

On Cartesian Products of Any Finite Number of Orthogonal Double Covers

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Abstract

We are concerned with the cartesian products of any finite number of symmetric starter vectors of orthogonal double covers of the complete bipartite graphs and use this method to construct ODCs by new infinite classes of disjoint unions of complete bipartite graphs.

Keywords: *Graph decomposition; Orthogonal double cover; Symmetric starter; Cartesian product.*

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1 INTRODUCTION

A generalization of notion of an ODC to arbitrary underlying graphs is as follows. Let H be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_0, G_1, \dots, G_{n-1}\}$ be a collection of n spanning subgraphs of H . $\mathcal{G}$ is called an ODC of H if there exists a bijective mapping $\phi : (V(H)) \rightarrow \mathcal{G}$ such that:

(i) Every edge of H is contained in exactly two of the graphs $G_0, G_1, \dots, G_{n-1}$.

(ii) For every choice of different vertices a, b of H ,

$$|E(\phi(a)) \cap E(\phi(b))| = \begin{cases} 1 & \text{if } \{a, b\} \in H, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we assume $H = K_{n,n}$, the complete bipartite graph with partition sets of size n each. A graph G is called a spanning subgraph of H if G contains all the vertices of H and the edges of G is a sub set of the edges of H .

We use the usual notation: $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n , P_n for the path on n vertices, C_n for the cycle on n vertices, K_n for the complete graph on n vertices, K_1 for an isolated vertex, $G \cup F$ for the disjoint union of G and F , and lG for l disjoint copies of G .

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An algebraic construction of ODCs via *symmetric starters* has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$: a few exceptions apart, all graphs G are found by this way (see [1], Table 1). This method has been applied in [1, 2] to detect some infinite classes of graphs G for which there are ODCs of $K_{n,n}$ by G .

All graphs here are finite, simple, and undirected. Let $\Gamma = \{\gamma_0, \dots, \gamma_{n-1}\}$ be an (additive) abelian group of order n . The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v, i) \in \Gamma \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{a_i, b_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $a, b \in \Gamma$ and $i, j \in \mathbb{Z}_2$. If there is no chance of confusion (a, b) will be written instead of $\{a_0, b_1\}$ for the edge between the vertices a_0, b_1 .

Let G be a spanning subgraph of $K_{n,n}$ and let $x \in \Gamma$. Then the graph $G + x$ with $E(G + x) = \{(u + x, v + x) : (u, v) \in E(G)\}$ is called the x -translate of G . The length of an edge $e = (u, v) \in E(G)$ is defined by $d(e) = v - u$.

G is called a half starter with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are mutually distinct; that is, $\{d(e) : e \in E(G)\} = \Gamma$. The following three results were proved in [1].

(i) If G is a *half starter*, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e. $\cup_{x \in \Gamma} E(G + x) = E(K_{n,n})$.

Hereafter, a half starter G will be represented by the vector $v(G) = (v_{\gamma_0}, \dots, v_{\gamma_{n-1}}) \in \Gamma^n$, where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i in G .

Example 1 *An edge decomposition of $K_{4,4}$ by $2K_{1,2}$ can be obtained by using the following half starter $v(2K_{1,2}) = (0, 1, 2, 3)$.*

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_{\gamma}(G_0) - v_{\gamma}(G_1) : \gamma \in \Gamma\} = \Gamma$.

(ii) If two half starter vectors $v(G_0)$ and $v(G_1)$ are orthogonal, then $\mathcal{G} = \{G_{x,i} : (x, i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{x,i} = G_i + x$ is an ODC of $K_{n,n}$.

Example 2 *An ODC of $K_{4,4}$ by $2K_{1,2}$ can be obtained by using the following two orthogonal half starters $v(G_0) = (0, 1, 2, 3)$ and $v(G_1) = (0, 2, 0, 2)$.*

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{(u_0, v_1) : (v_0, u_1) \in E(G)\}$ is called the symmetric graph of G . Note that if G is a half starter, then G_s is also a half starter. A half starter G is called a symmetric starter with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

(iii) Let n be a positive integer and let G be a half starter represented by the vector $v(G) = (v_{\gamma_0}, \dots, v_{\gamma_{n-1}})$. Then G is symmetric starter if and only if $\{v_{\gamma} - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.

