

# Original Research Article

# Image Normalization Using the Decomposition of Centroaffine Transformation Matrix

## ABSTRACT

Recognition is the main task of image processing, but before directly addressing recognition, it will be necessary to convert the image of interest into a recognizable form. The decomposition and transformation of the centroaffine matrix into a simple one-dimensional geometric form can help us to solve the problem of images normalization which will lead to increase images effectiveness. In this paper the problem of image normalization by joint use of decomposition of centroaffine transformation matrix and one-dimensional normalization methods is proposed. centroaffine transformation matrix to superposition of simple geometrical transformations are obtained and the result are verified.

*Keywords: Centroaffine, image normalization, transformation group, centroaffine transformation*

## 1. INTRODUCTION

The normalization can be defined as a procedure of compensation of geometric transformations that linking benchmark image, actual images and the set of geometric transformations, known as transformation group [1], that forming the difference between the benchmark and the actual image.

Typically, when processing real images, it may not be known beforehand which geometric transformations it could be. Since all simple geometric transformations and their combination  $x$  contain an affine transformation, so we will assume that investigating the actual image is influenced by the affine transformation group  $G_A$ .

Displacement compensation is quite simple to accomplish [1, 2], for example, we can find the image gravity center, then carry out the operation of centering, after combining the center of gravity of image with the geometric center of the field of the sight. After centering the image it can be investigated under the influence of only centroaffine group  $G_C$  which can be represented as a real square matrix [3]:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (1)$$

Where  $a_{11}a_{22} - a_{21}a_{12} \neq 0$ .

Thus, to normalize the image, it is necessary to determine in advance the unknown parameters of centroaffin transformation, after which the process of normalization can be presented by the reverse mapping centroaffin described by the inverse matrix  $A^{-1}$ .

## 2. DECOMPOSITION OF CENTROAFFINE TRANSFORMATION MATRIX

The parameters of centroaffin matrix can be determined by the decomposition of the matrix  $A$  in product of matrices of simple groups of geometric transformations [4,5], where each of

which is a subgroup of  $G_2^E$ . This combined transformation must satisfy the following requirements:

- It should be a group,
- Between the parameters of the combined transformation and the parameters  $a_{11}, a_{12}, a_{21}, a_{22}$  must exist unambiguous, i.e., the parameters  $a_{11}, a_{12}, a_{21}, a_{22}$  must be expressed through the parameters of the combined transformation, and vice versa

In general, any mapping can be described by a real square matrix  $A$ , represents a combination of self-adjoint (in our case - symmetric) and orthogonal mapping [2,5,6].

$$A = A_s A_{ort} \quad (2)$$

Where

$A_s$  – Symmetric mapping matrix;

$A_{ort}$  – Orthogonal mapping matrix.

After the transition to the basis of the eigenvectors of a symmetric mapping  $A_s$  then we can get the following:

$$H^{-1} A_s H = L \quad (3)$$

Where

$H$  – Orthogonal matrix consisting of eigenvectors matrix  $A_s$

$L$  – Real diagonal matrix,

Rewrite formula (3) in another form, we obtain:

$$A_s = H L H^{-1} \quad (4)$$

Where  $A = H L H^{-1} A_{ort}$

Since the matrix  $H$  is orthogonal and, consequently,  $H^{-1}$  also orthogonal, then, if we denote  $H = K, H^{-1} A_{ort} = L$ , then we will get the following:

$$A = K L P \quad (5)$$

Where

$L$  – Real diagonal matrix

$R, K$  – Orthogonal matrix

Let us take into consideration the case when  $K, L$  – pure rotation matrix  $U_1$  and  $U_2$ , in addition to  $L$  – matrix of an inhomogeneous rescaling. Then  $A = U_1 D U_1$ , which can be represented by the following:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix} \quad (6)$$

Where

$\alpha$  – Parameter transformation rotation  $U_2$

$\beta$  – Parameter transformation rotation  $U_1$

$\gamma, \mu$  – Parameters of the diagonal transformation  $D$ .

