

Interval-valued 2-tuple prioritized aggregation operators and their application to multiple attribute group decision making

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ABSTRACT

Aims: The aim of this paper is to present some interval-valued 2-tuple prioritized aggregation operators for handling the multiple attribute group decision making problems where there exists a prioritization relationship over the attributes and decision makers.

Study design: We first develop two interval-valued 2-tuple prioritized aggregation operators called the interval-valued 2-tuple prioritized weighted average (IV2TPWA) operator and the interval-valued 2-tuple prioritized weighted geometric (IV2TPWG) operator.

Place and Duration of Study: We examine their desirable properties.

Methodology: The significant feature of these operators is that they not only deal with the linguistic and interval linguistic information but also take the prioritization relationship among the arguments into account.

Results: Next, an approach to multiple attribute group decision making with interval linguistic information is developed based on the IV2TPWA and the IV2TPWG operators.

Conclusion: Finally, a practical example is provided to illustrate our method.

Keywords: Multiple attribute group decision making; interval-valued 2-tuple prioritized weighted average (IV2TPWA) operator; interval-valued 2-tuple prioritized weighted geometric (IV2TPWG) operator.

1. INTRODUCTION

Multiple attribute group decision making (MAGDM) consists of finding the most desirable alternative(s) from a given alternative set according to the preferences provided by a group of experts [16]. For some MAGDM problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking [2,9,19]; thus, the decision information cannot be precisely assessed in a quantitative form. However, it may be appropriate and sufficient to assess the information in a qualitative form rather than a quantitative form. For example, when evaluating a house's cost, linguistic terms such as "high", "medium", and "low" are usually used, and when evaluating a house's design, linguistic terms like "good", "medium", and "bad" can be frequently used. Up to now, some methods have been developed for coping with linguistic information. These methods can be summarized as follows [22]: (1) The approximative computational model based on the Extension Principle [5]. This model transforms linguistic assessment information into fuzzy numbers and uses fuzzy arithmetic to make computations over these fuzzy numbers. The use of fuzzy arithmetic increases the vagueness. The results obtained by the fuzzy arithmetic are fuzzy numbers that usually do not match any linguistic term in the initial term set. (2) The ordinal linguistic computational model [7]. This model is also called symbolic

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model which makes direct computations on labels using the ordinal structure of the linguistic term sets. But symbolic method easily results in a loss of information caused by the use of the round operator. (3) The 2-tuple linguistic computational model [12,13,14,15]. This model uses the 2-tuple linguistic representation and computational model to make linguistic computations. (4) The model, which computes with words directly [24,26,27,28,29,30,31,32,33].

The 2-tuple fuzzy linguistic representation model represents the linguistic information by means of a pair of values called 2-tuple, composed by a linguistic term and a number. Meanwhile, the model also gives a computational technique to deal with the 2-tuples without loss of information. Since its introduction, the 2-tuple fuzzy linguistic representation model has received more and more attention. In a MAGDM problem involving the 2-tuple fuzzy linguistic representation model, in order to aggregate the individual linguistic preference information into the overall linguistic preference information, 2-tuple aggregation operators are most widely used. In the past few decades, many scholars have developed a variety of 2-tuple aggregation operators, such as 2-tuple arithmetic mean operator [12,14], 2-tuple weighted averaging operator [12], 2-tuple OWA operator [12], 2-tuple weighted geometric averaging (TWGA) operator [34], 2-tuple ordered weighted geometric averaging (TOWGA) operator [34], 2-tuple hybrid geometric averaging (THGA) operator [34], 2-tuple arithmetic average (TAA) operator [12], 2-tuple weighted average (TWA) operator [12], 2-tuple ordered weighted average (TOWA) operator [12], extended 2-tuple weighted average (ET-WA) operator [12], 2-tuple ordered weighted geometric (TOWG) operator [18], extended 2-tuple weighted geometric (ET-WG) operator [21], extended 2-tuple ordered weighted geometric (ET-OWG) operator [21], generalized 2-tuple weighted average (G-2TWA) operator [22], generalized 2-tuple ordered weighted average (G-2TOWA) operator [22], induced generalized 2-tuple ordered weighted average (IG-2TOWA) operator [22], 2-tuple linguistic power average (2TLPA) operator [36], 2-tuple linguistic weighted power average (2TLWPA) operator [36], and 2-tuple linguistic power ordered weighted average (2TLPOWA) operator [36].

The above MAGDMs are under the assumption that the attribute and the decision makers are at the same priority level respectively. In this situation, we have the ability to trade off between attributes. For instance, if C_i and C_j are two attributes with the weights w_i and w_j respectively, then we can compensate for a decrease of q in satisfaction to attribute C_i by

gain $\frac{w_j}{w_i}q$ in satisfaction to attribute C_j . However, in some MAGDM problems, we do not

want to allow this kind of compensation between attributes. For example, consider the situation in which we are buying a car based on the safety and cost of cars. We may not allow a benefit with respect to cost to compensate for a loss in safety. In this case we have a kind of prioritization of the attributes, i.e., safety has a higher priority than cost. Additionally, decision making in a company, general manager has a higher priority than vice manager. Yager [38] first investigated criteria aggregation problems where there is a prioritization relationship over the criteria. Then, Yager [39] and Yager et al. [40] gave much deeper insights into on this issue. Motivated by the ideas of Yager [38,39] and Yager et al. [40], Wei [23] generalized prioritized aggregation operators to hesitant fuzzy environment, proposed some hesitant fuzzy prioritized aggregation operators, and applied these operators to develop some models for hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level. Yu and Xu [42] investigated the prioritization relationship of attributes in multi-attribute decision making with intuitionistic fuzzy information and developed some prioritized intuitionistic fuzzy aggregation operators by extending the prioritized aggregation operators. Yu et al. [41] proposed some interval-valued intuitionistic fuzzy prioritized aggregation operators and investigated the application of these operators in the group decision making under interval-valued intuitionistic fuzzy environment in which the attributes and experts are in different priority level. However, the existing interval-valued ter2-

tuple aggregation operators are difficult to deal with the MAGDM where the attributes and the decision makers are in different priority level respectively. Moreover, we are conscious that there has been rather little work completed for using prioritized aggregation operators to solve the MAGDM with interval linguistic preference information. Thus, it is necessary to extend prioritized aggregation operators to the interval linguistic environment. To do it, in the current paper, we develop some interval valued 2-tuple prioritized aggregation operators. The prominent characteristic of these proposed operators is that they take prioritization among the attributes and the decision makers into account. Then, we utilize these operators to develop some approaches to the MAGDM where the attributes and the decision makers are in different priority level. Finally, some numerical examples are given to verify the practicality and effectiveness of the proposed operators and approaches.

