

An Efficient Variational Model for restoring Noisy Images with Gamma Multiplicative Noise

ABSTRACT

Multiplicative noise removal has been a focus of research in recent years. Aiming at solving the problem that the total variation regularization method can remove noise well but sometimes produce stair-case effect, this paper proposes an efficient variational model and gives the iterative algorithm to remove Gamma multiplicative noise. This paper also shows that the iterative sequence converges to the optimal solution of the model. Through simulation experiments the proposed model has proved highly effective. That is, this model can preserve the edges of the image well and significantly reduce the stair-case effect in the smooth regions while removing Gamma multiplicative noise effectively.

Keywords: Variational model; multiplicative noise; noise removal; stair-case effect

1. INTRODUCTION

It is well known that images deteriorate during formation, transmission, recording processing. Image denoising is of momentous significance in coherent imaging systems and various image processing applications [1]. Therefore, the image noise removal problems have attracted much attention in recent years [2-3].

Image noise can be roughly divided into additive noise and multiplicative noise. Models for removing additive noise have been studied extensively. Since additive noises are dominant for high frequencies, their effects can be removed using some kinds of low-pass filters [4]. Other kinds of methods, such as variational methods, have also been proposed. In [5], for example, Rudin, Osher and Fatemi proposed a total variation regularization model (ROF model), which has been widely accepted as a reliable tool for removing additive noise. Over the last few years, various research efforts have been devoted to studying, solving and extending the ROF model [6-9]. Multiplicative noises are commonly found in many real world image processing applications, such as in synthetic aperture radar, medical ultrasound images, and particle tomography [10, 11]. Therefore, how to effectively remove multiplicative noises becomes a significant study in recent years [12-15].

In general, the recorded image g , defined in $D \subset \mathbb{R}^2$, is the multiplication of an original image u and a noise v [16]:

$$g = uv. \quad (1.1)$$

In 2003, Rudin, Lions and Osher proposed the following denoising model in [17]:

$$\min_u J(u) = \min_u \left\{ \int_D |Du| dx dy + \lambda_1 \int_D \frac{g}{u} dx dy + \lambda_2 \int_D \left(\frac{g}{u} - 1 \right)^2 dx dy \right\}. \quad (1.2)$$

This model is called *RLO* model, it was first used to remove Gaussian noise, but the experimental results show that the texture detail of the restored image has been destroyed. When the multiplicative noise is out of the Gauss distribution, the above *RLO* model is no longer available.

In 2008, Aubert and Aujol [13] used the maximum a posteriori (MAP) regularization approach and derived a functional whose minimizer corresponds to the denoised image to be recovered. Their model is called *AA* model which can be described as follows:

$$\min_u J(u) := \min_u \left\{ \int_D |Du| dx dy + \lambda \int_D \left(\frac{g}{u} + \log u \right) dx dy \right\}, \quad (1.3)$$

where λ is the weighted parameter. Although the functional is not convex, the related theory, such as the existence of a minimizer and the associated evolution problem and so forth, have been still set up, and the capability of their model on some numerical examples was shown. Since the computed solutions by some optimization methods are not necessary to be a global optimal solution for (1.3), the quality of the restored image solution may not be good.

In 2009, Huang, Ng and Wen [14] added a fitting term $\int_D (z - w)^2 dx dy$ based on AA model, used a logarithmic transformation $z = \log u$ in the fidelity term of AA model and proposed the following model (Named *HNW* model):

$$\min_{z,w} J(z, w) := \min_{z,w} \left\{ \int_D |Dw| dx dy + \lambda_1 \int_D (z - w)^2 dx dy + \lambda_2 \int_D (z + ge^{-z}) dx dy \right\} \quad (1.4)$$

HNW model got over the ill-posedness of the restoration problem tactfully, and some experimental results also showed the quality of the images restored was better than that of the previous main models. However, its denoising effect is not ideal enough, and the restored image used by the model results in stair-case effect in smooth regions of the image.

