

Bounds for the Convex Combination of the First Seiffert and Logarithmic Means in Terms of Generalized Heronian Mean

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Abstract

In this paper, the bounds for convex combination of the first Seiffert and Logarithmic means by general Heronian mean

$$H_{p(\alpha)}(a, b) < \alpha P(a, b) + (1 - \alpha)L(a, b) < H_{q(\alpha)}(a, b)$$

are proved, where $a, b > 0$, $a \neq b$, $\alpha \in [0, 1]$. The left bound is the best possible.

Keywords: Seiffert mean; Heronian mean; Logarithmic mean; convex combination; bounds

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1 Introduction

In the paper [Gao, Guo and Li (2012)], authors proved the following optimal inequalities:

Let $a, b > 0$, $a \neq b$ then

$$H_\delta(a, b) < P(a, b) \quad \text{for} \quad \delta \geq \pi - 2 \quad \text{and} \quad P(a, b) < H_\beta(a, b) \quad \text{for} \quad \beta \leq 1 \quad (1.1)$$

and $\delta = \pi - 2$, $\beta = 1$ are the best constants,

$$H_\gamma(a, b) < L(a, b) \quad \text{for} \quad \gamma = +\infty \quad \text{and} \quad L(a, b) < H_\tau(a, b) \quad \text{for} \quad \tau \leq 4 \quad (1.2)$$

and $\gamma = +\infty$, $\tau = 4$ are the best constants.

$P(a, b)$ is the first Seiffert mean introduced by [Seiffert (2001)]

$$P(a, b) = \frac{a - b}{4 \arctan\left(\sqrt{\frac{a}{b}}\right) - \pi} = \frac{a - b}{2 \arcsin\left(\frac{a-b}{a+b}\right)} \quad \text{for} \quad a, b > 0, \quad a \neq b. \quad (1.3)$$

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$L(a, b)$ is the Logarithmic mean

$$L(a, b) = \frac{a - b}{\log a - \log b} \quad \text{for } a, b > 0, a \neq b. \quad (1.4)$$

$H_\omega(a, b)$ is the Heronian mean introduced by [Janous (2001)]

$$H_\omega(a, b) = \frac{a + \omega\sqrt{ab} + b}{\omega + 2} \quad \text{for } 0 \leq \omega < +\infty, \quad (1.5)$$

$$= \sqrt{ab} \quad \text{for } \omega = +\infty.$$

Recently, the Logarithmic, Seiffert and Heronian means has been the subject of intensive research. In particular, many remarkable inequalities for the means can be found in the literature [see all references].

In [Liu, Meng (2011)], authors proved interesting bounds. They found the greatest values α_1, α_2 and the least values β_1, β_2 such that the double inequalities

$$\alpha_1 C(a, b) + (1 - \alpha_1)G(a, b) < P(a, b) < \beta_1 C(a, b) + (1 - \beta_1)G(a, b)$$

and

$$\alpha_2 C(a, b) + (1 - \alpha_2)G(a, b) < P(a, b) < \beta_2 C(a, b) + (1 - \beta_2)G(a, b)$$

hold for all $a, b > 0$ with $a \neq b$, where $C(a, b)$, $G(a, b)$, $H(a, b)$ and $P(a, b)$ denote the Contraharmonic, Geometric, Harmonic, and Seifferts means of two positive numbers a and b , respectively.

Similarly, in [Xia, Chu and Wang (2010)] it was shown that for $\alpha \in (0, 1)$, the double inequality

$$M_{\log 2 / (\log 2 - \log \alpha)}(a, b) \leq \alpha A(a, b) + (1 - \alpha)L(a, b) \leq M_{(1+2\alpha)/3}(a, b)$$

holds for all $a, b > 0$, each inequality becomes an equality if and only if $a = b$, and the given parameters $\log 2 / (\log 2 - \log \alpha)$ and $(1 + 2\alpha)/3$ in each inequality are best possible. Here $M_p(a, b)$, $L(a, b)$, $A(a, b)$ are Power, Logarithmic, Arithmetic means, respectively.