This paper is organized as follows. Section 2 introduces some basic concepts of the 2-tuple fuzzy linguistic approach and the prioritized average operator. In Section 3, we propose the interval-valued 2-tuple prioritized weighted average (IV2TPWA) operator and the interval-valued 2-tuple prioritized weighted geometric (IV2TPWG) operator. Furthermore, we develop a method for MAGDM based on the proposed operators under the interval linguistic environment. In Section 4, an example concerning talent introduction is provided to demonstrate the practicality and effectiveness of the developed approach. The final section offers some concluding remarks.

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110 2. PRELIMINARIES

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112 In this section, we will introduce the basic notions of the 2-tuple fuzzy linguistic approach, the interval-valued 2-tuple fuzzy linguistic approach, and the prioritized average operator.

114 2.1. The 2-tuple fuzzy linguistic representation model

115 Let $S = \{s_i | i = 0, 1, 2, \dots, g\}$ be a finite and totally ordered discrete linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [12,13,14].

- 118 (1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;
- 119 (2) There is the negation operator: $neg(s_i) = s_j$ such that $j = g - i$;
- 120 (3) Max operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$;
- 121 (4) Min operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$.

122 For example, a set of seven terms S could be given as follows [1,6,8,10,11,20,25]:

123 $S = \{s_0 = \text{nothing}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{medium}, s_4 = \text{high}, s_5 = \text{very high}, s_6 = \text{perfect}\}.$

124 Based on the concept of symbolic translation, Herrera and Martinez [12,13] introduced a 2-tuple fuzzy linguistic representation model for dealing with linguistic information. This model represents the linguistic assessment information by means of a 2-tuple (s_i, α) , where $s_i \in S$ represents a linguistic label from the predefined linguistic term set S and $\alpha \in [-0.5, 0.5)$ is the value of symbolic translation.

129 **Definition 2.1 [12,13].** Let β be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set S , i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g+1$ the cardinality of S . Let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$ then α is called a symbolic translation, where $\text{round}(\cdot)$ is the usual round operation.

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Definition 2.2 [12,13]. Let $S = \{s_i | i = 0, 1, 2, \dots, g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta: [0, g] \rightarrow S \times [-0.5, 0.5]$$

(1)

$$\Delta(\beta) = (s_i, \alpha) \quad , \quad \text{with} \quad \begin{cases} s_i, & i = \text{round}(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5] \end{cases}$$

(2)

where s_i has the closest index label to β and α is the value of the symbolic translation.

Theorem 2.1 [12,13]. Let $S = \{s_i | i = 0, 1, 2, \dots, g\}$ be a linguistic term set and (s_i, α) be a 2-tuple. There is always a Δ^{-1} function such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, g] \subset R$, where

$$\Delta^{-1}: S \times [-0.5, 0.5] \rightarrow [0, g]$$

(3)

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$$

(4)

It is obvious that the conversion of a linguistic term into a linguistic 2-tuple consist of adding a value zero as symbolic translation

$$s_i \in S \Rightarrow (s_i, 0).$$

Definition 2.3 [12,13]. The comparison of linguistic information represented by 2-tuples is carried out according to an ordinary lexicographic order. Let (s_k, α_k) and (s_l, α_l) be two 2-tuples, with each one representing a counting of information as follows.

(1) If $k < l$ then (s_k, α_k) is smaller than (s_l, α_l) .

(2) If $k = l$ then

- if $\alpha_k = \alpha_l$ then $(s_k, \alpha_k), (s_l, \alpha_l)$ represents the same information;
- if $\alpha_k < \alpha_l$ then (s_k, α_k) is smaller than (s_l, α_l) ;
- if $\alpha_k > \alpha_l$ then (s_k, α_k) is bigger than (s_l, α_l) .

2.2. The interval-valued 2-tuple fuzzy linguistic representation model

Chen and Tai [3] proposed the generalized 2-tuple linguistic variable and translation function as follows:

Definition 2.4 [3]. Suppose a linguistic term set, $S = \{s_i | i = 0, 1, 2, \dots, g\}$ and the crisp value

β ($\beta \in [0, 1]$) can be

transformed into the 2-tuple linguistic variable by the following function:

$$\Delta(\beta) = (s_i, \alpha) \quad , \quad \text{with} \quad \begin{cases} s_i, & i = \text{round}(\beta * g) \\ \alpha = \beta - \frac{i}{g}, & \alpha \in \left[\frac{-0.5}{g}, \frac{0.5}{g} \right] \end{cases}$$

(5)

On the contrary, the 2-tuple linguistic variable can be converted into the crisp value β ($\beta \in [0, 1]$) as follows:

$$\Delta^{-1}(s_i, \alpha) = \frac{i}{g} + \alpha = \beta$$

(6)

Different from the value of β in Definition 2.2, the value of β in Definition 2.4 ranges from 0 to 1. That is, the 2-tuple linguistic variable of Definition 2.4 is standardized, which brings big convenience to comparison between 2-tuples from different multi-granularity linguistic term sets [43]. In this section, without further notification, both the linguistic term set and the 2-tuple linguistic variable satisfy the requirements of Definition 2.4.

The existing aggregation operators involving 2-tuples mainly focus on usual 2-tuples. If the 2-tuples are from different linguistic term sets, they cannot be aggregated directly and tedious transformation should be conducted before aggregation operation. To avoid tedious calculation, Zhang [43] introduced the concept of an interval-valued 2-tuple linguistic variable and developed some aggregation operators with interval-valued 2-tuple linguistic information.

Definition 2.5 [43]. Suppose a linguistic term set, $S = \{s_i | i = 0, 1, 2, \dots, g\}$. An interval-valued 2-tuple is composed of two linguistic terms and two numbers, denoted by $[(s_i, \alpha_1), (s_j, \alpha_2)]$, where $i \leq j$, $s_i(s_j)$ and $\alpha_1(\alpha_2)$ represent the linguistic label of the predefined linguistic term set S and symbolic translation, respectively. An interval-valued 2-tuple linguistic variable can be converted into an interval value $[\beta_1, \beta_2]$ ($\beta_1, \beta_2 \in [0, 1], \beta_1 \leq \beta_2$) as follows:

$$\Delta^{-1}([(s_i, \alpha_1), (s_j, \alpha_2)]) = [i/g + \alpha_1, j/g + \alpha_2] = [\beta_1, \beta_2]$$

(7)

On the contrary, the interval value $[\beta_1, \beta_2]$ ($\beta_1, \beta_2 \in [0, 1], \beta_1 \leq \beta_2$) can be transformed into the interval-valued 2-tuple linguistic variable by the following function:

$$\Delta([\beta_1, \beta_2]) = [(s_i, \alpha_1), (s_j, \alpha_2)] \quad \text{with} \quad \begin{cases} s_i, & i = \text{round}(\beta_1 * g) \\ s_j, & j = \text{round}(\beta_2 * g) \\ \alpha_1 = \beta_1 - i/g, & \alpha_1 \in [-0.5/g, 0.5/g) \\ \alpha_2 = \beta_2 - j/g, & \alpha_2 \in [-0.5/g, 0.5/g). \end{cases}$$

(8)

Specially, if $s_i = s_j$ and $\alpha_1 = \alpha_2$, then the interval-valued 2-tuple linguistic variable reduces to a 2-tuple linguistic variable.

