Original Research Article

Multi-parametric deformations of Peregrine breathers solutions to the NLS equation.

Abstract

The structure of the solutions to the one dimensional focusing nonlinear Schrödinger equation (NLS) for the order N in terms of quasi rational functions is given here. We first give the proof that the solutions can be expressed as a ratio of two wronskians of order $2N$ and then two determinants by an exponential depending on t with $2N-2$ parameters. It also is proven that for the order N, the solutions can be written as the product of an exponential depending on t by a quotient of two polynomials of degree $N(N+1)$ in x and t. The solutions depend on $2N-2$ parameters and give when all these parameters are equal to 0, the analogue of the famous Peregrine breather P_N. It is fundamental to note that in this representation at order N, all these solutions can be seen as deformations with $2N-2$ parameters of the famous Peregrine breather P_N. With this method, we already built Peregrine breathers until the $N = 10$ order, and their deformations depending on $2N-2$ parameters. We present here Peregrine breather of order 11 constructed for the first time.

1 Introduction

The term of rogue wave was introduced in the scientific community by Draper in 1964 [1]. The usual criteria for rogue waves in the ocean, is that the vertical distance from trough to crest is two or more times greater than the average wave height among one third of the highest waves in a time series (10 to 30 min). The first rogue wave recorded by scientific measurement in North Sea was made on the oil platform of Draupner in 1995, located between Norway and Scotland. Rogue waves in the ocean have led to many marine catastrophes; it is one of the reasons why these rogue waves turn out to be so important for the scientific community. It becomes a challenge to get a better understanding of their mechanisms of formation.
The rogue waves phenomenon currently exceed the strict framework of the study

of ocean's waves and play a significant role in other fields; in nonlinear optics [2], Bose-Einstein condensate [3], atmosphere [4] and even finance [5].
Here, we consider the one dimensional focusing nonlinear Schrödinger equation (NLS) to describe the phenomena of rogue waves. The first results concerning the NLS equation date back the works of Zakharov and Shabat in 1968 who solved it using the inverse scattering method [6, 7]. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [8, 9]. In 1977 Kuznetsov found the first breather type solution of the NLS equation [13]; a simular result was given by Ma [12] in 1979. The first quasi rational solutions to NLS equation were constructed in 1983 by Peregrine [10]. In 1986 Akhmediev, Eleonski and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather [11]. Other analogues of Peregrine breathers of order 3 were constructed and initial data corresponding to orders 4 and 5 were described in a series of articles by Akhmediev et al., in particular in [14, 15] using Darboux transformations.
Quite recently, many works about NLS equation have been published using different methods. In 2010, rational solutions to the NLS equation were written as a quotient of two wronskians [16]. In 2011, the present author constructed in [17] another representation of the solutions to the NLS equation in terms of a ratio of two wronskians of even order $2N$ composed of elementary functions using truncated Riemann theta functions depending on two parameters; rational solutions were obtained when some parameter tended to 0. In 2012, Guo, Ling and Liu founded another representation of the solutions as a ratio of two determinants [18] using generalized Darboux transform; a new approach was proposed by Ohta and Yang in [19] using Hirota bilinear method; finally, the present author has obtained rational solutions in terms of determinants which do not involve limits in [20] depending on two parameters.
With this extended method, we present multi-parametric families of quasi rational solutions to the focusing NLS equation of order N in terms of determinants (determinants of order $2N$) dependent on $2N-2$ real parameters. With this representation, at the same time the well-known ring structure, but also the triangular shapes also found by Ohta and Yang [19], Akhmediev et al. [21] are found.
The aim of this paper is to prove the representation of the solutions to the focusing NLS equation depending this time on $2N-2$ parameters; the proof presented in this paper with $2N-2$ parameters has been never published. This is the first task of the paper; then we deduce its particular degenerate representations in terms of a ratio of two determinants of order $2N$. The second task of the paper is to give the proof of the structure of the solution at the order N as the ratio of two polynomials of order $N(N+1)$ in x and t by an exponential depending on t. This representation makes possible to get all the possible patterns for the solutions to the NLS equation. It is important to stress that contrary to other methods, these solutions depending on $2N-2$ parameters give the Peregrine breather as particular case when all the parameters are equal to 0 : for this reason, these solutions will be called $2N-2$ parameters deformations

of the Peregrine of order N.
The paper is organized as follows. First of all, we express the solutions of the NLS equation using Fredholm determinants from these expressed in terms of truncated functions theta of Riemann first obtained by Its, Rybin and Salle [9]; the representation given in theorem 2.1 is different from those given in [9]. From that, we prove the representation of the solutions of the NLS equation in terms of wronskians depending on $2N-2$ parameters. We deduce a degenerate representation of solutions to the NLS equation depending a priori on $2N-2$ parameters at the order N.
Then we prove a theorem which states the structure of the quasi-rational solutions to the NLS equation. It was only conjectured in preceding works [17, 20]. Families depending on $2N-2$ parameters for the N-th order as a ratio of two polynomials of x and t multiplied by an exponential depending on t are obtained; it is proved that each of these polynomials have a degree equal to $N(N+1)$.
Finally, to prove the efficiency of this method, we construct the Peregrine breather of order 11. To the best of my knowledge, it is the first time that this solution is presented.