In 2012, reference [18] proposed the following model (called *HZJ* model)

$$\min_u J(u) := \min_u \left\{ \int_\Omega |Du| \log(1 + |Du|) dx dy + \lambda_1 \int_\Omega ge^{-u} dx dy + \lambda_2 \int_\Omega (ge^{-u} - 1)^2 dx dy \right\}. \quad (1.5)$$

The model (1.5) has improved distinctly the denoising effect and preserved the details of images. Inspired by [14] and [18], Hu and Lou proposed a new total variational model as follows (called *HL* model) [19]:

$$\min_{z,w} J(z, w) := \min_{z,w} \left\{ \int_D |Dw| \log(e + |Dw|) dx dy + \lambda_1 \int_D (z - w)^2 dx dy + \lambda_2 \int_D (z + ge^{-z}) dx dy \right\}. \quad (1.6)$$

The numerical experiments showed that the texture details in the images restored by *HL* model were kept and the ‘stair-case effect’ was suppressed at some level.

In order to obtain an efficient denoising method for Gamma noise, it is a natural choice to build a novel model by introducing an appropriate coordinating term and regularization term. In this paper we propose a novel denoising model which not only ensures that our model removes Gamma noise very well, but also it can effectively protect the edge details and texture features of the image. Our experimental results show that the quality of images restored by the proposed method is better than that by several popular models.

The rest of this paper is organized as follows. In next section, we introduce our model briefly. In Section 3, the minimizing algorithm and the convergence of the proposed method are given. In Section 4, we show experimental results to demonstrate the performance of our proposed model. Finally, concluding remarks are given in section 5.

2. THE PROPOSED MODEL

What is usually referred to as multiplicative noise removal is of course nothing but the estimation of the reflectance of the underlying scene in imaging systems. This is an inverse problem calling for regularization, which usually consists in assuming that the underlying reflectance image is piecewise smooth. This assumption has been formalized, in a Bayesian estimation framework and variational approaches. Both the variational and the Bayesian MAP formulations to image denoising (under multiplicative, Gaussian, or other noise models) lead to optimization problems with two terms: a data fidelity term and a regularizer.

For the noise model (1.1), invoking the Bayesian rules and the MAP criterion, and using logarithm transformation, the fidelity term $\int (z + ge^{-z})$ for removing Gamma noise can be obtained [20]. In this paper, we consider the following multiplicative noise model which provides an accurate description of many special imaging systems [10]

$$g = u + \sqrt{uv}. \quad (2.1)$$

81 That is to say, $v = (g - u)^2 / u$. We have known that the fidelity term of AA is $\int_D \left(\frac{g}{u} + \log u \right) dx dy$, and
 82 many existing models used a logarithmic transformation $z = \log u$ in AA model to get a new
 83 fidelity term $\int_D (z + g e^{-z}) dx dy$. In this paper, we can get $\int_D \left(\frac{(g - u)^2}{u} + \log u \right) dx dy$ by using the new
 84 v to replace the original $v = \frac{g}{u}$. Now, we also use logarithmic transformation $z = \log u$, and then
 85 get a new the fidelity term $\int (z + g^2 e^{-z} - 2g + e^z)$ for removing Gamma noise.

86 Inspired by [19], this paper uses the new hybrid measure $\int |Dw| \ln(e + |Dw|)$ as energy functional
 87 regularization term, and invoke the coordination term $\int (z - w)^2$ to propose the new hybrid
 88 variational model for removing Gamma multiplicative noise as follows

$$89 \min_{z, w} J(z, w) := \min_{z, w} \left\{ \int_D (z + g^2 e^{-z} + e^z - 2g) dx dy + \lambda_1 \int_D (z - w)^2 dx dy + \lambda_2 \int_D |Dw| \ln(e + |Dw|) dx dy \right\}. \quad (2.2)$$

90 The first term of model (2.2) is called the loyalty term which ensures recovering image u to retain
 91 the main features from the virtual image $\log g$. The second term is the coordination term which
 92 measures the influence between the fitting term and the regularization term and plays a role in
 93 undertaking and tradeoffs. The final term is the regularization term which ensures the smooth of
 94 the denoising image w , and prevents stair-case effects.
 95 Before removing image noise by using our model, we first do a convolution operation for the
 96 original degraded image so as to avoid mistaking some noise gradient for the edge gradient.

