

APPENDIX: Proof of **Theorem 3.1**

According to (3.1) and (3.2), there

$$w^{(m)} = T(S(w^{(m-1)})) = R(w^{(m-1)}).$$

To prove this theorem, we give the following four lemmas, and the appropriate proof.

Lemma 1 The operator S is non-expansive; T is $\frac{1}{2}$ -averaged non-expansive; R is non-expansive. (If for any $x, y \in R^n$, we have $\|Tx - Ty\|_2^2 \leq \langle Tx - Ty, x - y \rangle$, the operator T is called non-expansive; if there exists $\alpha \in (0, 1)$, such that $T_1 = (1 - \alpha)I + \alpha T$, then T_1 is called α -averaged non-expansive)

Proof. We know that S is a non-expansive operator from (3.1) formula and literature [19]. Now we define

$$\begin{aligned} S(w^{(m-1)}) &= \arg \min (z + g^2 e^{-z} - 2g + e^z) + \lambda_1 (z - w^{(m-1)})^2 \\ &= \arg \min 2\lambda_1 \left[\frac{1}{2} \|z - w^{(m-1)}\|^2 + \frac{1}{2\lambda_1} (z + g^2 e^{-z} - 2g + e^z) \right]. \end{aligned}$$

and $\phi = \frac{1}{2\lambda_1} (z + g^2 e^{-z} - 2g + e^z)$. According to the definition of the neighboring operator [19], we know that S is a neighboring operator as respect to ϕ , strictly convex function on z and under semi-continuous. We can get the consequence from Proposition 2.4 in the literature [8] in which shows

$$\|S(w_1) - S(w_2)\|_2^2 \leq \langle S(w_1) - S(w_2), w_1 - w_2 \rangle.$$

It proves that S is non-expansive.

Now, we define $M = 2T - I$, then

$$\begin{aligned} &\|M(z_1) - M(z_2)\|_2^2 \\ &= \|(2T - I)(z_1) - (2T - I)(z_2)\|_2^2 \\ &= \|2(T(z_1) - T(z_2)) - (z_1 - z_2)\|_2^2 \\ &= 4\|T(z_1) - T(z_2)\|_2^2 + \|z_1 - z_2\|_2^2 - 4\langle T(z_1) - T(z_2), z_1 - z_2 \rangle. \end{aligned} \tag{4.1}$$

With the similar way to prove S , we can get the following function

$$T(z^{(m)}) = \arg \min_w 2\lambda_1 \left[\frac{1}{2} \|z^{(m)} - w\|^2 + \frac{\lambda_2}{2\lambda_1} \int_{\Omega} |Dw| \log(e + |Dw|) \right],$$

Where ϕ is given by

$$\phi = \frac{\lambda_2}{2\lambda_1} \int_{\Omega} |Dw| \log(e + |Dw|).$$

On the other hand, ϕ is a convex function and under semi-continuous. So T satisfies

$$\|T(z_1) - T(z_2)\|_2^2 \leq \langle T(z_1) - T(z_2), z_1 - z_2 \rangle. \tag{4.2}$$

Combined with (4.1) and (4.2), we obtain

$$\|M(z_1) - M(z_2)\|_2^2 \leq \|z_1 - z_2\|_2^2.$$

According to the definition of the non-expansive operator and the average non-expansive, operator M is non-expansive. However, $T = (1 - 1/2)I + 1/2M$, so the operator R is a $\frac{1}{2}$ -averaged non-expansive. $\square$

Lemma 2 $\sum_{m=1}^{\infty} \|w^{(m-1)} - w^{(m)}\|_2^2$ is convergence.