2 Expression of solutions to the NLS equation in terms of wronskians

2.1 Solutions to the NLS equation in terms of θ functions

For $r = 1, 3$, we define

$$\theta_r(x,t) = \sum_{k\in\{0;1\}^{2N}} \exp\left\{ \sum_{\mu>\nu,\,\mu,\nu=1}^{2N} \ln\left(\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right)^2 k_\mu k_\nu \right.$$
$$\left. + \left(\sum_{\nu=1}^{2N} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + \sum_{\mu=1,\,\mu\neq\nu}^{2n} \ln\left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right| + \pi i \epsilon_\nu + e_\nu \right) k_\nu \right\}, \tag{1}$$

In this formula, the symbol $\sum_{k\in\{0;1\}^{2N}}$ denotes summation over all 2N-dimensional vectors k whose coordinates k_ν are either 0 or 1.
The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of the parameters λ_ν, $1 \leq \nu \leq 2N$; they are defined by the formulas :

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu\lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}};$$
$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1, 3. \tag{2}$$

The parameters $-1 < \lambda_\nu < 1$, $\nu = 1, \ldots, 2N$, are real numbers such that

$$-1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1$$
$$\lambda_{N+j} = -\lambda_j, \quad j = 1, \ldots, N. \tag{3}$$

The condition (3) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1, \ldots, N. \tag{4}$$

Complex numbers e_ν $1 \leq \nu \leq 2N$ are defined in the following way :

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad a, b \in \mathbf{R}. \tag{5}$$

$\epsilon_\nu \in \{0; 1\}$, φ, $\nu = 1 \dots 2N$ are arbitrary real numbers.
With these notations, the solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0, \tag{6}$$

can be expressed as ([9])

$$v(x,t) = \frac{\theta_3(x,t)}{\theta_1(x,t)} \exp(2it - i\varphi), \tag{7}$$

2.2 From θ functions to Fredholm determinants

To get Fredholm determinants, we have to express the functions θ_r defined in (1) in terms of subsets of $[1, .., 2N]$

$$\theta_r(x,t) = \sum_{J \subset \{1,..,2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J,\, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \times \exp\left(\sum_{\nu \in J} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} \ + e_\nu \right). \tag{8}$$

In (8), the symbol $\sum_{J \subset \{1,..,2N\}}$ denotes summation over all subsets J of indices of the set $\{1, .., 2N\}$.
Let I be the unit matrix and $C_r = (c_{jk})_{1 \leq j,k \leq 2N}$ the matrix defined by :

$$c_{\nu\mu} = (-1)^{\epsilon_\nu} \frac{\prod_{\eta \neq \mu} |\gamma_\nu + \gamma_\eta|}{\prod_{\eta \neq \nu} |\gamma_\nu - \gamma_\eta|} \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu), \tag{9}$$

$$\epsilon_j = j \quad 1 \leq j \leq N, \quad \epsilon_j = N + j, \quad N + 1 \leq j \leq 2N. \tag{10}$$

Then $\det(I + C_r)$ has the following form

$$\det(I + C_r) = \sum_{J \subset \{1,...,2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J\, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{11}$$

Comparing this last expression (11) with the formula (8) at the beginning of this section, we have clearly the identity

$$\theta_r = \det(I + C_r). \tag{12}$$

We can give another representation of the solutions to NLS equation. To do this, let's consider the matrix $D_r = (d_{jk})_{1 \leq j,k \leq 2N}$ defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{13}$$

We have the equality $\det(I + D_r) = \det(I + C_r)$, and so the solution of NLS equation takes the form

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} \exp(2it - i\varphi). \tag{14}$$

Theorem 2.1 *The function v defined by*

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} \exp(2it - i\varphi). \tag{15}$$

is a solution of the focusing NLS equation with the matrix $D_r = (d_{jk})_{1 \leq j,k \leq 2N}$ defined by

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

where κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in(2), (3) and (5).

2.3 From Fredholm determinants to wronskians

We want to express solutions to NLS equation in terms of wronskian determinants. For this, we need the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \tag{16}$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \leq \nu \leq 2N. \tag{17}$$

We denote $W_r(y)$ the wronskian of the functions $\phi_{r,1}, \ldots, \phi_{r,2N}$ defined by

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu,\, \mu \in [1,\ldots,2N]}]. \tag{18}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\, \mu \in [1,\ldots,2N]}$ defined in (13). Then we have the following statement

Theorem 2.2

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \ldots, \phi_{r,2N})(0), \tag{19}$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

Proof : We start to remove the factor $(2i)^{-1} e^{i\Theta_{r,\nu}}$ in each row ν in the wronskian $W_r(y)$ for $1 \leq \nu \leq 2N$.
Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_{r,\nu}} (2i)^{-N} (2)^{-N} \times \tilde{W}_r, \tag{20}$$

with

$$\tilde{W}_r = \begin{vmatrix} (1-e^{-2i\Theta_{r,1}}) & i\gamma_1(1+e^{-2i\Theta_{r,1}}) & \dots & (i\gamma_1)^{2N-1}(1+(-1)^{2N}e^{-2i\Theta_{r,1}}) \\ (1-e^{-2i\Theta_{r,2}}) & i\gamma_2(1+e^{-2i\Theta_{r,2}}) & \dots & (i\gamma_2)^{2N-1}(1+(-1)^{2N}e^{-2i\Theta_{r,2}}) \\ \vdots & \vdots & \vdots & \vdots \\ (1-e^{-2i\theta_{r,2N}}) & i\gamma_{2N}(1+e^{-2i\Theta_{r,2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1+(-1)^{2n}e^{-2i\Theta_{r,2N}}) \end{vmatrix}$$

The determinant $\tilde{W}_r$ can be written as

$$\tilde{W}_r = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$, $e_j = e^{-2i\Theta_{r,j}}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $1 \leq j \leq N$, $1 \leq k \leq 2N$,
$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$, $e_j = e^{-2i\Theta_{r,j}}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $N+1 \leq j \leq 2N$, $1 \leq k \leq 2N$.
We want to calculate $\tilde{W}_r$. To do this, we use the following Lemma

Lemma 2.1 *Let* $A = (a_{ij})_{i,\,j\in[1,\dots,N]}$, $B = (b_{ij})_{i,\,j\in[1,\dots,N]}$, $(H_{ij})_{i,\,j\in[1,\dots,N]}$, *the matrix formed by replacing in* A *the jth row of* A *by the ith row of* B *Then*