Example 3 *An ODC of $K_{5,5}$ can be obtained by using the following symmetric starter $K_{1,5}$ represented by the vector $v(K_{1,5}) = (0, 0, 0, 0, 0)$.*

The vertices of the partition sets of $K_{n,n}$ take the labelling $\{0_0, 1_0, \dots, (n-1)_0\}$ and $\{0_1, 1_1, \dots, (n-1)_1\}$. The length of an edge x_0y_1 of $K_{n,n}$ is defined

to be the difference $y - x$, where $x, y \in \mathbb{Z}_n$. Note that sums and differences are calculated modulo n .

Much research on this subject focused on the detection of ODCs with pages isomorphic to a given graph G . For a summary of results on ODCs, see [2, 3]. In [4], the other terminology not defined here can be found.

The above results on ODCs of graphs motivated us to consider ODCs of $K_{n_1 n_2 \dots n_m, n_1 n_2 \dots n_m}$ if we have the ODCs of K_{n_j, n_j} by G^j , where G^j symmetric starters and $1 \leq j \leq m$. El Shanawany et al. In [5] got some results for ODCs by using the Cartesian product of two symmetric starter vectors. In this paper, we have settled the existence problem of ODCs of $K_{n_1 n_2 \dots n_m, n_1 n_2 \dots n_m}$ by a few infinite families of graphs which presented in the next section.

All our following results based on two major points :

a) The Cartesian product construction in Theorem 4 which is introduced in

the following section, and

b) The existence of symmetric starters for a few classes of graphs that can be used as ingredients for Cartesian product construction to obtain new symmetric starters. These are

1) $K_{1,n}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is

$v(K_{1,n}) = (\overbrace{0, 0, 0, \dots, 0}^{n \text{ times}}) \in \mathbb{Z}_n^n$, see Corollary 2.2.7 in [6].

2) $mK_{2,2}$ which is a symmetric starter of an ODC of $K_{4m,4m}$ whose vector is $v(mK_{2,2}) = (0, 1, 2, \dots, 2m-1, 0, 1, 2, \dots, 2m-1) \in \mathbb{Z}_{4m}^{4m}$, see Lemma 2.2.13 in [6].

3) $K_{1,2} \cup K_{1,2(n-1)}$ which is a symmetric starter of an ODC of $K_{2n,2n}$ whose vector is $v(K_{1,2} \cup K_{1,2(n-1)}) = (0, \overbrace{n, n, \dots, n, n}^{(n-1) \text{ times}}, 0, \overbrace{n, n, \dots, n, n}^{(n-1) \text{ times}}) \in \mathbb{Z}_{2n}^{2n}$, $n \geq 2$, it is easily checked that $v_i(K_{1,2} \cup K_{1,2(n-1)}) = v_{-i}(K_{1,2} \cup K_{1,2(n-1)})$, and hence $\{v_i - v_{-i} + i : i \in \mathbb{Z}_{2n}\} = \mathbb{Z}_{2n}$.

4) $nK_{1,2}$ which is a symmetric starter of an ODC of $K_{2n,2n}$ whose vector is $v(nK_{1,2}) = (0, 1, 2, \dots, 2n-1) \in \mathbb{Z}_{2n}^{2n}$, see Lemma 2.2.11 in [6].

5) nK_2 which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is $v(nK_2) = (0, 1, 2, \dots, 2n-1) \in \mathbb{Z}_n^n$, $n \equiv 1, 5 \pmod{6}$, see corollary 2.2.9 in [6].

6) $K_{1,2} \cup K_{2,n-1}$ which is a symmetric starter of an ODC of $K_{2n,2n}$ whose vector is $v(K_{1,2} \cup K_{2,n-1}) = (0, \overbrace{1, 1, 1, \dots, 1, 1}^{(n-1) \text{ times}}, n, \overbrace{n+1, n+1, \dots, n+1}^{(n-1) \text{ times}}) \in \mathbb{Z}_{2n}^{2n}$, $n \geq 2$, see [5].

7) $2K_2 \cup K_{1,n-2}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is $v(2K_2 \cup K_{1,n-2}) = (0, 0, 0, \dots, 0, 1, n-1) \in \mathbb{Z}_n^n$, see Lemma 2.2.17 in [6].

2 MAIN RESULTS

In the following, if there is no danger of ambiguity, if $(i_1, i_2, \dots, i_m) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_m}$ we can write $(i_1, i_2, \dots, i_m)$ as $(i_1 i_2 \dots i_m)$.