Zhang [43] extended the comparison of two 2-tuples to the interval-valued case, and proposed the score and accuracy functions [4, 17] for comparison between interval-valued 2-tuples.

Definition 2.6 [43]. For an interval-valued 2-tuple $A = [(s_i, \alpha_1), (s_j, \alpha_2)]$, its score function is expressed by the following formula

$$S(A) = \frac{i+j}{2g} + \frac{\alpha_1 + \alpha_2}{2}$$

(9)

where $g+1$ is the cardinality of S , $S = \{s_i | i = 0, 1, 2, \dots, g\}$.

It is easy to prove that $0 \leq S(A) \leq 1$. The score function is regarded as a basis to compare two interval-valued 2-tuples. For two interval-valued 2-tuples, the one with a larger score function corresponds to a larger interval-valued 2-tuple. However, it is really possible that

two different interval-valued 2-tuples may have an identical score value. In that event, accuracy function should be brought into consideration [43].

Definition 2.7 [43]. For an interval-valued 2-tuple $A = [(s_i, \alpha_1), (s_j, \alpha_2)]$, its accuracy function is expressed by the following function

$$H(A) = \frac{j-i}{g} + \alpha_2 - \alpha_1$$

(10)

It is obvious that $0 \leq H(A) \leq 1$. For two interval-valued 2-tuples with the same score function, the smaller the accuracy function, the larger the corresponding interval-valued 2-tuple.

Theorem 2.2 [43]. Let $A = [(s_i, \alpha_1), (s_j, \alpha_2)]$ and $B = [(s'_i, \alpha'_1), (s'_j, \alpha'_2)]$ be two interval-valued 2-tuples:

If $S(A) > S(B)$, then $A > B$;

If $S(A) < S(B)$, then $A < B$;

If $S(A) = S(B)$, then:

(1) If $H(A) > H(B)$, then $A < B$;

(2) If $H(A) < H(B)$, then $A > B$;

(3) If $H(A) = H(B)$, then $A = B$.

2.3. Prioritized average (PA) operators

The prioritized average (PA) operator was originally introduced by Yager [37,38], which was defined as follows:

Definition 2.8 [38]. Let $C = \{C_1, C_2, \dots, C_n\}$ be a collection of criteria and that there is a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ C_3 \dots \succ C_n$, indicate criteria C_j has a higher priority than C_k if $j < k$. The value $C_j(x)$ is the performance of any alternative x under criteria C_j , and satisfies $C_j(x) \in [0, 1]$. If

$$PA(C_i(x)) = \sum_{j=1}^n w_j C_j(x) \quad ,$$

(11)

where $w_j = \frac{T_j}{\sum_{j=1}^n T_j}$, $T_j = \prod_{k=1}^{j-1} C_k(x)$ ($j = 2, \dots, n$), $T_1 = 1$. Then PA is called the prioritized

average (PA) operator.

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3. INTERVAL-VALUED 2-TUPLE PRIORITIZED AGGREGATION OPERATORS

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3.1. Interval-valued 2-tuple prioritized weighted average (IV2TPWA) operators

In the following, we extend the PA operator to the interval-valued 2-tuple linguistic environment and define an IV2TPWA operator as follows:

Definition 3.1. Let $\{[(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]\}$ be a set of interval-valued 2-tuples, if

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$$\begin{aligned} & \text{IV2TPWA} \left(\left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right) \\ &= \Delta \left(\left[\sum_{i=1}^n \left(\frac{T_i}{\sum_{j=1}^n T_j} \Delta^{-1}(s_i, \alpha_i) \right), \sum_{i=1}^n \left(\frac{T'_i}{\sum_{j=1}^n T'_j} \Delta^{-1}(s'_i, \alpha'_i) \right) \right] \right) \end{aligned}$$

(12)

where $T_1 = T'_1 = 1$, $T_j = \prod_{k=1}^{j-1} \Delta^{-1}(s_k, \alpha_k)$ ($j = 2, \dots, n$), and $T'_j = \prod_{k=1}^{j-1} \Delta^{-1}(s'_k, \alpha'_k)$ ($j = 2, \dots, n$), then

IV2TPWA is called an interval-valued 2-tuple prioritized weighted average (IV2TPWA) operator.

Theorem 3.1 (Boundedness). Let $\left\{ \left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right\}$ be a set of interval-valued 2-tuples, then

$$\begin{aligned} & \left[\min_{1 \leq i \leq n} \{ (s_i, \alpha_i) \}, \min_{1 \leq i \leq n} \{ (s'_i, \alpha'_i) \} \right] \\ & \leq \text{IV2TPWA} \left(\left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right) \\ & \leq \left[\max_{1 \leq i \leq n} \{ (s_i, \alpha_i) \}, \max_{1 \leq i \leq n} \{ (s'_i, \alpha'_i) \} \right]. \end{aligned}$$

(13)

Theorem 3.2 (Idempotency). Let $\left\{ \left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right\}$ be a set of interval-valued 2-tuples. If all $\left[(s_j, \alpha_j), (s'_j, \alpha'_j) \right]$ ($j = 1, 2, \dots, n$) are equal, i.e., $\left[(s_j, \alpha_j), (s'_j, \alpha'_j) \right] = \left[(s, \alpha), (s', \alpha') \right]$, for all j , then

$$\text{IV2TPWA} \left(\left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right) = \left[(s, \alpha), (s', \alpha') \right].$$

(14)

Theorem 3.3 (Monotonicity). Let $\left\{ \left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right\}$ and $\left\{ \left[(s_1^*, \alpha_1^*), (s'_1, \alpha'_1) \right], \left[(s_2^*, \alpha_2^*), (s'_2, \alpha'_2) \right], \dots, \left[(s_n^*, \alpha_n^*), (s'_n, \alpha'_n) \right] \right\}$ be two set of interval-valued 2-tuples, if $(s_j^*, \alpha_j^*) \geq (s_j, \alpha_j)$ and $(s'_j, \alpha'_j) \geq (s'_j, \alpha'_j)$, for all j , then

$$\begin{aligned} & \text{IV2TPWA} \left(\left[(s_1^*, \alpha_1^*), (s'_1, \alpha'_1) \right], \left[(s_2^*, \alpha_2^*), (s'_2, \alpha'_2) \right], \dots, \left[(s_n^*, \alpha_n^*), (s'_n, \alpha'_n) \right] \right) \\ & \geq \text{IV2TPWA} \left(\left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right). \end{aligned}$$