In [Matejíčka (2010)], it was proved. Let $\alpha \in (0, 1/2) \cup (1/2, 1)$, $a \neq b$, $a, b > 0$. Let $p(\alpha)$ be a solution of

$$\frac{1}{p} \log(1 + p) + \log \alpha / 2 = 0 \quad \text{in } (-1, 1).$$

Then

$$\text{if } \alpha \in (0, 1/2), \text{ then } \alpha A(a, b) + (1 - \alpha)G(a, b) < L_p(a, b) \text{ for } p \geq p(\alpha)$$

and $p(\alpha)$ is the best constant,

$$\text{if } \alpha \in (1/2, 1), \text{ then } \alpha A(a, b) + (1 - \alpha)G(a, b) > L_p(a, b) \text{ for } p \leq p(\alpha)$$

and $p(\alpha)$ is the best constant.

Here $L_p(a, b)$, $A(a, b)$, $G(a, b)$ are Generalized logarithmic, Arithmetic, Geometric means, respectively.

It might be surprising that the means have applications in physics, economics, and even in meteorology.

It is well known, that $H_\omega(a, b)$ is a decreasing continuous function of the argument ω . From this and from results of [Gao, Guo and Li (2012)] it follows that there exist optimal functions $p(\alpha)$, $q(\alpha)$, $0 \leq \alpha \leq 1$ such that

$$H_{p(\alpha)}(a, b) < \alpha P(a, b) + (1 - \alpha)L(a, b) < H_{q(\alpha)}(a, b).$$

Therefore, it is natural to ask what are the optimal functions. The purpose of this paper is to find or establish suitable bounds for the optimal functions. The inequalities we obtained are new and improve the existing corresponding results. The left inequality is optimal.

2 Main Results

In this section we prove the following theorem.

Theorem 2.1. Let $a, b > 0$, $a \neq b$. Let $P(a, b)$, $L(a, b)$, $H_\omega(a, b)$, be the first Seiffert mean, the Logarithmic mean and the Heronian mean. Let $\alpha \in [0, 1]$. Then

$$H_p(a, b) < \alpha P(a, b) + (1 - \alpha)L(a, b) \quad (2.1)$$

if and only if $p < p(\alpha) = \frac{\pi}{\alpha} - 2$.

If

$$\alpha P(a, b) + (1 - \alpha)L(a, b) < H_q(a, b) \quad (2.2)$$

then $q < 4 - 3\alpha$. Moreover, if $q \leq \frac{4}{1+3\alpha}$ then (2.2) holds.

Prof. (2.1), (2.2) are equivalent to

$$\frac{1}{b} H_p(a, b) < \frac{\alpha}{b} P(a, b) + \frac{(1 - \alpha)}{b} L(a, b) < \frac{1}{b} H_q(a, b). \quad (2.3)$$

Without loss of generality we can suppose that $0 < b < a$. Denote $t = \sqrt{b/a}$ and put

$$F(t, \alpha, \omega) = \alpha \frac{1 - t^2}{\pi - 4 \arctan t} + (1 - \alpha) \frac{1 - t^2}{-2 \ln t} - \frac{1}{\omega + 2} (t^2 + \omega t + 1). \quad (2.4)$$

Then $0 < t < 1$. First we prove the left inequality. This inequality is equivalent to $R(t, \alpha) = F(t, \alpha, \pi/\alpha - 2) > 0$ for $0 < t, \alpha < 1$. Some computation gives

$$R(t, \alpha) = \alpha \frac{1 - t^2}{\pi - 4 \arctan t} + (1 - \alpha) \frac{1 - t^2}{-2 \ln t} - \frac{\alpha t^2}{\pi} - \left(1 - \frac{2\alpha}{\pi}\right) t - \frac{\alpha}{\pi}. \quad (2.5)$$

Because of $R(t, \alpha)$ is a linear function of argument α , it suffices to show that $R(t, 0) > 0$ and $R(t, 1) > 0$ for $t \in (0, 1)$. Denote $F_1(t) = R(t, 0)$ and $F_2(t) = R(t, 1)$. Then

$$F_1(t) = \frac{s(t)}{2 \ln t} = \frac{t^2 - 1 - 2t \ln t}{2 \ln t}. \quad (2.6)$$