Theorem 4 *The Cartesian product of any finite number of symmetric starter vectors is a symmetric starter vector with respect to the Cartesian product of the corresponding groups.*

Proof. Let $v(G^j) = (v_0^j, v_1^j, \dots, v_{n_j-1}^j) \in \mathbb{Z}_{n_j}^{n_j}$ be a symmetric starter vector of an ODC of K_{n_j, n_j} by G^j with respect to $\mathbb{Z}_{n_j}$, where $v_k^j, k \in \mathbb{Z}_{n_j}, 1 \leq j \leq m$ then

$$\{v_{i_j}^j - v_{n_j-i_j}^j + i_j : i_j \in \mathbb{Z}_{n_j}\} = \mathbb{Z}_{n_j}. \quad (1)$$

Then $v(G^1) \times v(G^2) \times \dots \times v(G^m) = ((v_0^1 v_0^2 \dots v_0^m) (v_0^1 v_0^2 \dots v_0^{m-1} v_1^m) \dots (v_1^1 v_1^2 \dots v_1^m) \dots (v_{n_1-1}^1 v_{n_2-1}^2 \dots v_{n_m-1}^m))$ where $i_j \in \mathbb{Z}_{n_j}$ and $1 \leq j \leq m$. From (1) we conclude

$\{(v_{i_1}^1 v_{i_2}^2 \dots v_{i_m}^m) - (v_{n_1-i_1}^1 v_{n_2-i_2}^2 \dots v_{n_m-i_m}^m) + (i_1 i_2 \dots i_m) : (i_1 i_2 \dots i_m) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_m}\} = \{\{v_{i_1}^1 - v_{n_1-i_1}^1 + i_1\} \times \{v_{i_2}^2 - v_{n_2-i_2}^2 + i_2\} \times \dots \times \{v_{i_m}^m - v_{n_m-i_m}^m + i_m\} : i_1 \in \mathbb{Z}_{n_1}, i_2 \in \mathbb{Z}_{n_2}, \dots, i_m \in \mathbb{Z}_{n_m}\} = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_m}$. Then $v(G^1) \times \dots \times v(G^m)$ is a symmetric starter vector of an ODC of $K_{n_1 n_2 \dots n_m, n_1 n_2 \dots n_m}$ by a new graph G with respect to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_m}$. The graph G can be described as follows $E(G) = \{((v_{i_1}^1 v_{i_2}^2 \dots v_{i_m}^m), (v_{i_1}^1 v_{i_2}^2 \dots v_{i_m}^m) + (i_1 i_2 \dots i_m)) : (i_1 i_2 \dots i_m) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_m}\}$. ■

Lemma 5 *Let m be a positive integer with $\gcd(m, 3)=1$, then there is an ODC of $K_{64m, 64m}$ by $mK_{8,8} \cup 112mK_1$.*

Proof. From the fact that the vectors $v(K_{2,2})$, $v(K_{2,2})$, and $v(mK_{2,2})$ are symmetric starter vectors in $\mathbb{Z}_4$, $\mathbb{Z}_4$, and $\mathbb{Z}_{4m}$ respectively, by applying Theorem 4, the vector $v(K_{2,2}) \times v(K_{2,2}) \times v(mK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{4m}$ of an ODC of $K_{64m, 64m}$ by $mK_{8,8} \cup 112mK_1$, $E(mK_{8,8}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{4m}, v_{i_1}^1 \in v(K_{2,2}), v_{i_2}^2 \in v(K_{2,2}), v_{i_3}^3 \in v(mK_{2,2})\}$. ■

Lemma 6 *Let m be a positive integer with $\gcd(m, 3)=1$, then there is an ODC of $K_{64m^2, 64m^2}$ by $m^2 K_{8,8} \cup 112m^2 K_1$.*

Proof. From the fact that the vectors $v(K_{2,2})$, $v(mK_{2,2})$, and $v(mK_{2,2})$ are symmetric starter vectors in $\mathbb{Z}_4$, $\mathbb{Z}_{4m}$, and $\mathbb{Z}_{4m}$ respectively, by applying Theorem 4, the vector $v(K_{2,2}) \times v(mK_{2,2}) \times v(mK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_4 \times \mathbb{Z}_{4m} \times \mathbb{Z}_{4m}$ then $v(K_{2,2}) \times v(mK_{2,2}) \times v(mK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_4 \times \mathbb{Z}_{4m} \times \mathbb{Z}_{4m}$ of an ODC of $K_{64m^2, 64m^2}$ by $m^2 K_{8,8} \cup 112m^2 K_1$, $E(m^2 K_{8,8}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_4 \times \mathbb{Z}_{4m} \times \mathbb{Z}_{4m}, v_{i_1}^1 \in v(K_{2,2}), v_{i_2}^2 \in v(mK_{2,2}), v_{i_3}^3 \in v(mK_{2,2})\}$. ■