Proof. Denote $J_1(z, w) = \|z - w\|_2^2$ and $J_2(w) = \int_D |Dw| \ln(e + |Dw|) dx dy$. Hence, we have

$$\begin{aligned} & J(z^{(m)}, w^{(m)}) - J(z^{(m)}, w^{(m+1)}) \\ &= \lambda_1 \left[J_1(z^{(m)}, w^{(m)}) - J_1(z^{(m)}, w^{(m+1)}) \right] \\ & \quad + \lambda_2 \left[J_2(w^{(m)}) - J_2(w^{(m+1)}) \right] \end{aligned} \quad (4.3)$$

That is to say $\frac{\partial J}{\partial w} = \lambda_1 \frac{\partial J_1}{\partial w} + \lambda_2 \frac{\partial J_2}{\partial w}$.

We consider the Taylor series expansion of $J_1(z^{(m)}, w^{(m)})$ in the second variable,

$$\begin{aligned} & J_1(z^{(m)}, w^{(m)}) \\ &= J_1(z^{(m)}, w^{(m+1)}) + (w^{(m+1)} - w^{(m)})^t \frac{\partial J_1}{\partial w}(z^{(m)}, w^{(m+1)}) \\ & \quad + \frac{1}{2} (w^{(m+1)} - w^{(m)})^t \frac{\partial^2 J_1}{\partial w^2}(z^{(m)}, w^{(m+1)}) (w^{(m+1)} - w^{(m)}) \end{aligned} \quad (4.4)$$

Here we note that J_1 is quadratic in w and x^t denotes a transpose of x . As J_2 is a convex function, we have

$$J_2(w^{(m)}) \geq J_2(w^{(m+1)}) + (w^{(m)} - w^{(m+1)})^t \frac{\partial J_2}{\partial w}(w^{(m+1)}) \quad (4.5)$$

By considering $\frac{\partial^2 J_1}{\partial w^2} = I$ and substituting (4.4) and (4.5) into (4.3), we obtain

$$\begin{aligned} & J(z^{(m)}, w^{(m)}) - J(z^{(m)}, w^{(m+1)}) \\ & \geq (w^{(m+1)} - w^{(m)})^t \left(\lambda_1 \frac{\partial J_1}{\partial w}(z^{(m)}, w^{(m+1)}) + \lambda_2 \frac{\partial J_2}{\partial w}(w^{(m+1)}) \right) \\ & \quad + \frac{\lambda_1}{2} \|w^{(m+1)} - w^{(m)}\|_2^2 \end{aligned}$$

As $w^{(m+1)}$ is the minimizer of $J(z^{(m)}, w)$, we obtain

$$\frac{\partial J}{\partial w}(z^{(m)}, w^{(m+1)}) = \lambda_1 \frac{\partial J_1}{\partial w}(z^{(m)}, w^{(m+1)}) + \frac{\partial J_2}{\partial w}(w^{(m+1)}) = 0$$

It is easy to prove the following fact $J(z^{(m)}, w^{(m+1)}) \geq J(z^{(m+1)}, w^{(m+1)})$, and we use it to show

$$\begin{aligned} & J(z^{(m)}, w^{(m)}) - J(z^{(m+1)}, w^{(m+1)}) \geq J(z^{(m)}, w^{(m)}) - J(z^{(m)}, w^{(m+1)}) \\ & \geq \frac{\lambda_1}{2} \|w^{(m+1)} - w^{(m)}\|_2^2 \end{aligned}$$

So we can say that $\sum_{m=0}^{\infty} \|w^{(m+1)} - w^{(m)}\|_2^2$ is bounded and converges. $\square$

Lemma 3 Model (2.2) corresponds to the objective function $J(z, w)$ is compulsory.