From $s''(t) = \frac{2t-2}{t} < 0$, $s'(1) = 0$, where $s'(t) = 2t - 2 \ln t - 2$ we have $s'(t) > 0$ and from $s(1) = 0$ we obtain $s(t) < 0$ and so $R(t, 0) > 0$. $F_2(t) = R(t, 1) > 0$ is equivalent to

$$h(t) = \pi - 4 \arctan t - \pi \frac{1 - t^2}{t^2 + (\pi - 2)t + 1} < 0. \quad (2.7)$$

Because of $h(0) = 0$, $h(1) = 0$ and $h'(0) = \pi^2 - 2\pi - 4 < 0$, where

$$h'(t) = -\frac{4}{1 + t^2} - \pi \left(\frac{-2t(t^2 + (\pi - 2)t + 1) - (1 - t^2)(2t + \pi - 2)}{(t^2 + (\pi - 2)t + 1)^2} \right) \quad (2.8)$$

it suffices to show that $h'(t)$ has only one root in $(0, 1)$. From $h'(t) = p(t)/((1+t^2)((t^2+(\pi-2)t+1)^2))$, where

$$p(t) = (\pi^2 - 2\pi - 4)t^4 + (16 - 4\pi)t^3 + (-2\pi^2 + 12\pi - 24)t^2 + (16 - 4\pi)t + \pi^2 - 2\pi - 4. \quad (2.9)$$

it suffices to show that $p(t)$ has only one root in $(0, 1)$.

Some computation gives that It is easy to show that

$$p(t) = (1 - t^2)q(t) = (1 - t^2) \left((\pi^2 - 2\pi - 4)t^2 + (2\pi^2 - 8\pi + 8)t + \pi^2 - 2\pi - 4 \right). \quad (2.10)$$

From $q(0) = \pi^2 - 2\pi - 4 < 0$ and $q(1) = 4\pi(\pi - 3) > 0$ we have $R(t, 1) > 0$.
From

$$\lim_{t \rightarrow 0^+} F(t, \alpha, \omega) = \frac{\alpha}{\pi} - \frac{1}{\omega + 2}$$

we obtain that $p(\alpha)$ is the best possible function.

Now we prove the right bounds.

To show $q(\alpha) > 4/(\alpha + 3\alpha)$ we need to prove $F(t, \alpha, 4/(\alpha + 3\alpha)) < 0$ for all $t, \alpha \in (0, 1)$. This is equivalent to

$$G(t, \alpha) = \alpha \frac{1-t^2}{\pi - 4 \arctan t} + (1-\alpha) \frac{1-t^2}{-2 \ln t} - \frac{1+3\alpha}{6+6\alpha} (t^2 + \frac{4}{1+3\alpha} t + 1) < 0. \quad (2.11)$$

Some computation leads to

$$G''_{\alpha\alpha}(t, \alpha) = 6 \frac{1-t^2}{(\pi - 4 \arctan t)(\ln t)} \{2 \ln t - 4 \arctan t + \pi\}. \quad (2.12)$$

Denote $u(t) = 2 \ln t - 4 \arctan t + \pi$. Because $u(1) = 0$ and

$$u'(t) = \frac{2t^2 - 4t + 2}{t(1+t^2)} > 0$$

we have $u(t) < 0$ and $G''_{\alpha\alpha}(t, \alpha) > 0$. It implies $G(t, \alpha)$ is a convex function in argument α for each $t \in (0, 1)$. If we show that $G(t, 0) < 0$ and $G(t, 1) < 0$ for all $t \in (0, 1)$ then $G(t, \alpha) < 0$. $G(t, 0) < 0$ is equivalent to $s(t) = (3t^2 - 3)/(t^2 + 4t + 1) - \ln t > 0$. Simple computation leads to

$$s'(t) = \frac{\xi(t)}{t(t^2 + 4t + 1)^2} = \frac{-t^4 + 4t^3 - 6t^2 + 4t - 1}{t(t^2 + 4t + 1)^2}.$$