(15)

3.2. Interval-valued 2-tuple prioritized weighted geometric (IV2TPWG) operators

Based on the IV2TPWA operator and the geometric mean, here we define an interval-valued 2-tuple prioritized weighted geometric (IV2TPWG) operator:

Definition 3.2. Let $\left\{ \left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right\}$ be a set of interval-valued 2-tuples, if

$$\begin{aligned} & \text{IV2TPWG} \left(\left[(s_1, \alpha_1), (s'_1, \alpha'_1) \right], \left[(s_2, \alpha_2), (s'_2, \alpha'_2) \right], \dots, \left[(s_n, \alpha_n), (s'_n, \alpha'_n) \right] \right) \\ &= \Delta \left(\left[\prod_{i=1}^n \left(\Delta^{-1}(s_i, \alpha_i) \right)^{T_i / \sum_{j=1}^n T_j}, \prod_{i=1}^n \left(\Delta^{-1}(s'_i, \alpha'_i) \right)^{T'_i / \sum_{j=1}^n T'_j} \right] \right), \end{aligned}$$

(16)

where $T_1 = T'_1 = 1$, $T_j = \prod_{k=1}^{j-1} \Delta^{-1}(s_k, \alpha_k)$ ($j = 2, \dots, n$), and $T'_j = \prod_{k=1}^{j-1} \Delta^{-1}(s'_k, \alpha'_k)$ ($j = 2, \dots, n$), then IV2TPWG is called an interval-valued 2-tuple prioritized weighted geometric (IV2TPWG) operator.

Theorem 3.4 (Boundedness). Let $\{[(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]\}$ be a set of interval-valued 2-tuples, then

$$\begin{aligned} & \left[\min_{1 \leq i \leq n} \{(s_i, \alpha_i)\}, \min_{1 \leq i \leq n} \{(s'_i, \alpha'_i)\} \right] \\ & \leq \text{IV2TPWG}([(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]) \\ & \leq \left[\max_{1 \leq i \leq n} \{(s_i, \alpha_i)\}, \max_{1 \leq i \leq n} \{(s'_i, \alpha'_i)\} \right]. \end{aligned}$$

(17)

Theorem 3.5 (Idempotency). Let $\{[(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]\}$ be a set of interval-valued 2-tuples. If all $[(s_j, \alpha_j), (s'_j, \alpha'_j)]$ ($j = 1, 2, \dots, n$) are equal, i.e., $[(s_j, \alpha_j), (s'_j, \alpha'_j)] = [(s, \alpha), (s', \alpha')]$, for all j , then

$$\text{IV2TPWG}([(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]) = [(s, \alpha), (s', \alpha')].$$

(18)

Theorem 3.6 (Monotonicity). Let $\{[(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]\}$ and $\{[(s_1^*, \alpha_1^*), (s'^*_1, \alpha'^*_1)], [(s_2^*, \alpha_2^*), (s'^*_2, \alpha'^*_2)], \dots, [(s_n^*, \alpha_n^*), (s'^*_n, \alpha'^*_n)]\}$ be two set of interval-valued 2-tuples, if $(s_j^*, \alpha_j^*) \geq (s_j, \alpha_j)$ and $(s'^*_j, \alpha'^*_j) \geq (s'_j, \alpha'_j)$, for all j , then

$$\begin{aligned} & \text{IV2TPWG}([(s_1^*, \alpha_1^*), (s'^*_1, \alpha'^*_1)], [(s_2^*, \alpha_2^*), (s'^*_2, \alpha'^*_2)], \dots, [(s_n^*, \alpha_n^*), (s'^*_n, \alpha'^*_n)]) \\ & \geq \text{IV2TPWG}([(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]). \end{aligned}$$

(19)

Theorem 3.7. Let $\{[(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]\}$ be a set of interval-valued 2-tuples, then we have

$$\begin{aligned} & \text{IV2TPWG}([(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]) \\ & \leq \text{IV2TPWA}([(s_1, \alpha_1), (s'_1, \alpha'_1)], [(s_2, \alpha_2), (s'_2, \alpha'_2)], \dots, [(s_n, \alpha_n), (s'_n, \alpha'_n)]). \end{aligned}$$

(20)

Theorem 3.7 shows that the values obtained by the IV2TPWG operator are not bigger than the ones obtained by the IV2TPWA operator.

3.3. An approach to multiple attribute group decision making based on interval-valued 2-tuple prioritized aggregation operators

In the present subsection, we aim to solve a multiple attribute group decision making problem, where the attribute values are represented by interval linguistic variables and there exists the prioritization relationships over the attributes and decision makers.

A group decision making problem with interval linguistic preference information in which the attributes and decision makers are in different priority level can be described as follows: Let $X = \{x_1, x_2, \dots, x_m\}$ be the set of alternatives. Let $C = \{c_1, c_2, \dots, c_n\}$ be a collection of attributes and that there is a prioritization between the attributes expressed by the linear ordering $c_1 \succ c_2 \succ c_3 \succ \dots \succ c_n$, indicate attribute c_j has a higher priority than c_k if $j < k$. Let

303 $D = \{d_1, d_2, \dots, d_l\}$ is the set of decision makers and that there is a prioritization between the
 304 decision makers expressed by the linear ordering $d_1 \succ d_2 \succ d_3 \succ \dots \succ d_l$, indicate decision
 305 maker d_p has a higher priority than d_q if $p < q$. The decision makers $d_k \in D$ ($k = 1, 2, \dots, l$)
 306 provide their preference values for each alternative on each attribute and construct the
 307 interval linguistic decision matrices $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n}$ ($k = 1, 2, \dots, l$), where $\tilde{r}_{ij}^{(k)}$ takes the form of
 308 interval linguistic variable $[s_{ij}^{(k)}, s_{ij}'^{(k)}]$, with $s_{ij}^{(k)}, s_{ij}'^{(k)} \in S$, $S = \{s_i | i = 0, 1, 2, \dots, g\}$, and $s_{ij}^{(k)} \leq s_{ij}'^{(k)}$.