97 3. THE ITERATIVE ALGORITHM

98 Inspired by the idea of the literature [14, 19], we use an adaptive alternating iterative algorithm to
 99 solve the model (2.2). It is divided into two optimization problems. Starting from the initial data
 100 $w^{(0)}$, we solve the following variational problems

$$101 \begin{cases} z^{(m)} = S(w^{(m-1)}) := \arg \min_z \left\{ \int_D (z + g^2 e^{-z} - 2g + e^z) + \lambda_1 \int_D (z - w^{(m-1)})^2 \right\} & (3.1) \\ w^{(m)} = T(z^{(m)}) := \arg \min_w \left\{ \lambda_1 \int_D (z^{(m)} - w)^2 + \lambda_2 \int_D |Dw| \ln(e + |Dw|) \right\} & (3.2) \end{cases}$$

102 Cycling alternately, we can get the following iterative sequence

$$103 w^{(0)}, z^{(1)}, w^{(1)}, z^{(2)}, w^{(2)}, \dots, z^{(m)}, w^{(m)}, \dots$$

104 Step 1. To solve the variational problem (3.1). we need to solve its discretization:

$$105 z^{(m)} = \arg \min_z \left\{ \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} (z(i, j) + g(i, j)^2 e^{-z(i, j)} - 2g(i, j) + e^{g(i, j)}) + \lambda_1 \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} (z(i, j) - w^{(m-1)}(i, j))^2 \right\}. \quad (3.3)$$

106 Letting $f(z(i, j)) := (z(i, j) + g(i, j)^2 e^{-z(i, j)} - 2g(i, j) + e^{g(i, j)}) + \lambda_1 (z(i, j) - w^{(m-1)}(i, j))^2$. Since f is
 107 continuous and derivable in the specified range, this problem (3.3) is equivalent to solving the
 108 regular system with n^2 equations:

$$109 f'(z(i, j)) := 1 - g(i, j)^2 e^{-z(i, j)} + e^{z(i, j)} + 2\lambda_1 (z(i, j) - w^{(m-1)}(i, j)) = 0, \quad i, j = 1, 2, \dots, n \quad (3.4)$$

110 Noticing that $f''(z(i, j)) := g(i, j)^2 e^{-z(i, j)} + e^{z(i, j)} + 2\lambda_1 > 0$, $i, j = 1, 2, \dots, n$, the function f is convex
 111 strictly with respect to z , so we can find has a unique solution of the system (3.4) by using
 112 Newton iterative method

$$113 z^{(m)}(i, j) = z^{(m-1)}(i, j) - f'(z^{(m-1)}(i, j)) / f''(z^{(m-1)}(i, j)). \quad (3.5)$$

114 This is the solution of the problem (3.1).

115 Step 2. To find the solution of the variational problem (3.2). Letting

$$116 H(x, y, w, w_x, w_y) = \lambda_1 \int_D (z - w)^2 + \lambda_2 \int_D |Dw| \log(e + |Dw|),$$

117 we get the corresponding Euler-Lagrange equation of the variational problem (3.2) as follows

$$\lambda_2 \operatorname{div} \left(\frac{|Dw| + (e + |Dw|) \log(e + |Dw|)}{(e + |Dw|)|Dw|} Dw \right) + 2\lambda_1(z - w) = 0. \quad (3.6)$$

Let $\psi(x) = (x + (e + x) \log(e + x)) / ((e + x)x)$, (3.6) is simplified as

$$\lambda_2 \operatorname{div}(\psi(|Dw|)Dw) + 2\lambda_1(z - w) = 0, \quad (3.7)$$

In this paper, $Dw(i, j)$ is the gradient at the location (i, j) , $Dw(i, j) = (w_x(i, j), w_y(i, j))$ and

$$|Dw(i, j)| = \sqrt{w_x(i, j)^2 + w_y(i, j)^2}, \quad i, j = 1, 2, \dots, n.$$

where

$$w_x(i, j) = \begin{cases} w(i+1, j) - w(i, j), & i < n, \\ 0, & i = n. \end{cases}$$

and

$$w_y(i, j) = \begin{cases} w(i, j+1) - w(i, j), & j < n, \\ 0, & j = n. \end{cases}$$

