right|^{p+q}\right)^{r+1} \|x\|^p \cdot \|y\|^q, \qquad x, y \in X \setminus \{0\}.$$

Thus, by induction we show that (2.3) holds for all $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (2.3), we obtain that

$$A_m(x) = A_m\left(\frac{x+y}{2}x\right) + A_m\left(\frac{x-y}{2}x\right), \qquad x, y \in X \setminus \{0\}.$$

So, we obtain a sequence $\{A_m\}_{m \geq m_0}$ of Cauchy-Jensen functions on $X \setminus \{0\}$ such that

$$\|f(x) - A_m(x)\| \leq \frac{cm^q\|x\|^{p+q}}{1 - \left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q}}, \qquad x \in X \setminus \{0\}.$$

It follows, with $m \longrightarrow \infty$, that f is Cauchy-Jensen on $X \setminus \{0\}$.

□

In a similar way we can prove the following two theorems.

Theorem 2.2. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p, q \in \mathbb{R}$, $p+q > 0$ and let $f : X \longrightarrow Y$ satisfies (2.1). Then f is Cauchy-Jensen on $X \setminus \{0\}$.*

Proof. Replacing y by $\frac{1}{m}x$ in (2.1), where $m \in \mathbb{N}$, we get

$$\left\| f\left(\frac{m+1}{2m}x\right) + f\left(\frac{m-1}{2m}x\right) - f(x)\right\| \leq \frac{c}{m^q}\|x\|^{p+q} \tag{2.4}$$

for all $x \in X \setminus \{0\}$. Define operators $\mathcal{T}_m : Y^{X\setminus\{0\}} \to Y^{X\setminus\{0\}}$ and $\Lambda_m : \mathbb{R}_+^{X\setminus\{0\}} \to \mathbb{R}_+^{X\setminus\{0\}}$ by

$$\mathcal{T}_m\xi(x) := \xi\left(\frac{m+1}{2m}x\right) + \xi\left(\frac{m-1}{2m}x\right), \qquad x \in X \setminus \{0\}, \quad \xi \in Y^{X\setminus\{0\}},$$
$$\Lambda_m\delta(x) := \delta\left(\frac{m+1}{2m}x\right) + \delta\left(\frac{m-1}{2m}x\right), \qquad x \in X \setminus \{0\}, \quad \delta \in \mathbb{R}_+^{X\setminus\{0\}}.$$

Then it is easily seen that Λ_m has the form described in (**H3**) with $k = 2$ and

$$f_1(x) = \frac{m+1}{2m}x, \quad f_2(x) = \frac{m-1}{2m}x, \quad L_1(x) = L_2(x) = 1$$

for $x \in X \setminus \{0\}$. Further, (2.4) can be written in the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X \setminus \{0\},$$

with

$$\varepsilon_m(x) := \frac{c}{m^q}\|x\|^{p+q}.$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$ and $x \in X \setminus \{0\}$, we have

$$\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| = \left\|\xi\left(\frac{m+1}{2m}x\right) + \xi\left(\frac{m-1}{2m}x\right) - \mu\left(\frac{m+1}{2m}x\right) - \mu\left(\frac{m-1}{2m}x\right)\right\|$$

$$\leq \left\|(\xi-\mu)\left(\frac{m+1}{2m}x\right)\right\| + \left\|(\xi-\mu)\left(\frac{m-1}{2m}x\right)\right\| = \sum_{i=1}^{2} L_i(x)\|(\xi-\mu)(f_i(x))\|$$

and hypothesis (**H2**) holds, too. We can find $m_0 \in \mathbb{N}$ such that

$$\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q} < 1 \quad \text{for all} \quad m \geq m_0.$$

Note yet that we have

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= \frac{c}{m^q}\|x\|^{p+q}\sum_{n=0}^{\infty}\left(\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q}\right)^n \\
&= \frac{c\|x\|^{p+q}}{m^q(1-\left|\frac{1+m}{2}\right|^{p+q} - \left|\frac{1-m}{2}\right|^{p+q})}, \qquad x \in X\setminus\{0\}, m \geq m_0.
\end{aligned}$$

Consequently, in view of Theorem 1.1, for each $m \geq m_0$ there exists a unique solution $A_m : X \setminus \{0\} \to Y$ of the equation

$$A_m(x) = A_m\left(\frac{m+1}{2m}x\right) + A_m\left(\frac{m-1}{2m}x\right), \qquad x \in X \setminus \{0\}$$

such that

$$\|f(x) - A_m(x)\| \leq \frac{c\|x\|^{p+q}}{m^q(1-\left|\frac{m+1}{2m}\right|^{p+q} - \left|\frac{m-1}{2m}\right|^{p+q})}, \qquad x \in X\setminus\{0\}, m \geq m_0.$$

Moreover,

$$A_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x), \qquad x \in X \setminus \{0\}.$$

Then we show

$$\left\|\mathcal{T}_m^n f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^n f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^n f(x)\right\| \leq c\left(\left|\frac{m+1}{2m}\right|^{p+q} - \left|\frac{m-1}{2m}\right|^{p+q}\right)^n \|x\|^p\|y\|^q \tag{2.5}$$

for every $x, y \in X \setminus \{0\}$ and every $n \in \mathbb{N}_0$.