309 To get the best alternative(s), we next develop an approach based on the interval-valued
 310 2-tuple prioritized aggregation operators to multiple attribute group decision making with
 311 interval linguistic preference information. The proposed approach involves the following
 312 steps:

313 **Step 1.** Transform the interval linguistic decision matrix $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n} = ([s_{ij}^{(k)}, s_{ij}'^{(k)}])_{m \times n}$
 314 ($k = 1, 2, \dots, l$) into the interval-valued 2-tuple linguistic decision matrix
 315 $\bar{\tilde{R}}^{(k)} = (\bar{\tilde{r}}_{ij}^{(k)})_{m \times n} = ([([s_{ij}^{(k)}, 0], [s_{ij}'^{(k)}, 0])])_{m \times n}$ ($k = 1, 2, \dots, l$).

316 **Step 2.** Calculate the matrices $T^{(p)} = (T_{ij}^{(p)})_{m \times n}$ ($p = 1, 2, \dots, l$) and
 317 $T'^{(p)} = (T'_{ij}{}^{(p)})_{m \times n}$ ($p = 1, 2, \dots, l$) based on the following equations:

$$318 \quad T_{ij}^{(p)} = \prod_{k=1}^{p-1} (\Delta^{-1}(s_{ij}^{(k)}, 0)) \quad , \quad p = 2, \dots, l \quad , \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad ,$$

319 (21)

$$320 \quad T'_{ij}{}^{(p)} = \prod_{k=1}^{p-1} (\Delta^{-1}(s_{ij}'^{(k)}, 0)) \quad , \quad p = 2, \dots, l \quad , \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad ,$$

321 (22)

$$322 \quad T_{ij}^{(1)} = T'_{ij}{}^{(1)} = 1 \quad , \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad .$$

323 (23)

324 **Step 3.** Utilize the IV2TPWA operator:

$$325 \quad \text{IV2TPWA} \left([([s_{ij}^{(1)}, 0], [s_{ij}'^{(1)}, 0])], [([s_{ij}^{(2)}, 0], [s_{ij}'^{(2)}, 0])], \dots, [([s_{ij}^{(l)}, 0], [s_{ij}'^{(l)}, 0])]\right) \\ = \Delta \left(\left[\sum_{k=1}^l \frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \Delta^{-1}(s_{ij}^{(k)}, 0) \right], \left[\sum_{k=1}^l \frac{T'_{ij}{}^{(k)}}{\sum_{p=1}^l T'_{ij}{}^{(p)}} \Delta^{-1}(s_{ij}'^{(k)}, 0) \right] \right)$$

326 (24)

327 or the IV2TPWG operator:

$$328 \quad \text{IV2TPWG} \left([([s_{ij}^{(1)}, 0], [s_{ij}'^{(1)}, 0])], [([s_{ij}^{(2)}, 0], [s_{ij}'^{(2)}, 0])], \dots, [([s_{ij}^{(l)}, 0], [s_{ij}'^{(l)}, 0])]\right) \\ = \Delta \left(\left[\prod_{k=1}^l \left(\Delta^{-1}(s_{ij}^{(k)}, 0) \right)^{T_{ij}^{(k)} / \sum_{p=1}^l T_{ij}^{(p)}} \right], \prod_{k=1}^l \left(\Delta^{-1}(s_{ij}'^{(k)}, 0) \right)^{T'_{ij}{}^{(k)} / \sum_{p=1}^l T'_{ij}{}^{(p)}} \right]$$

329 (25)

330 to aggregate all the individual interval-valued 2-tuple linguistic decision matrices

331 $\bar{\tilde{R}}^{(k)} = (\bar{\tilde{r}}_{ij}^{(k)})_{m \times n} = ([([s_{ij}^{(k)}, 0], [s_{ij}'^{(k)}, 0])])_{m \times n}$ ($k = 1, 2, \dots, l$) into the collective interval-valued 2-tuple

332 linguistic decision matrix $\bar{\tilde{R}} = (\bar{\tilde{r}}_{ij})_{m \times n} = ([([s_{ij}, \alpha_{ij}], [s_{ij}', \alpha_{ij}']])_{m \times n}$.

333 **Step 4.** Calculate the matrix $T = (T_{ij})_{m \times n}$ and $T' = (T'_{ij})_{m \times n}$ based on following equations:

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$$T_{ij} = \prod_{k=1}^{j-1} (\Delta^{-1}(s_{ik}, \alpha_{ik})) \quad (i = 1, 2, \dots, m, j = 2, \dots, n) \quad ,$$

335 (26)

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$$T'_{ij} = \prod_{k=1}^{j-1} (\Delta^{-1}(s'_{ik}, \alpha'_{ik})) \quad (i = 1, 2, \dots, m, j = 2, \dots, n) \quad ,$$

337 (27)

338
$$T_{i1} = T'_{i1} = 1 \quad (i = 1, 2, \dots, m) \quad .$$

339 (28)

340 **Step 5.** Utilize the IV2TPWA operator:

341
$$\begin{aligned} \bar{\tilde{r}}_i &= [(s_i, \alpha_i), (s'_i, \alpha'_i)] \\ &= \text{IV2TPWA}([(s_{i1}, \alpha_{i1}), (s'_{i1}, \alpha'_{i1})], [(s_{i2}, \alpha_{i2}), (s'_{i2}, \alpha'_{i2})], \dots, [(s_{in}, \alpha_{in}), (s'_{in}, \alpha'_{in})]) \\ &= \Delta \left(\left[\frac{\sum_{k=1}^n T_{ik} \Delta^{-1}(s_{ik}, \alpha_{ik})}{\sum_{j=1}^n T_{ij}}, \frac{\sum_{k=1}^n T'_{ik} \Delta^{-1}(s'_{ik}, \alpha'_{ik})}{\sum_{j=1}^n T'_{ij}} \right] \right) \end{aligned}$$

342 (29)

343 or the IV2TPWG operator:

344
$$\begin{aligned} \bar{\tilde{r}}_i &= [(s_i, \alpha_i), (s'_i, \alpha'_i)] \\ &= \text{IV2TPWG}([(s_{i1}, \alpha_{i1}), (s'_{i1}, \alpha'_{i1})], [(s_{i2}, \alpha_{i2}), (s'_{i2}, \alpha'_{i2})], \dots, [(s_{in}, \alpha_{in}), (s'_{in}, \alpha'_{in})]) \\ &= \Delta \left(\left[\prod_{k=1}^n \left(\Delta^{-1}(s_{ik}, \alpha_{ik}) \right)^{T_{ik} / \sum_{j=1}^n T_{ij}}, \prod_{k=1}^n \left(\Delta^{-1}(s'_{ik}, \alpha'_{ik}) \right)^{T'_{ik} / \sum_{j=1}^n T'_{ij}} \right] \right) \end{aligned}$$

345 (30)

346 to derive the collective overall preference value $\bar{\tilde{r}}_i = [(s_i, \alpha_i), (s'_i, \alpha'_i)]$ of the alternative x_i .