From $\xi(t) = -4t^3 + 12t^2 - 12t + 4$, and $\xi''(t) = -12t^2 + 24t - 12 < 0$, $\xi'(1) = 0$, $\xi(1) = 0$ we have $\xi'(t) > 0$ and $\xi(t) < 0$. From this and from $s(1) = 0$ we obtain $G(t, 0) < 0$. $G(t, 1) < 0$ is equivalent to $v(t) = 3(t^2 - 1)/(t^2 + t + 1) - 4 \arctan t + \pi > 0$. From $v(1) = 0$ it suffices to show that $v'(t) < 0$. It follows from

$$v'(t) = \frac{\xi(t)}{(1+t^2)(t^2+t+1)^2} = \frac{-t^4 + 4t^3 - 6t^2 + 4t - 1}{(1+t^2)(t^2+t+1)^2}.$$

To show $q(\alpha) < 4 - 3\alpha$ we need to prove $F(t, \alpha, 4 - 3\alpha) > 0$ for each α and some $t_\alpha \in (0, 1)$. Put $t_\alpha = 1 - \alpha + \alpha^2$. We show that $F(1 - \alpha + \alpha^2, \alpha, 4 - 3\alpha) > 0$ for $\alpha \in (0, 1)$. We use the following inequality

$$\frac{1-t^2}{-\ln t} > \frac{1-t^2}{\left(1-t + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3t}\right)}.$$

This inequality follows from the Taylor's series for $\ln t$, $t \in (0, 1)$

$$-\ln t < 1 - t + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3} + \frac{(1-t)^4}{3} + \dots$$

To prove $F(t, \alpha, 4 - 3\alpha) > 0$ for $t_\alpha = 1 - \alpha + \alpha^2$, $\alpha \in (0, 1)$ it suffices to show that

$$(6\alpha - 3\alpha^2) \frac{1-t_\alpha^2}{\pi - 4 \arctan t_\alpha} + (3\alpha^2 - 9\alpha + 6) \frac{1-t_\alpha^2}{2 \left(1-t_\alpha + \frac{(1-t_\alpha)^2}{2} + \frac{(1-t_\alpha)^3}{3t_\alpha}\right)} \quad (2.13)$$

$$-t_\alpha^2 - (4 - 3\alpha)t_\alpha - 1 > 0.$$

Some calculation gives (17) is equivalent to

$$(6\alpha - 3\alpha^2) \frac{1 - t_\alpha^2}{\pi - 4 \arctan t_\alpha} > \frac{\varphi(t_\alpha)}{\psi(t_\alpha)}, \quad (2.14)$$

where

$$\varphi(t_\alpha) = t_\alpha^4 - (1 + 3\alpha)t_\alpha^3 + (9\alpha^2 - 12\alpha - 3)t_\alpha^2 + (9\alpha^2 - 21\alpha + 5)t_\alpha - 2, \quad (2.15)$$

$$\psi(t_\alpha) = t_\alpha^2 - 5t_\alpha - 2. \quad (2.16)$$

It is easy to show that

$$\begin{aligned} t_\alpha^2 &= 1 - 2\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4, \\ t_\alpha^3 &= 1 - 3\alpha + 6\alpha^2 - 7\alpha^3 + 6\alpha^4 - 3\alpha^5 + \alpha^6, \\ t_\alpha^4 &= 1 - 4\alpha + 10\alpha^2 - 16\alpha^3 + 19\alpha^4 - 16\alpha^5 + 10\alpha^6 - 4\alpha^7 + \alpha^8 \\ \varphi(t_\alpha) &= \alpha \chi(\alpha) = \alpha(\alpha^7 - 7\alpha^6 + 27\alpha^5 - 61\alpha^4 + 91\alpha^3 - 105\alpha^2 + 72\alpha - 36). \end{aligned} \quad (2.17)$$

It implies

$$\chi(\alpha) < -36(1 - \alpha)^2 - 46\alpha^2(1 - \alpha)^2 + \alpha^2(-23 - 15\alpha^2 + 27\alpha^3 - 7\alpha^4 + \alpha^5). \quad (2.18)$$