Invoking the numerical method in [19], we can obtain the numerical solution of (3.7) as follows

$$w_{-t} = \lambda_2 \sum_{p \in K} [\psi(|Dw(p)|_\varepsilon)(w(p) - w(i, j))] + 2\lambda_1(w - z^{(m)}) \quad (3.8)$$

and iterative formula

$$w^{(m)} = w^{(m-1)} + dt \times w_{-t}. \quad (3.9)$$

Where $K = \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$, $|Dw(p)|_\varepsilon \approx \sqrt{\varepsilon + w_x^2(p) + w_y^2(p)}$, ε is an arbitrary small positive constant, dt is a constant, we take $dt = 0.12$ in this paper.

Step 3. Repeat the above process until a stopping criterion is satisfied. The cyclic condition is as follows

$$\frac{\|w^{(m+1)} - w^{(m)}\|}{\|w^{(m)}\|} \leq 10^{-4}.$$

Slightly modifying the corresponding procedures in e.g. [19], we can prove the following theorem easily.

Theorem 3.1 For any initial data $w^{(0)}$, iterative sequence $\{w^{(m)}\}$ converges to the optimal solution of the problem (2.2).

4. EXPERIMENT RESULTS AND ANALYSIS

In this section, numerical results are presented to demonstrate the performance of our proposed model. The results are compared with several popular models such as AA model, HNW model and HL model. We used three images with the size 256×256 , Lena, Barbara and Butterfly, in our experiments; see Fig. 1-7. Image Lena is a good test image because it has a nice mixture of detail, flat regions, shading area, and texture. Both Barbara and Butterfly also consist of complex components in different scales, with different patterns and under inhomogeneous illuminations. The three images are suited for our experiments.

In the tests, each pixel of an original image is degraded by a Gamma noise with mean one, and the noise level is controlled by the value of L in the experiments. The experiments are performed in MATLAB under the same software and hardware conditions.

The original image Lena and Barbara are shown in Fig.1. The Lena image in Fig.1 (a) is distorted by a Gamma noise with $L = 20$ and $L = 10$, respectively. The noisy images are shown in Fig. 2 (a) and Fig. 3 (a), respectively. The Barbara image in Fig. 1 (b) is distorted by a Gamma noise with $L = 5$ and $L = 10$ respectively. The noisy images are shown in Fig. 4 (a) and Fig. 5 (a), respectively. As L is smaller, the pictures are more noisy.



(a) Lena (b) Barbara

Fig.1. The original images

From the denoising contrast experiments shown in Fig.2-5, we see that the denoising results obtained by the proposed model and the *HL* model are visually much better than those by the others. Obviously, the restoration result by the proposed model is the best in the four models.



(a)The noisy image for $L=20$ (b) the restored image by AA (c) the restored image by HNW



(d) the restored image by HL (e) the restored image by ours

Fig.2. The noisy image Lena and denoising contrast experiment 1.

In Fig.2, we show how the four models behave with a small noisy image. Notice that in this case all the models perform well. Besides, the proposed model is superior to the other three models for preserving the textures and fine details of the images. In Fig.3-5, we see that the proposed model gets a very good visual effect and works well in the case of heavy noise.



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(a)The noisy image for L=10 (b) the restored image by AA (c) the restored image by HNW



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(d) the restored image by HL (e) the restored image by ours
Fig.3. The noisy image Lena and denoising contrast experiment 2.



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(a)The noisy image for L=10 (b) the restored image by AA (c) the restored image by HNW



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(d) the restored image by HL (e) the restored image by ours
Fig.4. The noisy image Barbara and denoising contrast experiment 3.



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(a)The noisy image for L=5 (b) the restored image by AA (c) the restored image by HNW



(d) the restored image by HL (e) the restored image by ours
Fig.5. The noisy image Barbara and denoising contrast experiment 4.