347 **Step 6.** According to Theorem 2.2, rank all the alternatives x_i ($i = 1, 2, \dots, m$) and select the

348 best one(s) in accordance with the collective overall preference values $\bar{\tilde{r}}_i = [(s_i, \alpha_i), (s'_i, \alpha'_i)]$

349 ($i = 1, 2, \dots, m$).

350 **Step 7.** End.

351

352 4. AN ILLUSTRATIVE EXAMPLE

353

354 In this section, let us consider an illustrative example adapted from Herrera et al. [15], and
355 Herrera and Herrera-Viedma [13].

356 **Example 4.1 [13,15].** Suppose that an investment company wants to invest a sum of money
357 in the best option. There is a panel with four possible alternatives in which to invest the
358 money: (1) x_1 is a car industry; (2) x_2 is a food company; (3) x_3 is a computer company;
359 and (4) x_4 is an arms industry. The investment company must make a decision according to
360 the following four attributes: (1) c_1 is the risk analysis; (2) c_2 is the growth analysis; (3) c_3 is
361 the social-political impact analysis; and (4) c_4 is the environmental impact analysis. The four
362 possible alternatives x_i ($i = 1, 2, 3, 4$) are to be evaluated using the linguistic term set

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$$S = \left\{ \begin{array}{l} s_0 = \text{extremely poor, } s_1 = \text{very poor, } s_2 = \text{poor, } s_3 = \text{slightly poor, } s_4 = \text{fair, } \\ s_5 = \text{slightly good, } s_6 = \text{good, } s_7 = \text{very good, } s_8 = \text{extremely good} \end{array} \right\}$$

	c_1	c_2	c_3	c_4
x_1	$[s_7, s_8]$	$[s_5, s_6]$	$[s_3, s_4]$	$[s_3, s_5]$
x_2	$[s_3, s_6]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_6, s_8]$
x_3	$[s_1, s_3]$	$[s_2, s_3]$	$[s_3, s_4]$	$[s_4, s_5]$
x_4	$[s_4, s_5]$	$[s_1, s_3]$	$[s_1, s_2]$	$[s_6, s_8]$

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ers d_k ($k = 1, 2, 3$) under the above four attributes, and construct, respectively, the interval linguistic decision matrices $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n} = ([s_{ij}^{(k)}, s_{ij}^{r(k)}])_{m \times n}$ ($k = 1, 2, 3$) as shown in Tables 1-3.

The decision maker d_1 has the absolute priority for decision making, the decision maker d_2 comes next. That is, there is a prioritization between three decision makers expressed by the linear ordering $d_1 \succ d_2 \succ d_3$. In three decision makers' opinion, there exists the prioritization relationship among these attributes, for example, the risk analysis of the candidate is the most important, but the environmental impact analysis of the candidate is not so important comparing with other attributes. Therefore, the prioritization relationship can be denoted by: $c_1 \succ c_2 \succ c_3 \succ c_4$.

Table 1. Decision matrix $\tilde{R}^{(1)}$ provided by d_1 .

Table 2. Decision matrix $\tilde{R}^{(2)}$ provided by d_2 .

1	c_1	c_2	c_3	c_4
x_1	$[s_1, s_4]$	$[s_2, s_3]$	$[s_2, s_3]$	$[s_5, s_7]$
x_2	$[s_1, s_2]$	$[s_3, s_4]$	$[s_3, s_6]$	$[s_6, s_8]$
x_3	$[s_5, s_8]$	$[s_1, s_2]$	$[s_1, s_3]$	$[s_5, s_6]$
x_4	$[s_3, s_4]$	$[s_1, s_2]$	$[s_1, s_3]$	$[s_7, s_8]$

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Table 3. Decision matrix $\tilde{R}^{(3)}$ provided by d_3 .

Step 1.
Transform the interval-valued linguistic uisti

3	c_1	c_2	c_3	c_4
x_1	$[s_1, s_3]$	$[s_1, s_2]$	$[s_1, s_3]$	$[s_5, s_7]$
x_2	$[s_6, s_7]$	$[s_6, s_8]$	$[s_5, s_6]$	$[s_3, s_4]$
x_3	$[s_7, s_8]$	$[s_1, s_2]$	$[s_7, s_8]$	$[s_2, s_3]$
x_4	$[s_3, s_5]$	$[s_5, s_8]$	$[s_5, s_7]$	$[s_6, s_8]$

convert decision matrices $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{4 \times 4} = ([s_{ij}^{(k)}, s_{ij}'^{(k)}])_{4 \times 4}$ ($k=1,2,3$) given in Tables 1-3 into the interval-valued 2-tuple linguistic decision matrices $\bar{\tilde{R}}^{(k)} = (\bar{\tilde{r}}_{ij}^{(k)})_{4 \times 4} = ([(s_{ij}^{(k)}, 0), (s_{ij}'^{(k)}, 0)])_{4 \times 4}$ ($k=1,2,3$) which are given in Tables 4-6.

Table 4. Interval-valued 2-tuple linguistic decision matrix $\bar{\tilde{R}}^{(1)}$.

4	c_1	c_2	c_3	c_4
x_1	$[(s_1, 0), (s_4, 0)]$	$[(s_2, 0), (s_3, 0)]$	$[(s_2, 0), (s_3, 0)]$	$[(s_5, 0), (s_7, 0)]$
x_2	$[(s_1, 0), (s_2, 0)]$	$[(s_3, 0), (s_4, 0)]$	$[(s_3, 0), (s_6, 0)]$	$[(s_6, 0), (s_8, 0)]$
x_3	$[(s_5, 0), (s_8, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_1, 0), (s_3, 0)]$	$[(s_5, 0), (s_6, 0)]$
x_4	$[(s_3, 0), (s_4, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_1, 0), (s_3, 0)]$	$[(s_7, 0), (s_8, 0)]$

Table 5. Interval-valued 2-tuple linguistic decision matrix $\bar{\tilde{R}}^{(2)}$.**Table 6.** Interval-valued 2-tuple linguistic decision matrix $\bar{\tilde{R}}^{(3)}$.