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \tag{21}$$

Proof : We use the classical notations : $\tilde{A} = (\tilde{a}_{ji})_{i,\,j\in[1,\dots,N]}$ the transposed matrix in cofactors of A. We have the well known formula $A \times \tilde{A} = \det A \times I$.
So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.
The general term of the product $(c_{ij})_{i,j\in[1,..,N]} = (a_{ij}x_i + b_{ij})_{i,j\in[1,..,N]} \times (\tilde{a}_{ji})_{i,j\in[1,..,N]}$ can be written as
$c_{ij} = \sum_{s=1}^{N}(a_{is}x_i + b_{is}) \times \tilde{a}_{js}$
$= x_i \sum_{s=1}^{n} a_{is}\tilde{a}_{js} + \sum_{s=1}^{n} b_{is}\tilde{a}_{js}$
$= \delta_{ij}\det(A)x_i + \det(H_{ij})$.
We get
$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
Thus $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
$\square$.
We denote $U = (\alpha_{ij})_{i,\,j\in[1,\dots,2N]}$, $V = (\beta_{ij})_{i,\,j\in[1,\dots,2N]}$.
By applying the previous lemma, one obtains :

$$\begin{aligned}\tilde{W}_r &= \det(\alpha_{ij}e_i + \beta_{ij}) \\ &= \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}) = \det(U) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(U)}),\end{aligned} \tag{22}$$

where $(H_{ij})_{i,\,j\in[1,\dots,N]}$ is the matrix formed by replacing in U the jth row of U by the ith row of V defined previously.
The determinant of U of Vandermonde type is clearly equal to

$$\det(U) = i^{N(2N-1)} \prod_{2N\geq l>m\geq 1} (\gamma_l - \gamma_m). \tag{23}$$

To calculate determinant $\tilde{W}_r$, we must compute now $\det(H_{ij})$. To do that, two cases must be studied :
1. For $1 \leq j \leq N$. The matrix H_{ij} is clearly of the VanderMonde type where the j-th row of U in U is replaced by the i-th row of V. Clearly, we have :

$$\det(H_{ij}) = (-1)^{N(2N+1)+N-1}(i)^{N(2N-1)} \times M, \tag{24}$$

where $M = M(m_1, \dots, m_{2N})$ is the Vandermonde determinant defined by $m_k = \gamma_k$ for $k \neq j$ and $m_j = -\gamma_i$. Thus we have :

$$\begin{aligned} &\det(H_{ij}) = -(i)^{N(2N-1)} \times \textstyle\prod_{2N \geq l > k \geq 1,} (m_l - m_k) \\ &= -(i)^{N(2N-1)} \times \textstyle\prod_{2N \geq l > m \geq 1,\, l \neq j,\, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l<j}(-\gamma_i - \gamma_l) \times \prod_{l>j}(\gamma_l + \gamma_i), \\ &= (-1)^j (i)^{N(2N-1)} \times \textstyle\prod_{2N \geq l > m \geq 1,\, l \neq j,\, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j}(\gamma_l + \gamma_i). \end{aligned} \tag{25}$$

To evaluate $\tilde{W}_r$, we must simplify the quotient $q_{ij} := \frac{\det(H_{ij})}{\det(U)}$:

$$\begin{aligned} q_{ij} &= \frac{(-1)^j (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1,\, l \neq j,\, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j}(\gamma_l + \gamma_i)}{i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m)} \\ &= \frac{(-1)^j \prod_{l \neq j}(\gamma_l + \gamma_i)}{\prod_{l<j}(\gamma_j - \gamma_l) \prod_{l>j}(\gamma_l - \gamma_j)} = \frac{(-1)^j \prod_{l \neq j}(\gamma_l + \gamma_i)}{(-1)^{j-1} \prod_{l \neq j}(\gamma_l - \gamma_j)} = -\frac{\prod_{l \neq j}(\gamma_l + \gamma_i)}{\prod_{l \neq j}(\gamma_l - \gamma_j)}. \end{aligned} \tag{26}$$

We can replace q_{ij} by r_{ij} defined by $-\frac{\prod_{l \neq j}(\gamma_l + \gamma_i)}{\prod_{l \neq i}(\gamma_l - \gamma_i)}$, because $\det(\delta_{ij} x_i + \frac{\det(q_{ij})}{\det(A)}) = \det(\delta_{ij} x_i + \frac{\det(r_{ij})}{\det(A)})$ (similar matrices).
We express r_{ij} in terms of absolute value; as $j \in [1; N]$ and $0 < \gamma_1 < \dots < \gamma_N < 1 < \gamma_{2N} < \dots < \gamma_{N+1}$, we have :

$$\textstyle\prod_{l \neq i}(\gamma_l - \gamma_i) = (-1)^{i-1} \prod_{l \neq i} |\gamma_l - \gamma_i|, \quad \prod_{l \neq j}(\gamma_l + \gamma_i) = \prod_{l \neq j} |\gamma_l + \gamma_i|. \tag{27}$$

So the term r_{ij} can be written as

$$r_{ij} = (-1)^i \frac{\prod_{l \neq j} |\gamma_l + \gamma_i|}{\prod_{l \neq i} |\gamma_l - \gamma_i|} = (-1)^{\epsilon(i)} \frac{\prod_{l \neq j} |\gamma_l + \gamma_i|}{\prod_{l \neq i} |\gamma_l - \gamma_i|} = c_{ij} e^{-2i\Theta_{r,i}(0)}, \tag{28}$$

with respect to the notations given in (10) and (13).

2. The same estimations for $N+1 \leq j \leq 2N$ are made; $\det H_{ij}$ is first

$$\det(H_{ij}) = (-1)^{N(2N+1)+N-1}(i)^{N(2N-1)} \times M, \tag{29}$$

with $M = M(m_1, \dots, m_{2N})$ the Vandermonde determinant defined by $m_k = \gamma_k$ for $k \neq j$ and $m_j = -\gamma_i$. Thus we have :

$$\begin{aligned} &\det(H_{ij}) = (i)^{N(2N-1)} \times \textstyle\prod_{2N \geq l > k \geq 1,} (m_l - m_k) \\ &= (i)^{N(2N-1)} \times \textstyle\prod_{2N \geq l > m \geq 1,\, l \neq j,\, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l<j}(-\gamma_i - \gamma_l) \times \prod_{l>j}(\gamma_l + \gamma_i), \\ &= (-1)^{j-1} (i)^{N(2N-1)} \times \textstyle\prod_{2N \geq l > m \geq 1,\, l \neq j,\, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j}(\gamma_l + \gamma_i). \end{aligned} \tag{30}$$