From the matrix relation (6) we can proceed to four equations:

$$\begin{aligned} a_{11} &= \gamma \cos \alpha \cos \beta - \mu \sin \alpha \sin \beta \\ a_{12} &= \gamma \cos \alpha \cos \beta + \mu \sin \alpha \cos \beta \\ a_{21} &= -\gamma \sin \alpha \cos \beta - \mu \cos \alpha \sin \beta \\ a_{22} &= -\gamma \sin \alpha \sin \beta - \mu \cos \alpha \cos \beta \end{aligned} \quad (7)$$

Equations (7) can be rewritten as:

$$\begin{aligned} \gamma \cos \beta &= a_{11} \cos \alpha - a_{21} \sin \alpha \\ \gamma \sin \beta &= a_{12} \cos \alpha - a_{22} \sin \alpha \\ -\mu \sin \beta &= a_{11} \sin \alpha + a_{21} \cos \alpha \\ \mu \cos \beta &= a_{12} \sin \alpha + a_{22} \cos \alpha \end{aligned} \quad (8)$$

Solving these equations for the parameters  $\alpha, \beta, \gamma$  and  $\mu$ , we obtain:

$$\begin{aligned} \tan 2\alpha &= \frac{-2(a_{11}a_{21} + a_{12}a_{22})}{a_{11}^2 + a_{12}^2 - a_{21}^2 - a_{22}^2}, \\ \tan 2\beta &= \frac{2(a_{12}a_{11} + a_{21}a_{22})}{a_{11}^2 + a_{12}^2 - a_{21}^2 - a_{22}^2}, \\ \gamma &= \frac{a_{11} \cos \alpha - a_{21} \sin \alpha}{\cos \beta}, \\ \mu &= \frac{a_{12} \sin \alpha + a_{22} \cos \alpha}{\cos \beta}. \end{aligned} \quad (9)$$

This verifies that the transformation  $U_2 D U_1$  is a group.

The condition of associativity will be executed on the basis of the properties of matrices, i.e.,  $(U_2 D) U_1 = U_2 (D U_1)$ .

Identity transformation is  $U_2 D U_1$  with the parameters  $\alpha = 0, \beta = 0, \gamma = \mu = 1$ .

$(U_2 D U_1)^{-1} = U_1^{-1} D^{-1} U_2^{-1}$ . Since  $\det U_1 \neq 0, \det D \neq 0, \det U_2 \neq 0$ , then the inverse exists.

As a result of superposition transformations of  $U_2 D U_1$  and  $U_4 D U_3$  should get a transformation consisting of rotation of  $U_1^*$ , diagonal  $U_2^*$  and another rotation  $U_3^*$  which can be represented in the following transformation:  $U_3^* D^* U_1^*$ , this shows that the parameters of the resulting transformation  $\alpha^*, \beta^*, \gamma^*$  and  $\mu^*$  are:

$$\begin{aligned} \tan 2\alpha^* &= \frac{-2(a_{11}^* a_{21}^* + a_{12}^* a_{22}^*)}{(a_{11}^*)^2 + (a_{12}^*)^2 - (a_{21}^*)^2 - (a_{22}^*)^2}, \\ \tan 2\beta^* &= \frac{-2(a_{11}^* a_{12}^* + a_{21}^* a_{22}^*)}{(a_{11}^*)^2 + (a_{12}^*)^2 - (a_{21}^*)^2 - (a_{22}^*)^2}, \\ \gamma^* &= \frac{a_{11}^* \cos \alpha^* - a_{21}^* \sin \alpha^*}{\cos \beta^*}, \\ \mu^* &= \frac{a_{12}^* \sin \alpha^* + a_{22}^* \cos \alpha^*}{\cos \beta^*}, \end{aligned} \quad (10)$$

Where

$a_{11}^*, a_{12}^*, a_{21}^*, a_{22}^*$  – Parameters of the matrix  $A^*$  equal to  $(U_2 D U_1) * (U_4 D U_3)$  i.e., it can be expressed by the terms  $\alpha_1, \beta_1, \gamma_1, \mu_1$  of  $(U_2 D U_1)$  transformation and the terms  $\alpha_2, \beta_2, \gamma_2, \mu_2$  of  $(U_4 D U_3)$  transformation.

As all conditions are satisfied then  $(U_4 D U_3)$  is a group and since between the parameters  $\alpha, \beta, \gamma, \mu$  and the parameters of the centroaffine matrix  $a_{11}, a_{12}, a_{21}, a_{22}$  there exists a mutual unambiguity, then the combined transformation  $U_2 D U_1$  is a centroaffine group and, therefore,

it contains all subgroups of centroaffine transformation, as well as all their combinations [7,8].