5	c_1	c_2	c_3	c_4
x_1	$[(s_7, 0), (s_8, 0)]$	$[(s_5, 0), (s_6, 0)]$	$[(s_3, 0), (s_4, 0)]$	$[(s_3, 0), (s_5, 0)]$
x_2	$[(s_3, 0), (s_6, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_6, 0), (s_8, 0)]$
x_3	$[(s_1, 0), (s_3, 0)]$	$[(s_2, 0), (s_3, 0)]$	$[(s_3, 0), (s_4, 0)]$	$[(s_4, 0), (s_5, 0)]$
x_4	$[(s_4, 0), (s_5, 0)]$	$[(s_1, 0), (s_3, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_6, 0), (s_8, 0)]$

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6	c_1	c_2	c_3	c_4
x_1	$[(s_1, 0), (s_3, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_1, 0), (s_3, 0)]$	$[(s_5, 0), (s_7, 0)]$
x_2	$[(s_6, 0), (s_7, 0)]$	$[(s_6, 0), (s_8, 0)]$	$[(s_5, 0), (s_6, 0)]$	$[(s_3, 0), (s_4, 0)]$
x_3	$[(s_7, 0), (s_8, 0)]$	$[(s_1, 0), (s_2, 0)]$	$[(s_7, 0), (s_8, 0)]$	$[(s_2, 0), (s_3, 0)]$
x_4	$[(s_3, 0), (s_5, 0)]$	$[(s_5, 0), (s_8, 0)]$	$[(s_5, 0), (s_7, 0)]$	$[(s_6, 0), (s_8, 0)]$

435 **Step 2.** Utilize Eqs. (21), (22), and (23) to calculate the matrices $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, $T^{r(1)}$, $T^{r(2)}$,
436 and $T^{r(3)}$ as follows:
437

$$438 \quad T^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad T^{(2)} = \begin{pmatrix} 0.1250 & 0.2500 & 0.2500 & 0.6250 \\ 0.1250 & 0.3750 & 0.3750 & 0.7500 \\ 0.6250 & 0.1250 & 0.1250 & 0.6250 \\ 0.3750 & 0.1250 & 0.1250 & 0.8750 \end{pmatrix},$$

$$439 \quad T^{(3)} = \begin{pmatrix} 0.1094 & 0.1563 & 0.0938 & 0.2344 \\ 0.0469 & 0.0469 & 0.0469 & 0.5625 \\ 0.0781 & 0.0313 & 0.0469 & 0.3125 \\ 0.1875 & 0.0156 & 0.0156 & 0.6563 \end{pmatrix},$$

$$440 \quad T^{r(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad T^{r(2)} = \begin{pmatrix} 0.5000 & 0.3750 & 0.3750 & 0.8750 \\ 0.2500 & 0.5000 & 0.7500 & 1.0000 \\ 1.0000 & 0.2500 & 0.3750 & 0.7500 \\ 0.5000 & 0.2500 & 0.3750 & 1.0000 \end{pmatrix},$$

$$441 \quad T^{r(3)} = \begin{pmatrix} 0.1094 & 0.1563 & 0.0938 & 0.2344 \\ 0.0469 & 0.0469 & 0.0469 & 0.5625 \\ 0.0781 & 0.0313 & 0.0469 & 0.3125 \\ 0.1875 & 0.0156 & 0.0156 & 0.6563 \end{pmatrix}.$$

442 **Step 3.** Utilize the IV2TPWA operator (Eq. (24) to aggregate all the individual interval-valued
443 2-tuple linguistic decision matrices $\bar{R}^{(k)} = (\bar{r}_{ij}^{(k)})_{4 \times 4} = ([(s_{ij}^{(k)}, 0), (s_{ij}^{r(k)}, 0)])_{4 \times 4}$ ($k=1,2,3$) into the
444 collective interval-valued 2-tuple linguistic decision matrix $\bar{\bar{R}} = (\bar{\bar{r}}_{ij})_{4 \times 4} = ([(s_{ij}, \alpha_{ij}), (s'_{ij}, \alpha'_{ij})])_{4 \times 4}$
445 (see Table 7).

446 **Table 7.** The collective interval-valued 2-tuple linguistic decision matrix $\bar{\bar{R}}$.

7	c_1	c_2	c_3	c_4
x_1	$[(s_2, -0.0491), (s_5, -0.0313)]$	$[(s_2, 0.0528), (s_4, -0.0613)]$	$[(s_2, 0.0145), (s_3, 0.0300)]$	$[(s_4, 0.0410), (s_6, 0.0347)]$
x_2	$[(s_1, 0.0517), (s_3, 0.0435)]$	$[(s_3, -0.0536), (s_4, -0.0385)]$	$[(s_3, -0.0577), (s_4, 0.0565)]$	$[(s_5, 0.0338), (s_7, -0.0417)]$
x_3	$[(s_4, -0.0470), (s_6, -0.0132)]$	$[(s_1, 0.0135), (s_2, 0.0233)]$	$[(s_1, 0.0567), (s_4, -0.0200)]$	$[(s_4, 0.0242), (s_5, 0.0035)]$
x_4	$[(s_3, 0.0300), (s_4, 0.0560)]$	$[(s_1, 0.0068), (s_3, -0.0494)]$	$[(s_1, 0.0068), (s_3, -0.0000)]$	$[(s_6, 0.0494), (s_8, 0)]$

447 **Step 4.** Calculate the matrix $T = (T_{ij})_{4 \times 4}$ and $T' = (T'_{ij})_{4 \times 4}$ based on Eqs. (26), (27), and (28):

448
$$T = \begin{pmatrix} 1 & 0.2009 & 0.0608 & 0.0161 \\ 1 & 0.1767 & 0.0568 & 0.0180 \\ 1 & 0.4530 & 0.0627 & 0.0114 \\ 1 & 0.4050 & 0.0534 & 0.0070 \end{pmatrix}, \quad T' = \begin{pmatrix} 1 & 0.5938 & 0.2605 & 0.1055 \\ 1 & 0.4185 & 0.1931 & 0.1075 \\ 1 & 0.7368 & 0.2013 & 0.0966 \\ 1 & 0.5560 & 0.1810 & 0.0679 \end{pmatrix}.$$

449 **Step 5.** Utilize the IV2TPWA operator (Eq. (29)) to derive the collective overall preference
450 value $\bar{r}_i = [(s_i, \alpha_i), (s'_i, \alpha'_i)]$ of the alternative x_i .

451
$$\bar{r}_1 = [(s_2, -0.0257), (s_4, 0.0320)], \bar{r}_2 = [(s_2, -0.0396), (s_4, -0.0296)],$$

452
$$\bar{r}_3 = [(s_3, -0.0259), (s_4, 0.0384)], \bar{r}_4 = [(s_3, -0.0535), (s_4, -0.0164)].$$

453 **Step 6.** According to Eq. (9), we calculate the score function $S(\bar{r}_i)$ ($i=1,2,3,4$) of $\bar{r}_i$
454 ($i=1,2,3,4$) as follows:

455
$$S(\bar{r}_1) = 0.3781, \quad S(\bar{r}_2) = 0.3404, \quad S(\bar{r}_3) = 0.4437, \quad S(\bar{r}_4) = 0.4025.$$

456 Because $S(\bar{r}_3) > S(\bar{r}_4) > S(\bar{r}_1) > S(\bar{r}_2)$, by Theorem 2.2, we have that $\bar{r}_3 > \bar{r}_4 > \bar{r}_1 > \bar{r}_2$, which
457 implies that $x_3 \succ x_4 \succ x_1 \succ x_2$. Thus, the best candidate is x_3 .