The quotient $q_{ij} := \frac{\det(H_{ij})}{\det(U)}$ equals :

$$
\begin{aligned}
q_{ij} &= \frac{(-1)^{j-1}(i)^{N(2N-1)} \times \prod_{2N\geq l>m\geq 1,\, l\neq j,\, m\neq j}(\gamma_l-\gamma_m)\times\prod_{l\neq j}(\gamma_l+\gamma_i)}{i^{N(2N-1)}\prod_{2N\geq l>m\geq 1}(\gamma_l-\gamma_m)} \\
&= \frac{(-1)^{j-1}\prod_{l\neq j}(\gamma_l+\gamma_i)}{\prod_{l<j}(\gamma_j-\gamma_l)\prod_{l>j}(\gamma_l-\gamma_j)} = \frac{(-1)^{j-1}\prod_{l\neq j}(\gamma_l+\gamma_i)}{(-1)^{j-1}\prod_{l\neq j}(\gamma_l-\gamma_j)} = \frac{\prod_{l\neq j}(\gamma_l+\gamma_i)}{\prod_{l\neq j}(\gamma_l-\gamma_j)}.
\end{aligned}
\tag{31}
$$

We replace q_{ij} by r_{ij} defined by $\frac{\prod_{l\neq j}(\gamma_l+\gamma_i)}{\prod_{l\neq i}(\gamma_l-\gamma_i)}$, for the same reason as previously exposed.
r_{ij} is expressed in terms of absolute value; as $j \in [N+1; 2N]$ and $0 < \gamma_1 < \ldots < \gamma_N < 1 < \gamma_{2N} < \ldots < \gamma_{N+1}$, we have :

$$
\prod_{l\neq i}(\gamma_l-\gamma_i) = (-1)^{2N-i+N}\prod_{l\neq i}|\gamma_l-\gamma_i|, \quad \prod_{l\neq j}(\gamma_l+\gamma_i) = \prod_{l\neq j}|\gamma_l+\gamma_i|. \tag{32}
$$

So the term r_{ij} can be written as

$$
r_{ij} = (-1)^{N+i}\frac{\prod_{l\neq j}|\gamma_l+\gamma_i|}{\prod_{l\neq i}|\gamma_l-\gamma_i|} = (-1)^{\epsilon(i)}\frac{\prod_{l\neq j}|\gamma_l+\gamma_i|}{\prod_{l\neq i}|\gamma_l-\gamma_i|} = c_{ij}e^{-2i\Theta_{r,i}(0)}, \tag{33}
$$

with respect to the notations given in (10) and (13).
Replacing e_i by $e^{-2i\Theta_{r,i}}$, $\det \tilde{W}_r$ can be expressed as

$$
\begin{aligned}
&\det \tilde{W}_r = \det(U)\times\det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(U)}) = \det(U)\times\det(\delta_{ij}e_i + r_{ij}) \\
&= \det(U)\prod_{i=1}^{2N}e^{-2i\Theta_i}\det(\delta_{ij} + (-1)^{\epsilon(i)}\prod_{l\neq i}\left|\frac{\gamma_l+\gamma_i}{\gamma_l-\gamma_i}\right|e^{2i\Theta_{r,i}}).
\end{aligned}
\tag{34}
$$

We estimate the two members of the last relation (34) in $y = 0$, and using (23) we obtain the following result

$$
\begin{aligned}
&\det \tilde{W}_r(0) = i^{N(2N-1)}\prod_{2N\geq l>m\geq 1}(\gamma_l-\gamma_m)\prod_{i=1}^{2N}e^{-2i\Theta_{r,i}(0)} \\
&\times\det(\delta_{ij} + (-1)^{\epsilon(i)}\prod_{l\neq i}\left|\frac{\gamma_l+\gamma_i}{\gamma_l-\gamma_i}\right|e^{2i\Theta_{r,i}(0)}) \\
&= i^{N(2N-1)}\prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j-\gamma_i)e^{-2i\sum_{i=1}^{2N}\Theta_{r,i}(0)}\det(\delta_{ij}+c_{ij}) \\
&= i^{N(2N-1)}\prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j-\gamma_i)e^{-2i\sum_{i=1}^{2N}\Theta_{r,i}(0)}\det(I+C_r) \\
&= i^{N(2N-1)}\prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j-\gamma_i)e^{-2i\sum_{i=1}^{2N}\Theta_{r,i}(0)}\det(I+D_r).
\end{aligned}
\tag{35}
$$

Therefore, the wronskian W_r given by (20) can be written as

$$
\begin{aligned}
&W_r(\phi_{r,1},\ldots,\phi_{r,2N})(0) = \prod_{j=1}^{2N}e^{i\Theta_{r,j}(0)}(2)^{-2N}(i)^{-N}\times\tilde{W}r \\
&= \prod_{j=1}^{2N}e^{i\Theta_{r,j}(0)}(2)^{-2N}(i)^{-N}i^{N(2N-1)}\prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j-\gamma_i)e^{-2i\sum_{i=1}^{2N}\Theta_{r,i}(0)}\det(I+D_r) \\
&= (2)^{-2N}\prod_{j=2}^{2N}\prod_{i=1}^{j-1}(\gamma_j-\gamma_i)e^{-i\sum_{i=1}^{2N}\Theta_{r,i}(0)}\det(I+D_r).
\end{aligned}
\tag{36}
$$

As a consequence

$$
\det(I+D_r) = k_r(0)W_r(\phi_1,\ldots,\phi_{2N})(0). \tag{37}
$$

□

2.4 Wronskian representation of solutions to the NLS equation

From the initial formulation (15) we have

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} \exp(2it - i\varphi).$$

Using (19), the following relation between Fredholm determinants and wronskians is obtained

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \ldots, \phi_{r,2N})(0)$$

and

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \ldots, \phi_{r,2N})(0).$$

As $\Theta_{3,j}(0)$ contains N terms $x_{3,j}$ $1 \leq j \leq N$ and N terms $-x_{3,j}$ $1 \leq j \leq N$, we have the equality $k_3(0) = k_1(0)$, and we get the following result :

Theorem 2.3 *The function v defined by*

$$v(x,t) = \frac{W_3(\phi_{3,1}, \ldots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \ldots, \phi_{1,2N})(0)} \exp(2it - i\varphi).$$

is a solution of the focusing NLS equation depending on two real parameters a and b with ϕ_ν^r defined in (16)

$$\begin{array}{l} \phi_{r,\nu} = \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad 1 \leq \nu \leq N, \\ \phi_{r,\nu} = \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \end{array}$$

κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in(2), (3) and (5).