In addition to visual comparison, we use the peak signal to noise ratio (PSNR) and relative error rate (ReErr) of the images to assess the quality of the restored images [14]. Greater PSNR or the smaller ReErr is, the denoising effect is better. From Tables 1, we can see that the PSNR obtained by our model is the biggest in those by all four models for the same noisy image. Similarly the ReErr obtained by the proposed model is smallest. Both denoising visual effect and objective index have showed that the proposed model behaves very well. The model can remove Gamma noise effectively, but also it can protect the edge details and texture features of the image, prevent from producing stair-case phenomenon at some level.

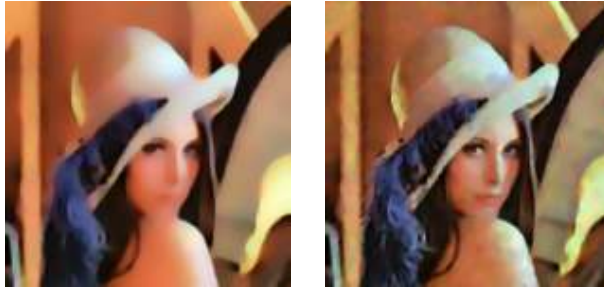
Table 1. The PSNRs and ReErrs of the restored images

Image	AA model	HNW model	HL model	Proposed model
"Lena" in Fig.2 (L=20)	PSNR=61.55 ReErr=0.091	PSNR=61.77 ReErr=0.087	PSNR=63.51 ReErr=0.084	PSNR=65.60 ReErr=0.074
"Lena" in Fig.3 (L=10)	PSNR=55.78 ReErr=0.129	PSNR=57.58 ReErr=0.115	PSNR=57.90 ReErr=0.113	PSNR=61.51 ReErr=0.092
"Barbara" in Fig.4 (L=10)	PSNR=48.31 ReErr=0.207	PSNR=48.65 ReErr=0.203	PSNR=49.36 ReErr=0.182	PSNR=49.42 ReErr=0.197
"Barbara" in Fig.5 (L=5)	PSNR=45.11 ReErr=0.257	PSNR=46.58 ReErr=0.201	PSNR=47.77 ReErr=0.200	PSNR=48.86 ReErr=0.197

Finally, we also display the restoration of the color Lena image and the Butterfly image. From Fig. 6-7, it is clear that the restoration results by the proposed method are visually better than those by *HL* model. Especially, our model is superior in avoiding or reducing the stair-case effect.



(a) The original color Lena image (b) The noisy image for L=10

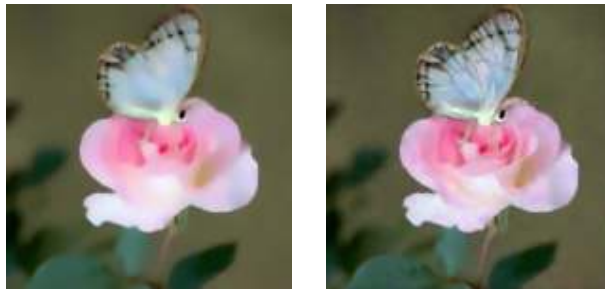


(c) the restored image by HL (d) the restored image by ours

Fig.6. The original color Lena image and denoising contrast experiment 5.



(a) The original color Butterfly image (b)The noisy image for L=50



(c) the restored image by HL (d) the restored image by ours

Fig.7. The original color Butterfly image and denoising contrast experiment 6.

5. COMCLUSION

In this paper, we propose an efficient total variation model for Gamma multiplicative noise removal. Strictly convex objective function not only ensures that the new model has a unique solution, but also improves the quality of the restored image. Our model can reduce stair-case effect significantly while removing Gamma multiplicative noise quite well. Our experimental results also show that the method is superior to some of existing total variation methods.

6. REFERENCES

1. Aubert G and Kornprobst P. Mathematical Problem in Image Processing. Springer, New York, NY, USA. 2002.
2. Buades A, Coll B, Morel JM. Image Denoising Methods: a New Nonlocal Principle. SIAM Review. 2010; 52(1):113-147.
3. Talebi H, Milanfar P. Global Image Denoising. IEEE Transactions on Image Processing. 2013; 23(2):755-768.