Proof. Denote $L = \begin{pmatrix} L_h \\ L_v \end{pmatrix}$. On one hand, L_h is a one-sided difference matrix on the horizontal direction, and L_v is on the vertical direction. On the other hand, L is not a full-rank matrix. The lower bound of the discrete TV is given by

$$\begin{aligned} \int_D |Dw| \ln(e + |Dw|) dx dy &= \sum_{1 \leq j, k \leq n} |\nabla w|_{j,k} \ln(e + |\nabla w|_{j,k}) \\ &= \sum_{1 \leq j, k \leq n} \sqrt{[(\nabla w)_{j,k}^x]^2 + [(\nabla w)_{j,k}^y]^2} \ln\left(e + \sqrt{[(\nabla w)_{j,k}^x]^2 + [(\nabla w)_{j,k}^y]^2}\right), \end{aligned} \quad (4.6)$$

where $\ln(e + |\nabla w|_{j,k}) \geq \ln(e) = 1$, so

$$(4.6) \geq \sum_{1 \leq j, k \leq n} \sqrt{[(\nabla w)_{j,k}^x]^2 + [(\nabla w)_{j,k}^y]^2} \geq \frac{1}{\sqrt{2}} \sum_{1 \leq j, k \leq n} |(\nabla w)_{j,k}^x| + |(\nabla w)_{j,k}^y| = \frac{1}{\sqrt{2}} \|Lw\|_1$$

The next proof refers to Lemma 3.8 in [19]. $\square$

Lemma 4 The operator R has at least one fixed point.

Proof. From Lemma 3, we can know that objective functional J is compulsory, so the set of fixed points of J is non-empty. Assume (z, w) is a minimizer of $J(z, w)$, i.e.,

$$\begin{aligned} \frac{\partial J}{\partial z}(z, w) &= 0, \quad \frac{\partial J}{\partial w}(z, w) = 0, \\ w &= T(z) = T(S(w)) = R(w). \end{aligned} \quad (4.7)$$

We can obtain that w is a fixed point of R .

proof of Theorem 1. Since objective functional (2.2) is strictly convex, and J is differentiable with respect to z , so the fixed point set of J is as well as its minimum point set. According to Lemma 4, the operator R exists fixed points, while strictly convex of the functional J ensures that R is at most one fixed point, so the operator R has a unique fixed point denoted w_0 . There, we can also get

$$w^{(m)} = R^m(w_0).$$

We have proved that R is non-expansive in Lemma 1, whereby $R^m(w_0)$ and $\{ \|R^m w - w_0\|_2 \}$ are also non-expansive. From the following inequality

$$\|R^{m+1}w - w_0\|_2 = \|R(R^m w) - R w_0\|_2 \leq \|R^m w - w_0\|_2, \quad m = 0, 1, \dots$$

we can get the below

$$d(w_0) = \lim_{m \rightarrow \infty} \|R^m w - w_0\|_2.$$

Next work is to prove $R^m w - w_0 \rightarrow 0 (m \rightarrow \infty)$.

(Proof by contradiction) Now if $\{R^{m_i} w\}$ subsequence of $\{R^m w\}$ has limits w' , and $w' \neq w_0$.

Known by the asymptotically regular of R , $\lim_{m_i \rightarrow \infty} \{(I - R)(R^{m_i} w)\} = 0$, or $(I - R)(w') = 0$, i.e.

$Rw' = w'$, so w' is also a fixed point of R . Which, however, is contradiction with the truth that R has a unique fixed point w_0 . So the limit is only one given by w_0 .

For further proof the Lemm3.1, we denote

$$w = \lim_{m \rightarrow \infty} R^m(w_0) = \lim_{m \rightarrow \infty} w^{(m)}.$$

Therefore

$$S(w) = \lim_{m \rightarrow \infty} S(w^{(m)}).$$

Using the non-expansive of S , we can obtain

$$\|z^{(m)} - S(w)\|_2 = \|S(w^{(m-1)}) - S(w)\|_2 \leq \|w^{(m-1)} - w\|_2.$$

So $\lim_{m \rightarrow \infty} w^{(m-1)} - w = 0$, $\lim_{m \rightarrow \infty} z^{(m)} = S(w)$, that is to say, $\{z^{(m)}\}$ converges to a unique fixed point,

when $m \rightarrow \infty$.

This completes the proof.

□