3 Families of multi-parametric solutions to the NLS equation in terms of a ratio of two determinants

Solutions to the NLS equation as a quotient of two determinants are constructed. Similar functions defined in a preceding work [20] are used, but modified as explained in the following. The following notations are needed :

$$\begin{aligned} X_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ Y_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{aligned}$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (2).
Parameters e_ν are defined by (5).
Here, is the crucial point : we choose the parameters a_j and b_j in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k} j^{2k+1} \epsilon^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b_k} j^{2k+1} \epsilon^{2k+1}, \quad 1 \leq j \leq N. \tag{38}$$

Below the following functions are used :

$$\begin{array}{ll}\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, & \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k,\\ \varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, & \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,\end{array} \tag{39}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll}\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, & \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k},\\ \varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, & \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k},\end{array} \tag{40}$$

for $1 \leq k \leq N$.
We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k.

$$\begin{array}{ll}\psi_{4j+1,k} = \gamma_k^{4j-1}\sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j}\cos Y_k,\\ \psi_{4j+3,k} = -\gamma_k^{4j+1}\sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2}\cos Y_k,\end{array} \tag{41}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll}\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin Y_{N+k},\\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin Y_{N+k},\end{array} \tag{42}$$

for $1 \leq k \leq N$.
Then it is clear that

$$q(x,t) := \frac{W_3(0)}{W_1(0)}$$

can be written as

$$q(x,t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,\,k\in[1,2N]}}{\det(\psi_{j,k})_{j,\,k\in[1,2N]}}. \tag{43}$$

We recall that $\lambda_j = 1-2j\epsilon^2$. All the functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ and can all be prolonged by continuity when $\epsilon = 0$.
Then the following expansions are used

$$\varphi_{j,k}(x,t,\epsilon) = \sum_{l=0}^{N-1}\frac{1}{(2l)!}\varphi_{j,1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,1}[l] = \frac{\partial^{2l}\varphi_{j,1}}{\partial\epsilon^{2l}}(x,t,0),$$

$$\varphi_{j,1}[0] = \varphi_{j,1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\varphi_{j,N+k}(x,t,\epsilon) = \sum_{l=0}^{N-1}\frac{1}{(2l)!}\varphi_{j,N+1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,N+1}[l] = \frac{\partial^{2l}\varphi_{j,N+1}}{\partial\epsilon^{2l}}(x,t,0),$$

$$\varphi_{j,N+1}[0] = \varphi_{j,N+1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1.$$

We have the same expansions for the functions $\psi_{j,k}$.

$$\psi_{j,k}(x,t,\epsilon) = \sum_{l=0}^{N-1}\frac{1}{(2l)!}\psi_{j,1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,1}[l] = \frac{\partial^{2l}\psi_{j,1}}{\partial\epsilon^{2l}}(x,t,0),$$

$$\psi_{j,1}[0] = \psi_{j,1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\psi_{j,N+k}(x,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \psi_{j,N+1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,N+1}[l] = \frac{\partial^{2l} \psi_{j,N+1}}{\partial \epsilon^{2l}}(x,t,0),$$

$$\psi_{j,N+1}[0] = \psi_{j,N+1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad N+1 \leq k \leq 2N..$$

Then we get the following result :

Theorem 3.1 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} \tag{44}$$

is a quasi-rational solution of the NLS equation (6)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$\begin{array}{l}
n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0), \\
n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0), \\
d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0), \\
d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0), \\
2 \leq k \leq N,\ 1 \leq j \leq 2N
\end{array}$$

The functions φ and ψ are defined in (39),(40), (41), (42).

Proof : The columns of the determinants appearing in $q(x,t)$ are combined successively to eliminate in each column k (and $N+k$) of them the powers of ϵ strictly inferior to $2(k-1)$; then each common term in numerator and denominator is factorized and simplified; finally we take the limit when ϵ goes to 0.
Precisely, first of all, the components j of the columns 1 and $N+1$ are respectively equal by definition to $\varphi_{j1}[0] + 0(\epsilon)$ for C_1, $\varphi_{jN+1}[0] + 0(\epsilon)$ for C_{N+1} of Δ_3, and $\psi_{j1}[0] + 0(\epsilon)$ for C'_1, $\psi_{jN+1}[0] + 0(\epsilon)$ for C'_{N+1} of Δ_1.
At the first step of the reduction, we replace the columns C_k by $C_k - C_1$ and C_{N+k} by $C_{N+k} - C_{N+1}$ for $2 \leq k \leq N$, for Δ_3; the same changes for Δ_1 are done. Each component j of the column C_k of Δ_3 can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \varphi_{j,1}[l](k^{2l}-1)\epsilon^{2l}$ and the column C_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \varphi_{j,N+1}[l](k^{2l}-1)\epsilon^{2l}$ for $2 \leq k \leq N$. For Δ_1, we have the same reductions, each component j of the column C'_k can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \psi_{j,1}[l](k^{2l}-1)\epsilon^{2l}$ and the column C'_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \psi_{j,N+1}[l](k^{2l}-1)\epsilon^{2l}$ for $2 \leq k \leq N$.
The term $\frac{k^2-1}{2}\epsilon^2$ for $2 \leq k \leq N$ can factorized in Δ_3 and Δ_1 in each column k and $N+k$, and so these common terms can be simplified in numerator and denominator.