Theorem 7 *Let α, m, n be positive integers with $\gcd(m, 3)=1$, then there is an ODC of $K_{8\alpha mn, 8\alpha mn}$ by $\alpha K_{2,4m} \cup \alpha K_{2,4m(n-1)} \cup 4\alpha(3mn - 1)K_1$.*

Proof. From the fact that the vectors $v(\alpha K_{2,2})$, $v(K_{1,m})$, and $v(K_{1,2} \cup K_{1,2(n-1)})$ are symmetric starter vectors in $\mathbb{Z}_{4\alpha}$, $\mathbb{Z}_m$, and $\mathbb{Z}_{2n}$ respectively, by applying Theorem 4, the vector $v(\alpha K_{2,2}) \times v(K_{1,m}) \times v(K_{1,2} \cup K_{1,2(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_{4\alpha} \times \mathbb{Z}_m \times \mathbb{Z}_{2n}$ of an ODC of $K_{8\alpha mn, 8\alpha mn}$ by $\alpha K_{2,4m} \cup \alpha K_{2,4m(n-1)} \cup 4\alpha(3mn-1)K_1$, $E(\alpha K_{2,4m} \cup \alpha K_{2,4m(n-1)}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_{4\alpha} \times \mathbb{Z}_m \times \mathbb{Z}_{2n}, v_{i_1}^1 \in v(\alpha K_{2,2}), v_{i_2}^2 \in v(K_{1,m}), v_{i_3}^3 \in v(K_{1,2} \cup K_{1,2(n-1)})\}$. ■

Theorem 8 Let α, m, n be positive integers with $\gcd(m, 3)=1$, then there is an ODC of $K_{8\alpha mn, 8\alpha mn}$ by $mnK_{2,4\alpha} \cup 2mn(6\alpha-1)K_1$.

Proof. From the fact that the vectors $v(K_{1,\alpha})$, $v(nK_{1,2})$ and $v(mK_{2,2})$ are symmetric starter vectors in $\mathbb{Z}_\alpha$, $\mathbb{Z}_{2n}$, and $\mathbb{Z}_{4m}$ respectively, by applying Theorem 4, the vector $v(K_{1,\alpha}) \times v(nK_{1,2}) \times v(mK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_\alpha \times \mathbb{Z}_{2n} \times \mathbb{Z}_{4m}$ of an ODC of $K_{8\alpha mn, 8\alpha mn}$ by $mnK_{2,4\alpha} \cup 2mn(6\alpha-1)K_1$, $E(mnK_{2,4\alpha}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_\alpha \times \mathbb{Z}_{2n} \times \mathbb{Z}_{4m}, v_{i_1}^1 \in v(K_{1,\alpha}), v_{i_2}^2 \in v(nK_{1,2}), v_{i_3}^3 \in v(mK_{2,2})\}$. ■

Theorem 9 Let α, m, n be positive integers with $\gcd(m, 3)=1$, then there is an ODC of $K_{2\alpha mn, 2\alpha mn}$ by $\alpha K_{1,2m} \cup \alpha K_{1,2m(n-1)} \cup 2\alpha(mn-1)K_1$.

Proof. From the fact that the vectors $v(\alpha K_2)$, $v(K_{1,m})$ and $v(K_{1,2} \cup K_{1,2(n-1)})$ are symmetric starter vectors in $\mathbb{Z}_\alpha$, $\mathbb{Z}_m$, and $\mathbb{Z}_{2n}$ respectively, by applying Theorem 4, the vector $v(\alpha K_2) \times v(K_{1,m}) \times v(K_{1,2} \cup K_{1,2(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_\alpha \times \mathbb{Z}_m \times \mathbb{Z}_{2n}$ of an ODC of $K_{2\alpha mn, 2\alpha mn}$ by $\alpha K_{1,2m} \cup \alpha K_{1,2m(n-1)} \cup 2\alpha(mn-1)K_1$, $E(\alpha K_{1,2m} \cup \alpha K_{1,2m(n-1)}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_\alpha \times \mathbb{Z}_m \times \mathbb{Z}_{2n}, v_{i_1}^1 \in v(\alpha K_2), v_{i_2}^2 \in v(K_{1,m}), v_{i_3}^3 \in v(K_{1,2} \cup K_{1,2(n-1)})\}$. ■