But the decomposition  $U_2 D U_1$  was not too convenient in practice, since there are ambiguities and uncertainties, even with some simple transformations [8,9,10]. Ambiguity, for example, occurs when  $\gamma = \mu$ , which means the same scale in all direction. In this case, there is no sense in the second rotation. Consequently, the angle of one of rotations must be equal to zero. But there is ambiguity, which consists in the fact that zero may be equal to the angle of first and second rotation. Moreover, with the transformations of rotation, uniform scale, their combination and some other transformations in the first two formulas arises uncertainty of the form  $\frac{0}{0}$ .

There are also restrictions on the parameters  $\alpha$  and  $\beta$  from the first two formulas, where  $\alpha$  and  $\beta$  must not equal to  $\frac{\pi}{4} \mp \frac{\pi}{2} n$ .

Therefore, the use of the specified decomposition for the normalization will cause difficulties [11,12]. Let's consider other decomposition, assume that the decomposition of the centroaffine set will be a combined transformation of  $DXU$ , where  $D$  – the matrix of the inhomogeneous scale,  $X$  – matrix of a slanting shift along the  $X$  axis and  $U$  – rotation matrix. Then  $A = DXU$ , which can be rewritten as follow:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \quad (11)$$

From formula (11) we can write four equations:

$$\begin{aligned} a_{11} &= \gamma(\cos \alpha - h \sin \alpha), \\ a_{12} &= \gamma(\sin \alpha - h \cos \alpha), \\ a_{21} &= -\sin \alpha, \\ a_{22} &= \mu \cos \alpha. \end{aligned} \quad (12)$$

Solving these equations regarding  $\alpha, h, \gamma, \mu$ , we obtain:

$$\begin{aligned} \alpha &= \arctg \left( -\frac{a_{21}}{a_{22}} \right) \\ \mu &= \frac{a_{22}}{\cos \alpha} = \frac{a_{22}}{\cos \left( \arctg \left( -\frac{a_{21}}{a_{22}} \right) \right)} \\ h &= \frac{a_{21} a_{11} + a_{22} a_{12}}{a_{11} a_{22} - a_{21} a_{21}} \\ \gamma &= \frac{\mu a_{11}}{a_{22} + a_{21} h} = \frac{a_{11} a_{22} - a_{12} a_{21}}{a_{22}} \cos \left( \arctg \left( -\frac{a_{21}}{a_{22}} \right) \right) \end{aligned} \quad (13)$$

Verifying the performance of requirements to the group, we conclude that the combined transformation  $DXU$  is also centroaffine. Therefore, all simple groups, for example, conversions of slanting shift on  $X$  axis, on  $Y$  axis, the transformation of rotation, transformation of inhomogeneous scale, as well as all their combinations will be subgroups of decomposition  $DXU$ , i.e.  $X = DXU, Y = DXU, U = DXU, D = DXU, \dots$  etc. at defined values of the parameters  $\alpha, \beta, \mu, h$ . So,  $Y = DXU$  only when the parameters of  $DXU$  are equal:

$$\begin{aligned}\mu &= \frac{1}{\cos(\arctg(-h^*))} \\ \alpha &= \arctg(-h^*) \\ h &= h^*\end{aligned}\quad (14)$$

$$\gamma = \cos(\arctg(-h^*))$$

Where

$h^*$  – Parameter of transformation of slanting shift along an X axis for which transformation matrix looks like  $\begin{bmatrix} 1 & 0 \\ h^* & 1 \end{bmatrix}$ .

Suppose further that  $A = YDX$ , then:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ h_2 & 1 \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & h_1 \\ 0 & 1 \end{bmatrix} \quad (15)$$

Rewrite (15) in the form:

$$\begin{aligned}a_{11} &= \gamma \\ a_{12} &= \gamma h_2 \\ a_{21} &= \gamma h_1 \\ a_{22} &= \gamma h_1 h_2 + \mu\end{aligned}\quad (16)$$

Express the parameters  $\gamma, h_1, h_2, \mu$  through  $a_{11}, a_{12}, a_{21}, a_{22}$ , we obtain:

$$\begin{aligned}\gamma &= a_{11} \\ h_1 &= \frac{a_{21}}{a_{11}} \\ h_2 &= \frac{a_{12}}{a_{11}} \\ \mu &= \frac{a_{11} a_{22} - a_{12} a_{21}}{a_{11}}\end{aligned}\quad (17)$$

Based on formula (17) and the results of performance requirements group for the decomposition of  $YDX$  we find that  $YDX$  is also a decomposition of the matrix A.