If we restrict the developments at order 1 in columns 2 and $N+2$, we get respectively $\varphi_{j1}[1]+0(\epsilon)$ for component j of C_2, $\varphi_{jN+1}[1]+0(\epsilon)$ for component j of C_{N+2} of Δ_3, and $\psi_{j1}[1]+0(\epsilon)$ for component j of C'_2, $\psi_{jN+1}[1]+0(\epsilon)$ for component j of C'_{N+2} of Δ_1. This algorithm can be continued up to the columns C_N, C_{2N} of Δ_3 and C'_N, C'_{2N} of Δ_1.
Then taking the limit when ϵ tends to 0, $q(x,t)$ can be replaced by $Q(x,t)$ defined by :

$$Q(x,t) := \frac{\begin{vmatrix} \varphi_{1,1}[0] & \dots & \varphi_{1,1}[N-1] & \varphi_{1,N+1}[0] & \dots & \varphi_{1,N+1}[N-1] \\ \varphi_{2,1}[0] & \dots & \varphi_{2,1}[N-1] & \varphi_{2,N+1}[0] & \dots & \varphi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{2N,1}[0] & \dots & \varphi_{2N,1}[N-1] & \varphi_{2N,N+1}[0] & \dots & \varphi_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} \psi_{1,1}[0] & \dots & \psi_{1,1}[N-1] & \psi_{1,N+1}[0] & \dots & \psi_{1,N+1}[N-1] \\ \psi_{2,1}[0] & \dots & \psi_{2,1}[N-1] & \psi_{2,N+1}[0] & \dots & \psi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2N,1}[0] & \dots & \psi_{2N,1}[N-1] & \psi_{2N,N+1}[0] & \dots & \psi_{2N,N+1}[N-1] \end{vmatrix}} \quad (45)$$

So the solution of the NLS equation takes the form :

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

So we get the result given in (44). □

4 Families of quasi-rational solutions of order N depending on $2N-2$ parameters

Here a theorem which states the structure of the quasi-rational solutions to the NLS equation is given. It was only conjectured in preceding works [17, 20]. Moreover we obtain here families depending on $2N-2$ parameters for the Nth-order Peregrine breather including families with 2 parameters constructed in preceding works and so we get other symmetries in these deformations than those were expected.
In this section we use the notations defined in the previous sections. The functions φ and ψ are defined in (39), (40), (41), (42).

Theorem 4.1 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \quad (46)$$

is a quasi-rational solution of the NLS equation (6) quotient of two polynomials $N(x,t)$ and $D(x,t)$ depending on $2N-2$ real parameters $\tilde{a_j}$ and $\tilde{b_j}$, $1 \leq j \leq N-1$.
N and D are polynomials of degrees $N(N+1)$ in x and t.

Proof : From the previous result (45), we need to analyze functions $\varphi_{k,1}$, $\psi_{k,1}$ and $\varphi_{k,N+1}$, $\psi_{k,N+1}$. Functions $\varphi_{k,j}$ and $\psi_{k,j}$ differ only by the term of the argument $x_{3,k}$, so we only the study of functions $\varphi_{k,j}$ will be carried out. Then the study of functions $\psi_{k,j}$ can be easily deduced from the analysis of $\varphi_{k,j}$.
The expansions of these functions in ϵ are studied. We denote $(l_{kj})_{k,j\in[1,2N]}$ the matrix defined by

$$l_{kj} = \frac{\partial^{2j-2}}{\partial\epsilon^{2j-2}}\varphi_{k1}, \quad l_{k,j+N} = \frac{\partial^{2j-2}}{\partial\epsilon^{2j-2}}\varphi_{k,1+N}, \quad 1 \leq j \leq N,\, 1 \leq k \leq 2N,$$

$\frac{\partial^0}{\partial x^0}\varphi$ meaning φ. Each coefficient of the matrix $(l_{kj})_{k,j\in[1,2N]}$ must be evaluated, the power of x and t in the coefficient of $\epsilon^{2(m-1)}$ for the column $m \in [1, 2N]$. We remark that with these notations, the matrix $(l_{kj})_{k,j\in[1,2N]}$ evaluated in $\epsilon = 0$ is exactly $(n_{kj})_{k,j\in[1,2N]}$ defined in (45). Four cases must be studied depending on the parity of k.
1. We study l_{k1} for k odd, $k = 2s + 1$.

$$\begin{aligned}
l_{k1} &= (-1)^s \sin(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x + 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)t \\
&\quad -i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_1) \times \epsilon^{k-2}(1-\epsilon^2)^{-\frac{k-2}{2}}
\end{aligned}$$

$$= (-1)^s \sin\epsilon(\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x + 2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)t + 2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$

$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= (-1)^s \sin\epsilon(\sum_{l=0}^{p}(c_{2l}x + d_{2l}t + f_{2l} + O(\epsilon^{p+1}))\epsilon^{2l}) \times \epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p}(c_{2n}x + d_{2n}t + f_{2n} + O(\epsilon^{p+1}))\epsilon^{2n})^{2l+1} \times \epsilon^{k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p} P_n(x,t)\epsilon^{2n})^{2l+1} \times \epsilon^{k-1}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

where $P_n(x, t)$ is a polynomial or order 1 in x and t.

$$l_{k,1} = \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l+1}\beta_{\alpha_0,\ldots,\alpha_p}P_0(x,t)^{\alpha_0}\ldots P_p(x,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

$$= \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l+1} Q_{\alpha_0,\ldots,\alpha_p}(x,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x, t)$ is a polynomial in order $2l + 1$ in x and t.
The terms in ϵ^0 are obtained for $l = 0$ in the two summations with $\alpha_0 = 1$.