Theorem 10 Let α, m, n be positive integers, then there is an ODC of $K_{8\alpha mn, 8\alpha mn}$ by $m\alpha K_{2,4} \cup m\alpha K_{2,4(n-1)} \cup 4\alpha m(3n-1)K_1$.

Proof. From the fact that the vectors $v(\alpha K_2)$, $v(K_{1,2} \cup K_{1,2(n-1)})$, and $v(K_{1,m})$ are symmetric starter vectors in $\mathbb{Z}_\alpha$, $\mathbb{Z}_{2n}$, and $\mathbb{Z}_m$ respectively, by applying Theorem 4, the vector $v(\alpha K_2) \times v(K_{1,2} \cup K_{1,2(n-1)}) \times v(K_{1,m})$ is a symmetric starter vector with respect to $\mathbb{Z}_\alpha \times \mathbb{Z}_{2n} \times \mathbb{Z}_m$ of an ODC of $K_{8\alpha mn, 8\alpha mn}$ by $m\alpha K_{2,4} \cup m\alpha K_{2,4(n-1)} \cup 4\alpha m(3n-1)K_1$, $E(m\alpha K_{2,4} \cup m\alpha K_{2,4(n-1)}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_\alpha \times \mathbb{Z}_{2n} \times \mathbb{Z}_m, v_{i_1}^1 \in v(\alpha K_2), v_{i_2}^2 \in v(K_{1,2} \cup K_{1,2(n-1)}), v_{i_3}^3 \in v(K_{1,m})\}$. ■

Theorem 11 Let α, m, n be positive integers with $\gcd(m, 3)=1$, then there is an ODC of $K_{2\alpha mn, 2\alpha mn}$ by $\alpha K_{2,m} \cup \alpha K_{2,m(n-1)} \cup \alpha(3mn-4)K_1$.

Proof. From the fact that the vectors $v(\alpha K_2)$, $v(K_{1,m})$, and $v(K_{1,2} \cup K_{2,(n-1)})$ are symmetric starter vectors in $\mathbb{Z}_\alpha$, $\mathbb{Z}_m$, and $\mathbb{Z}_{2n}$ respectively, by applying

Theorem 4, the vector $v(\alpha K_2) \times v(K_{1,m}) \times v(K_{1,2} \cup K_{2,(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_\alpha \times \mathbb{Z}_m \times \mathbb{Z}_{2n}$ of an ODC of $K_{2\alpha mn, 2\alpha mn}$ by $\alpha K_{2,m} \cup \alpha K_{2,m(n-1)} \cup \alpha(3mn-4)K_1$, $E(\alpha K_{2,m} \cup \alpha K_{2,m(n-1)}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_\alpha \times \mathbb{Z}_m \times \mathbb{Z}_{2n}, v_{i_1}^1 \in v(\alpha K_2), v_{i_2}^2 \in v(K_{1,m}), v_{i_3}^3 \in v(K_{1,2} \cup K_{2,(n-1)})\}$. ■

Theorem 12 Let α, m, n be positive integers with $n \geq 5$, then there is an ODC of $K_{4\alpha mn, 4\alpha mn}$ by $m\alpha K_{2,2(n-2)} \cup 2m\alpha K_{2,2} \cup 6m\alpha(n-1)K_1$.

Proof. From the fact that the vectors $v(\alpha K_2)$, $v(2K_2 \cup K_{1,(n-2)})$, and $v(K_{1,m})$ are symmetric starter vectors in $\mathbb{Z}_\alpha$, $\mathbb{Z}_{2n}$, and $\mathbb{Z}_m$

respectively, by applying Theorem 4, the vector $v(\alpha K_2) \times v(2K_2 \cup K_{1,(n-2)}) \times v(K_{1,m})$ is a symmetric starter vector with respect to $\mathbb{Z}_\alpha \times \mathbb{Z}_{2n} \times \mathbb{Z}_m$ of an ODC of $K_{4\alpha mn, 4\alpha mn}$ by $m\alpha K_{2,2(n-2)} \cup 2m\alpha K_{2,2} \cup 6m\alpha(n-1)K_1$, $E(m\alpha K_{2,2(n-2)} \cup 2m\alpha K_{2,2}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_\alpha \times \mathbb{Z}_{2n} \times \mathbb{Z}_m, v_{i_1}^1 \in v(\alpha K_2), v_{i_2}^2 \in v(2K_2 \cup K_{1,(n-2)}), v_{i_3}^3 \in v(K_{1,m})\}$. ■