Since the decomposition of  $DXU$  and  $YDX$  are centroaffine, then they are equivalent to each other, and the combined transformation of  $YDX$  can be decomposed into  $DXU$ , and vice versa, in other words  $YDX = DXU$  only when the parameters  $\alpha, \beta, \gamma, \mu$  of  $DXU$  decomposition equal to:

$$\begin{aligned}\alpha &= \arctg\left(-\frac{\gamma^* h_1^*}{\gamma^* h_1^* h_2^* + \mu^*}\right) \\ \mu &= -\frac{\gamma^* h_1^*}{\sin \alpha} \\ \gamma &= \frac{\cos \alpha - \sin \alpha h}{h_2^* \cos \alpha - \sin \alpha} \\ h &= \frac{\cos \alpha + h_2^* \sin \alpha}{\cos \alpha - \sin \alpha h}\end{aligned}\quad (18)$$

Where  $\gamma^*, \mu^*, h_1^*, h_2^*$  – parameters of  $YDX$  decomposition which has the form  $\begin{bmatrix} \gamma^* & \gamma^* h_1^* \\ \gamma^* h_2^* & \gamma^* h_1^* h_2^* + \mu^* \end{bmatrix}$ .

184 But  $DXU = YDX$  only when the parameters of  $YDX$  decomposition will have the following  
185 values:

$$\begin{aligned} \gamma^* &= \gamma(\cos \alpha - h \sin \alpha) \\ h_2^* &= \frac{\sin \alpha + h \cos \alpha}{\cos \alpha - h \sin \alpha} \\ h_1^* &= -\frac{\mu \sin \alpha}{\gamma(\cos \alpha - h \sin \alpha)} \\ \mu^* &= \frac{\mu}{\cos \alpha - h \sin \alpha} \end{aligned} \quad (19)$$

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188  
189 There are other decomposition centroaffine matrixes:  
190

$$\begin{aligned} A &= DYU, A = XDY, A = DXY, A = DYX \\ A &= UXD, A = UYD, A = UDY, A = UDX \end{aligned} \quad (20)$$

191  
192  
193 As mentioned in the beginning, not any superposition of elementary transformations  
194  $D, Y, X, U$  is suitable for the decomposition of the matrix A. Only the superposition [14,15],  
195 which is a group and contains four parameters, with one-one correspondence with the  
196 parameters  $a_{11}, a_{12}, a_{21}, a_{22}$  may be a centroaffine decomposition of the matrix A. If these  
197 requirements are not met, then such combined transformation is not a decomposition of the  
198 matrix A. For example, the combined transformation  $YXU, YUX$  are not a decomposition of  
199 matrix A.

200 Depending on the decomposition of matrix A and the properties of the affine transformation it  
201 is possible to find the parameters of a simple transformation which are included in the  
202 decomposition of matrix A by using one-dimensional normalization method.

203 This normalization cannot be used to calculate the normalization parameters for the whole  
204 image  $B(x, y)$ , it can be only used to calculate its restriction  $b(\xi)$  on some straight lines [1],  
205 which is much easier.

206 Calculating the parameters of one-dimensional normalization and using the decomposition of  
207 the matrix A, we obtain the parameters for the normalization of the entire image, i.e., the  
208 parameters of the inverse matrix  $A^{-1}$ .

209 We know that the centroaffine transformation of the center of gravity of the image remains  
210 fixed [1,10,11]. Since affine transformation translates straight line into straight line [10,17,18],  
211 then any line passing through the center coordinates under centroaffine transformation it will  
212 be translated into straight line.

213 Thus, each line passing through the coordinates (0, 0), corresponds to the real images  
214 obtained from the benchmark centroaffine transformation.

215 Appropriate straight lines of benchmark and real images will differ in the angle of slope, while  
216 the restriction  $b_D(\xi), b(\xi)$  of benchmark and real images on these straight lines will differ in  
217 the scale [19].