For column m, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l+k-1=2m-2$, which gives $l=m-s-1$.
The notations given in (44) are used. We get the following result

Proposition 4.1

$$\deg(n_{2s+1,m}) = 2(m-s)-1 \textit{ for } s \leq m-1, \quad n_{2s+1,m} = 0 \textit{ for } s \geq m. \quad (47)$$

2. We study l_{k1} for k even, $k=2s$.

$$l_{k1} = (-1)^{s+1}\cos(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x + 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)t$$
$$-i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_1) \times \epsilon^{k-2}(1-\epsilon^2)^{-\frac{k-2}{2}}$$

$$= (-1)^{s+1}\cos\epsilon(\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x + 2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)t + 2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$
$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= (-1)^{s+1}\cos\epsilon(\sum_{l=0}^{p}(c_{2l}x + d_{2l}t + f_{2l} + O(\epsilon^{p+1}))\epsilon^{2l}) \times \epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+d+1}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p}(c_{2n}x+d_{2n}t+f_{2n}+O(\epsilon^{p+1}))\epsilon^{2n})^{2l}\times\epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s+1}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p} P_n(x,t)\epsilon^{2n})^{2l} \times \epsilon^{2s-2}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

where $P_n(x,t)$ is a polynomial or order 1 in x and t.

$$l_{k,1} = \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l}\beta_{\alpha_0,\ldots,\alpha_p}P_0(x,t)^{\alpha_0}\ldots P_p(x,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)}\times\epsilon^{2s-2}\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^t)$$

$$= \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l}Q_{\alpha_0,\ldots,\alpha_p}(x,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{2s-2}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,t)$ is a polynomial in order $2l$ in x and t.
The terms in ϵ^0 are obtained for $l=0$ in the two summations with $\alpha_0=1$.
For column m, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l+k-2=2m-2$, which gives $l=m-s$.
With the notations given in (44), we have

Proposition 4.2

$$\deg(n_{2s,m}) = 2(m-s) \textit{ for } s \leq m, \quad n_{2s,m} = 0 \textit{ for } s > m. \quad (48)$$

3. We study $l_{k\frac{M}{2}+1}$ for k odd, $k=2s+1$.

$$l_{k\frac{M}{2}+1}=(-1)^s\cos(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x-4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)t+i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}-e_{\frac{M}{2}+1})$$

$$\times\epsilon^{M-k-1}(1-\epsilon^2)^{-\frac{M-k-1}{2}}$$

$$=(-1)^s(\cos\epsilon(\sum_{l=0}^{p}c_{2l}\epsilon^{2l}x-2i\sum_{l=0}^{p}c_{2l}\epsilon^{2l}(1-2\epsilon^2)t-2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$

$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l}+i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l}+O(\epsilon^{p+1}))\times\epsilon^{M-k-1}(\sum_{l=1}^{r}g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$=(-1)^s(\cos\epsilon(\sum_{l=0}^{p}(c_{2l}x+d_{2l}t+f_{2l})\epsilon^{2l}+O(\epsilon^{p+1}))\times\epsilon^{M-k-1}(\sum_{l=1}^{r}g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$=\sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p}(c_{2n}x+d_{2n}t+f_{2n}+O(\epsilon^{p+1}))\epsilon^{2n})^{2l}\times\epsilon^{M-k-1}(\sum_{l=1}^{r}g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$=\sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p}P_n(x,t)\epsilon^{2n}+O(\epsilon^{p+1}))^{2l}\times\epsilon^{M-2s-2}(\sum_{l=1}^{r}g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

where $P_n(x,t)$ is a polynomial or order 1 in x and t.

$$l_{k,\frac{M}{2}+1}=\sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l}\beta_{\alpha_0,\ldots,\alpha_p}P_0(x,t)^{\alpha_0}$$

$$\ldots P_p(x,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)}\times\epsilon^{M-2s-2}\sum_{l=1}^{r}g_{2l}\epsilon^{2l}+O(\epsilon^t)$$

$$=\sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l}Q_{\alpha_0,\ldots,\alpha_p}(x,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)}\times\epsilon^{M-2s-2}\sum_{l=1}^{r}g_{2l}\epsilon^{2l}+O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,t)$ is a polynomial in order $2l$ in x and t.
The terms in ϵ^0 (column $\frac{M}{2}+1$) are obtained for $l=0$ in the two summations with $\alpha_0=1$.
For column $\frac{M}{2}+m$, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l+2(N-s-1)=2m-2$, which gives $l=m+s-N$. Then we get the following result

Proposition 4.3

$$\deg(n_{2s+1,m+\frac{M}{2}})=2m+2s-M \textit{ for } s\geq\frac{M}{2}-m,\quad n_{2s+1,m}=0 \textit{ for } s<\frac{M}{2}-m. \quad (49)$$

4. We study $l_{k,1+\frac{M}{2}}$ for k even, $k=2s$.

$$l_{k\frac{M}{2}+1} = (-1)^s \sin(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x - 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)t + i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_{\frac{M}{2}+1})$$

$$\times \epsilon^{M-k-1}(1-\epsilon^2)^{-\frac{M-k-1}{2}}$$

$$= (-1)^s \sin\epsilon(\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x - 2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)t - 2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$

$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= (-1)^s \sin\epsilon(\sum_{l=0}^{p}(c_{2l}x + d_{2l}t + f_{2l})\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p}(c_{2n}x + d_{2n}t + f_{2n} + O(\epsilon^{p+1}))\epsilon^{2n})^{2l+1} \times \epsilon^{M-k}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p}P_n(x,t)\epsilon^{2n} + O(\epsilon^{p+1}))^{2l+1} \times \epsilon^{M-2s}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

where $P_n(x,t)$ is a polynomial or order 1 in x and t.

$$l_{k,1} = \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l+1}\beta_{\alpha_0,\ldots,\alpha_p}P_0(x,t)^{\alpha_0}$$

$$\ldots P_p(x,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

$$= \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l+1}Q_{\alpha_0,\ldots,\alpha_p}(x,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,t)$ is a polynomial in order $2l+1$ in x and t.
The terms in ϵ^0 are obtained for $l=0$ in the two summations with $\alpha_0=1$.
For column $\frac{M}{2}+m$, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l+M-k=2m-2$, which gives $l=m+s-N-1$.
Using the notations given in (44), we get the following result