458 If we deal with Example 4.1 using the IV2TPWG operator instead of the IV2TPWA
459 operator, then the main steps are shown as follows:

460 **Step 1'**: See Step 1.

461 **Step 2'**: See Step 2.

462 **Step 3'**. Utilize the IV2TPWG operator (Eq. (25)) to aggregate all the individual interval-
463 valued 2-tuple linguistic decision matrices $\bar{R}^{(k)} = (\bar{r}_{ij}^{(k)})_{4 \times 4} = \left([(s_{ij}^{(k)}, 0), (s_{ij}'^{(k)}, 0)] \right)_{4 \times 4}$ ($k=1,2,3$) into
464 the collective interval-valued 2-tuple linguistic decision matrix
465 $\bar{R}^* = (\bar{r}_{ij}^*)_{4 \times 4} = \left([(s_{ij}^*, \alpha_{ij}^*), (s_{ij}'^*, \alpha_{ij}'^*)] \right)_{4 \times 4}$ (see Table 8).

466 **Table 8.** The collective interval-valued 2-tuple linguistic decision matrix $\bar{R}^*$.

8	c_1	c_2	c_3	c_4
x_1	$[(s_1, 0.0272), (s_4, 0.0533)]$	$[(s_2, 0.0224), (s_3, 0.0345)]$	$[(s_2, 0.0069), (s_3, 0.0268)]$	$[(s_4, 0.0264), (s_6, 0.0248)]$
x_2	$[(s_1, 0.0260), (s_3, -0.0186)]$	$[(s_2, 0.0372), (s_3, 0.0511)]$	$[(s_2, 0.0354), (s_4, -0.0098)]$	$[(s_5, 0.0086), (s_6, 0.0437)]$
x_3	$[(s_3, -0.0234), (s_5, 0.0367)]$	$[(s_1, 0.0097), (s_2, 0.0196)]$	$[(s_1, 0.0269), (s_4, -0.0480)]$	$[(s_4, 0.0017), (s_5, -0.0159)]$
x_4	$[(s_3, 0.0268), (s_4, 0.0526)]$	$[(s_1, 0.0028), (s_2, 0.0470)]$	$[(s_1, 0.0028), (s_3, -0.0181)]$	$[(s_6, 0.0471), (s_8, 0)]$

467

468 **Step 4'**. Calculate the matrix $T^* = (T^*_{ij})_{4 \times 4}$ and $T'^* = (T'^*_{ij})_{4 \times 4}$ based on Eqs. (26), (27), and
469 (28):

470
$$T^* = \begin{pmatrix} 1 & 0.1522 & 0.0415 & 0.0107 \\ 1 & 0.1510 & 0.0434 & 0.0124 \\ 1 & 0.3516 & 0.0474 & 0.0072 \\ 1 & 0.4018 & 0.0513 & 0.0066 \end{pmatrix}, \quad T'^* = \begin{pmatrix} 1 & 0.5533 & 0.2266 & 0.0911 \\ 1 & 0.3564 & 0.1518 & 0.0744 \\ 1 & 0.6617 & 0.1784 & 0.0806 \\ 1 & 0.5526 & 0.1641 & 0.0586 \end{pmatrix}.$$

471 **Step 5'**. Utilize the IV2TPWG operator (Eq. (30)) to derive the collective overall preference
472 value $\bar{r}_i^* = [(s_i^*, \alpha_i^*), (s_i'^*, \alpha_i'^*)]$ of the alternative x_i .

$$\begin{aligned} \bar{r}_1^* &= [(s_1, 0.0437), (s_4, -0.0050)], \bar{r}_2^* = [(s_1, 0.0449), (s_3, 0.0222)], \\ \bar{r}_3^* &= [(s_2, 0.0194), (s_4, -0.0329)], \bar{r}_4^* = [(s_2, 0.0324), (s_4, -0.0539)]. \end{aligned}$$

Step 6'. According to Eq. (9), we calculate the score function $S(\bar{r}_i^*)$ ($i=1,2,3,4$) of $\bar{r}_i^*$ ($i=1,2,3,4$) as follows:

$$S(\bar{r}_1^*) = 0.3781, \quad S(\bar{r}_2^*) = 0.3404, \quad S(\bar{r}_3^*) = 0.4437, \quad S(\bar{r}_4^*) = 0.4025.$$

Because $S(\bar{r}_3^*) > S(\bar{r}_4^*) > S(\bar{r}_1^*) > S(\bar{r}_2^*)$, by Theorem 2.2, we have that $\bar{r}_3^* > \bar{r}_4^* > \bar{r}_1^* > \bar{r}_2^*$, which implies that $x_3 \succ x_4 \succ x_1 \succ x_2$. Thus, the best candidate is x_3 .

Through Example 4.1, we can see that there are different priority levels among four attributes and three decision makers respectively. For instance, if a candidate owns bad morality, then this candidate is impossible to be selected by three decision makers, no matter how well he/she performs on the other three attributes. If a candidate receives a bad evaluation from university president, then he/she is also impossible to be selected no matter how high evaluations he has received from the other two decision makers. Clearly, the existing interval-valued 2-tuple linguistic aggregation operators are difficult to deal with such cases due to the fact that these operators are usually used to solve MAGDM where the attributes and the decision makers are at the same priority level. However, the proposed operators in this paper not only accommodate the interval-valued linguistic preference information but also take the prioritization among the attributes and the decision makers into account; thus, our operators and approaches can effectively cope with the situations in which the attributes and the decision makers are at different priority levels.

5. CONCLUSIONS

Considering that there may exist a prioritization relationship over the attributes and decision makers in some multiple attribute group decision making problems with interval-valued linguistic information, this paper provides some interval-valued 2-tuple prioritized aggregation operators to deal with such situations. Furthermore, we apply the proposed operators to solve some multiple attribute group decision making problems. Finally, an illustrative example is employed to show that the proposed approaches are not only more reasonable but more efficient in practical applications due to the fact that these approaches consider the prioritization relationship among the attributes and decision makers.

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512 **COMPETING INTERESTS**

513

514 **AUTHORS' CONTRIBUTIONS**

515

516 'Zhiming Zhang' designed the study, performed the statistical analysis, wrote the protocol,
 517 and wrote the overall draft of the manuscript. The author read and approved the final
 518 manuscript.

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