218 To find the parameters of one-dimensional normalization it is necessary to calculate the  
219 angle of slope and the scaling factor between the appropriate straight lines.

### 220 221 3. EXAMPLE OF USAGE

222  
223 Here is an example of one-dimensional normalization and the resulting decomposition of  
224 centroaffine matrix into superposition simple transformation to find the parameters of  
225 normalized matrix.

226 Let's take  $DXU$  as a decomposition of normalized matrix A, and calculate the parameters of  
227 simple transformations, involved in this decomposition, using the methods of one-  
228 dimensional normalization.

229 Figure 1 shows the benchmark image, while Figure 2 shows the image obtained by acting on  
 230 the benchmark image using any centroaffine transformation of  $A$ .  
 231  
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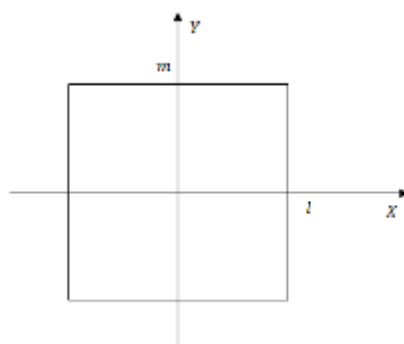


Fig 1. Benchmark image

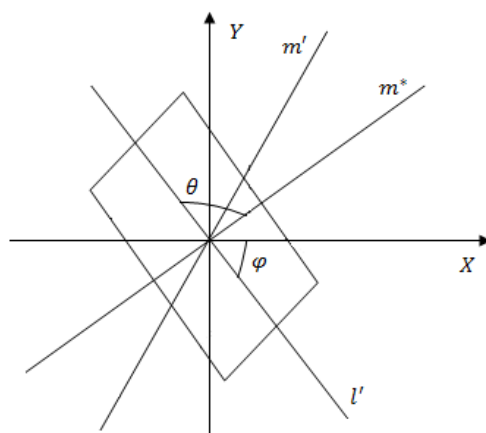


Fig 2. Images distorted by the centroaffine transformation of  $A$

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242 In Figure 1, for example, if we select two perpendicular lines  $l$  and  $m$ , coordinate  $X$  and  $Y$   
 243 axes, we must find the corresponding lines in Figure 2  $l'$  and  $m'$ , and calculate the following  
 244 four parameters:

- 245 ▪  $k_1$  – Scaling factor between the constrains on the lines  $l$  and  $l'$ ;
- 246 ▪  $k_2$  – Scaling factor between the constrains on the lines  $m$  and  $m'$ ;
- 247 ▪  $\varphi$  – Angle between the old position of line  $l$ , in our case, the  $X$  axis, and  $l'$  (new  
 248 position of line  $l$ );
- 249 ▪  $\theta$  – Angle between the line  $m^*$ , perpendicular to the lines  $l'$  and  $m'$ .

250  
 251 So we can use the found parameters  $\varphi, \theta, k_1, k_2$ , defined by the parameters  $a, b, \gamma, \mu$  of  
 252 simple transformations in the decomposition of  $DXU$ . If the perpendicular lines  $l$  and  $m$  on  
 253 the benchmark image were chosen coordinate axes  $X$  and  $Y$ , then:  
 254

$$\begin{aligned} \gamma &= \frac{l}{k_1} \\ \mu &= \frac{l}{(k_2 \cos \theta)} \\ \alpha &= -\varphi \\ h &= -tg\theta \end{aligned} \quad (21)$$

Substituting (21) in equation (12), we obtain the parameters of normalized matrix A.

#### 4. CONCLUSIONS

The normalization can be understood as the influence on the image obtained by the resulting matrix A. In other words, the simultaneous compensation of all transformations will form the difference between the benchmark and real images.

We can also apply a consistent normalization consistently to calculate the parameters  $\gamma, \mu, \alpha, \beta$ , and alternately having compensated transformation  $U, X, D$ .

In conclusion, we note that all the decompositions of matrix A are equivalent to each other, but in terms of some practical implementation, they are not equal. The criterion for selecting the most suitable can be: ease of implementation, restrictions on the allowed value of parameter of transformation, interference protection, etc.

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