Proposition 4.4

$$\begin{aligned}&\deg(n_{2s,m+\frac{M}{2}}) = 2m+2s-M-1 \text{ for } s \geq \tfrac{M}{2}+1-M,\\ &n_{2s,m+\frac{M}{2}} = 0 \text{ for } s < \tfrac{M}{2}+1-m.\end{aligned} \tag{50}$$

These results can be rewritten in the following way

Proposition 4.5

$$\begin{aligned}
&\deg(n_{j,k}) = 2k - j \textit{ for } j \leq 2k,\\
&n_{j,k} = 0 \textit{ for } j > 2k,\\
&\deg(n_{j,k}) = 2k + j - 2M - 1 \textit{ for } j \geq 2M + 1 - 2k,\\
&n_{j,k} = 0 \textit{ for } j < 2M + 1 - 2k.
\end{aligned} \tag{51}$$

The degree of the determinant of the matrix $(n_{kj})_{k,j\in[1,2N]}$ can now be evaluated.
From the previous analysis, we see that x and t have necessarily the same power in each n_{kj}. The maximal power in x and t, is successively taken in each column. It is realized by the following product

$$\prod_{j=1}^{N} n_{j,j} \prod_{j=1}^{N} n_{N+j,2N+1-j}.$$

Applying the result given in (51) we get

$$\deg(\det(n_{kj})_{k,j\in[1,2N]}) = \sum_{j=1}^{N} \deg(n_{j,j}) + \sum_{j=1}^{N} \deg(n_{N+j,2N+1-j})$$

$$= \sum_{j=1}^{N} 2j - j + \sum_{j=1}^{N} 2(M + 1 - j) - 2M - 1 + \frac{M}{2} + j$$

$$= \sum_{j=1}^{N} j + \sum_{j=1}^{N} N + 1 - j = N(N + 1).$$

It is the same for determinant $\det(d_{kj})_{k,j\in[1,2N]}$, we have $\deg(\det(d_{kj})_{k,j\in[1,2N]}) = N(N + 1)$.
Thus the quotient

$$\frac{\det((n_{kj})_{j,k\in[1,2N]})}{\det((d_{kj})_{j,k\in[1,2N]})}$$

defines a quotient of two polynomials, each of them of degree $N(N + 1)$, and this proves the result.
Parameters $a_1 = \sum_{k=1}^{N-1} \tilde{a}_k \epsilon_k$ and $a_1 = \sum_{k=1}^{N-1} \tilde{a}_k \epsilon_k$ must be chosen in the following way.
The term ϵ_k must be a power of ϵ to get a nontrivial solution; ϵ_k must be a strictly positive number a in order to have a finite limit when ϵ goes to 0. If the power of ϵ is superior to $2N - 2$, the derivations going up to $2N - 2$, then this coefficient becomes 0 when the limit is taken when ϵ goes to 0 and so has no relevance in the expression of the limit.
□

5 The Peregrine breather of order 11

It is important to say that, contrary to the P_1 breather, all higher ranks P_N breathers can be obtained by deformation of multi-parameters solutions. It is fundamental to stress that with this method, we can get very easily the P_N breather : it is sufficient to take all parameters equal to 0.
Actually $N = 11$ is a greatest rank for which the solution is given. We get explicitly solutions in terms of a ratio of two polynomials of degree 132 in x and t by an exponential depending on time. We don't have the space to present it here. We postpone to give a more precise study of this eleventh Peregrine breather and its 20 parameters deformations to another publication.

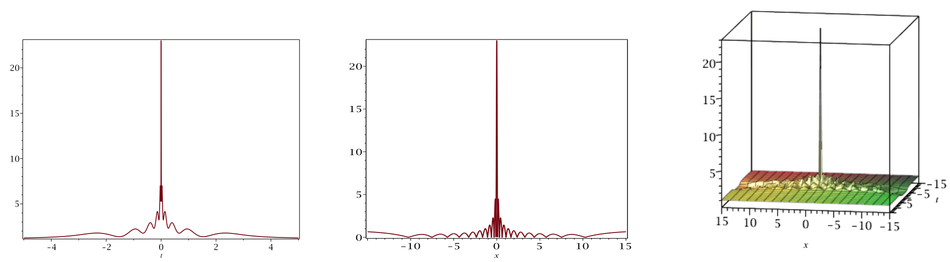

Figure 1. Solution to NLS, N=11, to the left $v(x, 0)$; in the center $v(0, t)$; to the right $v(x, t)$.

6 Conclusion

Here we proved the structure of quasi-rational solutions to the one dimensional focusing NLS equation at the order N. They can be expressed as a product of an exponential depending on t by a ratio of two polynomials of degree $N(N+1)$ in x and t.
If we choose $\tilde{a}_i = \tilde{b}_i = 0$ for $1 \leq i \leq N-1$, we obtain the classical (analogue) Peregrine breather. Thus these solutions appear as $2N-2$-parameters deformations of the Peregrine breather of order N.
Solutions of order 3 and 4 with respectively 4 and 6 parameters were first explicitly found in 2012 by Matveev using another method based on [23], but only published in 2013 in [22].
The solutions for orders 3 and 4 have also been explicitly found by the present author [24, 25]. The equivalence between with two types of solutions was made in [22] for the order 3; the equivalence between these solutions for the order 4 was made by the present author in [25].
We have also explicitly found the solutions at order 5 with 8 parameters [26]: these expressions are too extensive to be presented : it takes 14049 pages! For other orders 6, 7, 8, the solutions are also explicitly found but are too long to be published in any review. In the relative works [27, 28, 29] only the analysis has been done and figures of deformations of the Peregrine breathers has been

realized. The solutions for order 9 with 16 parameters and respectively for order 10 with 18 parameters are also completely found; we postpone to publish them in another publication in order not to weigh down the text of this present paper.
We can conclude that the method described in the present paper provides a very efficient and powerful tool to get explicit solutions to the NLS equation and to understand the behavior of rogue waves.
There are currently many applications in different fields as recent works by Akhmediev et al. [30] or Kibler et al. [31] attest it in particular.
This study leads to a better understanding of the phenomenon of rogue waves, and it would be relevant to go on with higher orders.

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