Theorem 13 For any positive integers $m \geq 2, n \geq 2$, and α with $\gcd(\alpha, 3) = 1$, there exists an ODC of $K_{16\alpha mn, 16\alpha mn}$ by $\alpha K_{2,8} \cup \alpha K_{2,8(n-1)} \cup \alpha K_{2,8(m-1)} \cup \alpha K_{2,8(m-1)(n-1)} \cup 8\alpha(3mn-1)K_1$.

Proof. From the fact that the vectors $v(\alpha K_{2,2})$, $v(K_{1,2} \cup K_{1,2(m-1)})$, and $v(K_{1,2} \cup K_{1,2(n-1)})$ are symmetric starter vectors in $\mathbb{Z}_{4\alpha}$, $\mathbb{Z}_{2m}$, and $\mathbb{Z}_{2n}$ respectively, by applying Theorem 4, the vector $v(\alpha K_{2,2}) \times v(K_{1,2} \cup K_{1,2(m-1)}) \times v(K_{1,2} \cup K_{1,2(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_{4\alpha} \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$ of an ODC of $K_{16\alpha mn, 16\alpha mn}$ by $\alpha K_{2,8} \cup \alpha K_{2,8(n-1)} \cup \alpha K_{2,8(m-1)} \cup \alpha K_{2,8(m-1)(n-1)} \cup 8\alpha(3mn-1)K_1$, $E(\alpha K_{2,8} \cup \alpha K_{2,8(n-1)} \cup \alpha K_{2,8(m-1)} \cup \alpha K_{2,8(m-1)(n-1)}) = \{((v_{i_1}^1 v_{i_2}^2 v_{i_3}^3), (v_{i_1}^1 v_{i_2}^2 v_{i_3}^3) + (i_1 i_2 i_3)) : (i_1 i_2 i_3) \in \mathbb{Z}_{4\alpha} \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}, v_{i_1}^1 \in v(\alpha K_{2,2}), v_{i_2}^2 \in v(K_{1,2} \cup K_{1,2(m-1)}), v_{i_3}^3 \in v(K_{1,2} \cup K_{1,2(n-1)})\}$. ■

The following conjecture generalizes the above results.

Conjecture 14 Let $\alpha_i, \beta_i, \delta_i$, and r be positive integers. Then there is an ODC of $K_{r,r}$ by $\bigcup_{i=0}^m \{\alpha_i K_{\beta_i, \delta_i} \cup (2r - \alpha_i(\beta_i + \delta_i))K_1\}$, where m is the number of the disjoint complete bipartite graphs.

3 CONCLUSION

In conclusion, known symmetric starters are used as ingredients for the Cartesian product construction to obtain new symmetric starters which are $mK_{8,8} \cup 112mK_1, m^2K_{8,8} \cup 112m^2K_1, \alpha K_{2,4m} \cup \alpha K_{2,4m(n-1)} \cup 4\alpha(3mn-1)K_1, mnK_{2,4\alpha} \cup 2mn(6\alpha-1)K_1, \alpha K_{1,2m} \cup \alpha K_{1,2m(n-1)} \cup 2\alpha(mn-1)K_1, m\alpha K_{2,4} \cup m\alpha K_{2,4(n-1)} \cup 4\alpha m(3n-1)K_1, \alpha K_{2,m} \cup \alpha K_{2,m(n-1)} \cup \alpha(3mn-4)K_1, m\alpha K_{2,2(n-2)} \cup 2m\alpha K_{2,2} \cup 6m\alpha(n-1)K_1$, and $\alpha K_{2,8} \cup \alpha K_{2,8(n-1)} \cup \alpha K_{2,8(m-1)} \cup \alpha K_{2,8(m-1)(n-1)} \cup 8\alpha(3mn-1)K_1$.

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COMPETING INTERESTS

No competing interests exist.

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