- 230 4. Coifman R, Donoho D, Translation-invariant de-noising, in Wavelets and Statistics,
231 Springer, New York. 1995; 125-150.
- 232 5. Rudin LI, Osher S, Fatemi E. Nonlinear total variation based noise removal algorithms.
233 Physica D. 1992; 60:259-268.
- 234 6. Malgouyres F. Minimizing the total variation under a general convex constraint for image
235 restoration. IEEE Trans. Image Process. 2002; 11(12):1450-1456.
- 236 7. Chang Q, Chern IL. Acceleration methods for total variation based image denoising.
237 SIAM J. SCI. COMPUT. 2003; 25(3):982-994.
- 238 8. Oshe S, Burger M, Goldfarb D, et al. An iterative regularization method for total
239 variation-based image restoration. Multiscale Modeling & Simulation. 2005;
240 4(2):460-489.
- 241 9. Dong Y, Hintermüller M, Rincon-Camacho MM. Automated regularization parameter
242 selection in multi-Scale total variation models for image restoration. J. Math. Imaging
243 Vis.2011; 40: 82-104.
- 244 10. Zheng MJ, Xiao PY. Analysis of a new variational model for multiplicative noise removal.
245 Journal of Mathematical Analysis and Applications.2010; 362:415-426.
- 246 11. Shih Y, Rei C, Wang H. A novel PDE based image restoration: Convection- diffusion
247 equation for image de-noising. Journal of Computational and Applied Mathematics. 2009;
248 231: 771-779.
- 249 12. Allard WK. Total variation regularization for image de-noising, I. Geometric theory. SIAM
250 Journal on Mathematical Analysis. 2007; 39:1150-1190.
- 251 13. Aubert G, Aujol JF. A variational approach to removing multiplicative noise. SIAM Journal
252 on Applied Mathematics. 2008; 68(4):925-946.
- 253 14. Huang YM, Ng MK, Y W Wen. A new total variation method for multiplicative noise
254 removal. SIAM Journal on Imaging Sciences.2009; 2(1):20-40.
- 255 15. Zheng MJ, Xiao PY. A variational model to remove the multiplicative noise in ultrasound
256 images. Journal of Mathematical Imaging and Vision. 2011; 39:62-74.
- 257 16. Shi J, Osher S. A nonlinear inverse scale space method for a convex multiplicative noise
258 model. SIAM Journal on Imaging Sciences. 2008; 1(3): 294-321.
- 259 17. Rudin LI, Lions PL, Osher S. Multiplicative de-noising and deblurring: theory and
260 algorithms. In: Osher, S., Paragios, N. (eds.) Geometric Level Sets in Imaging, Vision,
261 and Graphics. 2003; 103–119.
- 262 18. Hu XG, Zhang LT, Jiang W. Improved variational model to remove multiplicative noise
263 based on partial differential equation. Journal of Computer Applications. 2012;
264 32(7):1879-1881.(in Chinese)
- 265 19. Hu XG, Lou YF. A novel total variational model for Gamma multiplicative noise removal.
266 Journal of Sichuan University (Engineering Science Edition). 2014; 46(2):59-65. (in
267 Chinese)
- 268 20. José M, Mário A. Multiplicative noise removal using variable splitting and constrained
269 optimization. IEEE Transactions on Image Processing. 2010; 19(7): 1720-1730.
- 270 21. Lv XG, Jiang L. A fast high-order total variation minimization method for multiplicative
271 noise removal. Mathematical Problems in Engineering. 2013; 1-13.
- 272 22. Xu JL, Hao Y. An effective dual method for multiplicative noise removal.
273 J.Vis.Commun.Image R.2014; 306-312.
- 274 23. Chen LY, Zeng TY. A convex variational model for restoring blurred images with large
275 Rician noise. J Math Imaging Vis. 2014; 1-23.
- 276 24. Sanches JM, Afonso M. Image reconstruction under multiplicative speckle using total
277 variation. Contents lists available at ScienceDirect Neurocomputing. 2015; 200-213.