If $n = 0$, then (2.5) is simply (2.1). So, take $r \in \mathbb{N}_0$ and suppose that (2.5) holds for $n = r$. Then

$$\left\|\mathcal{T}_m^{r+1} f\left(\frac{x+y}{2}\right) + \mathcal{T}_m^{r+1} f\left(\frac{x-y}{2}\right) - \mathcal{T}_m^{r+1} f(x)\right\| = \left\|\mathcal{T}_m^{r} f\left(\frac{m+1}{2m}\frac{x+y}{2}\right) + \mathcal{T}_m^{r} f\left(\frac{m-1}{2m}\frac{x+y}{2}\right)\right.$$

$$\left. +\mathcal{T}_m^{r} f\left(\frac{m+1}{2m}\frac{x-y}{2}\right) + \mathcal{T}_m^{r} f\left(\frac{m-1}{2m}\frac{x-y}{2}\right) - \mathcal{T}_m^{r} f\left(\frac{m+1}{2m}x\right) - \mathcal{T}_m^{r} f\left(\frac{m-1}{2m}x\right)\right\|$$

$$\leq \left\|\mathcal{T}_m^{n} f\left(\frac{m+1}{2m}\frac{x+y}{2}\right) + \mathcal{T}_m^{n} f\left(\frac{m+1}{2m}\frac{x-y}{2}\right) - \mathcal{T}_m^{n} f(\frac{m+1}{2m}x)\right\|$$

$$+ \left\|\mathcal{T}_m^{n} f\left(\frac{m-1}{2m}\frac{x+y}{2}\right) + \mathcal{T}_m^{n} f\left(\frac{m-1}{2m}\frac{x-y}{2}\right) - \mathcal{T}_m^{n} f(\frac{m-1}{2m}x)\right\|$$

$$\leq c\left(\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q}\right)^{r}\left(\left\|\frac{m+1}{2m}x\right\|^{p}\cdot\left\|\frac{m+1}{2m}y\right\|^{q} + \left\|\frac{m-1}{2m}x\right\|^{p}\cdot\left\|\frac{m-1}{2m}y\right\|^{q}\right)$$

$$= c\left(\left|\frac{m+1}{2m}\right|^{p+q} + \left|\frac{m-1}{2m}\right|^{p+q}\right)^{r+1}\|x\|^p\cdot\|y\|^q, \quad x, y \in X\setminus\{0\}.$$

Thus, by induction we show that (2.5) holds for all $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (2.5), we obtain that

$$A_m(x) = A_m\left(\frac{x+y}{2}x\right) + A_m\left(\frac{x-y}{2}x\right), \quad x, y \in X\setminus\{0\}.$$

So, we obtain a sequence $\{A_m\}_{m\geq m_0}$ of Cauchy-Jensen functions on $X\setminus\{0\}$ such that

$$\|f(x) - A_m(x)\| \leq \frac{c\|x\|^{p+q}}{m^q(1 - \left|\frac{m+1}{2m}\right|^{p+q} - \left|\frac{m-1}{2m}\right|^{p+q})}, \quad x \in X\setminus\{0\}.$$

It follows, with $m \longrightarrow \infty$, that f is Cauchy-Jensen on $X\setminus\{0\}$. $\square$

Theorem 2.3. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x)\right\| \leq c(\|x\|^p + \|y\|^p) \tag{2.6}$$

for all $x, y \in X\setminus\{0\}$. Then f is Cauchy-Jensen on $X\setminus\{0\}$.

Proof. Replacing x by $(m+2)x$ and y by $-mx$, where $m \in \mathbb{N}$, in (2.6), we obtain

$$\|f(x) + f((m+1)x) - f((m+2)x)\| \leq c((m+2)^p + m^p)\|x\|^p \tag{2.7}$$

for all $x \in X\setminus\{0\}$. Write

$$\mathcal{T}_m\xi(x) := \xi((m+2)x) - \xi((m+1)x), \quad x \in X\setminus\{0\}, \xi \in Y^{X\setminus\{0\}},$$

$$\varepsilon_m(x) := c((m+2)^p + m^p)\|x\|^p.$$

Inequality (2.7) takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X \setminus \{0\}.$$

The following linear operator $\Lambda_m : \mathbb{R}_+^{X\setminus\{0\}} \longrightarrow \mathbb{R}_+^{X\setminus\{0\}}$ which is defined by

$$\Lambda_m \eta(x) := \eta((m+2)x) + \eta((m+1)x), \quad \eta \in \mathbb{R}_+^{X\setminus\{0\}}, x \in X \setminus \{0\}$$

has the form described in (**H3**) with $k = 2$ and $f_1(x) = (m+2)x$, $f_2(x) = (m+1)x$, $L_1(x) = L_2(x) = 1$, for $x \in X \setminus \{0\}$.

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$

$$\|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x)\| = \|\xi\left((m+2)x\right) - \xi\left((m+1)x\right) - \mu\left((m+2)x\right) + \mu\left((m+1)x\right)\|$$

$$\le \|(\xi - \mu)\left((m+2)x\right)\| + \|(\xi - \mu)\left((m+1)x\right)\| = \sum_{i=1}^{2} L_i(x)\|(\xi - \mu)(f_i(x))\|.$$

So, (**H2**) is valid. Now, we can find $m_0 \in \mathbb{N}$ such that

$$(m+2)^p + (m+1)^p < 1 \quad \text{for all} \quad m \ge m_0.$$

Therefore, we obtain that

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= c((m+2)^p + m^p) \sum_{n=0}^{\infty} \Lambda_m^n \left(\|(m+2)x\|^p + \|(m+1)x\|^p\right) \\
&= c((m+2)^p + m^p)\|x\|^p \sum_{n=0}^{\infty} \left((m+2)^p + (m+1)^p\right)^n \\
&= \frac{c((m+2)^p + m^p)\|x\|^p}{1 - (m+2)^p - (m+1)^p}, \qquad x \in X \setminus \{0\}, m \ge m_0.
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1.

□

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