Denote $k(\alpha) = -23 - 15\alpha^2 + 27\alpha^3 - 7\alpha^4 + \alpha^5$. We have $k(\alpha) < l(u) = -23 + 12u - 6u^2$, where $u = \alpha^2$. From $l(1) = -17$ and $l'(u) = 12(1 - u) > 0$ we have $l(u) < 0$ so $\varphi(t_\alpha) < 0$. Evidently $\psi(t_\alpha) < 0$. Denote

$$H(\alpha) = 4 \arctan(t_\alpha) - \pi + (6\alpha - 3\alpha^2)(1 - t_\alpha^2) \frac{\psi(t_\alpha)}{\varphi(t_\alpha)}. \quad (2.19)$$

We need to show $H(\alpha) > 0$. From $H(1) = H(0) = 0$ it suffices to show that $H'(\alpha)$ has only one root in $(0, 1)$, and $H'(0) > 0$. Some computation gives

$$\begin{aligned} H(\alpha) &= 4 \arctan(1 - \alpha + \alpha^2) - \pi - \\ &\frac{(6 - 3\alpha)(12\alpha - 24\alpha^2 + 25\alpha^3 - 14\alpha^4 - \alpha^5 + 5\alpha^6 - 4\alpha^7 + \alpha^8)}{-36 + 72\alpha - 105\alpha^2 + 91\alpha^3 - 61\alpha^4 + 27\alpha^5 - 7\alpha^6 + \alpha^7}. \end{aligned} \quad (2.20)$$

This can be rewritten as

$$H(\alpha) = 4 \arctan(1 - \alpha + \alpha^2) - \pi - \frac{a(\alpha)}{b(\alpha)}, \quad (2.21)$$

where

$$a(\alpha) = 72\alpha - 180\alpha^2 + 222\alpha^3 - 159\alpha^4 + 36\alpha^5 + 33\alpha^6 - 39\alpha^7 + 18\alpha^8 - 3\alpha^9, \quad (2.22)$$

$$b(\alpha) = -36 + 72\alpha - 105\alpha^2 + 91\alpha^3 - 61\alpha^4 + 27\alpha^5 - 7\alpha^6 + \alpha^7. \quad (2.23)$$

Some computation gives

$$\begin{aligned} H'(\alpha) &= \frac{8\alpha - 4}{2 - 2\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4} - \frac{a'(\alpha)b(\alpha) - a(\alpha)b'(\alpha)}{b^2(\alpha)} = \\ &\frac{v(\alpha)}{(2 - 2\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4)b(\alpha)^2}, \end{aligned} \quad (2.24)$$

where

$$v(\alpha) = \alpha^3(\alpha - 1)^3 s(\alpha) \quad (2.25)$$

$$\begin{aligned} s(\alpha) &= -2304 + 11232\alpha - 26976\alpha^2 + 46372\alpha^3 - 59080\alpha^4 + 59711\alpha^5 - 48129\alpha^6 + \\ &31498\alpha^7 - 16634\alpha^8 + 6980\alpha^9 - 2250\alpha^{10} + 513\alpha^{11} - 75\alpha^{12} + 6\alpha^{13}. \end{aligned} \quad (2.26)$$

We used

$$b^2(\alpha) = 1296 - 5184\alpha + 12744\alpha^2 - 21672\alpha^3 + 28521\alpha^4 - 29838\alpha^5 + 25483\alpha^6 \quad (2.27)$$

$$-17852\alpha^7 + 10249\alpha^8 - 4778\alpha^9 + 1765\alpha^{10} - 500\alpha^{11} + 103\alpha^{12} - 14\alpha^{13} + \alpha^{14},$$

$$a'(\alpha) = 72 - 360\alpha + 666\alpha^2 - 636\alpha^3 + 180\alpha^4 + 198\alpha^5 - 273\alpha^6 + 144\alpha^7 - 27\alpha^8, \quad (2.28)$$

$$b'(\alpha) = 72 - 210\alpha + 273\alpha^2 - 244\alpha^3 + 135\alpha^4 - 42\alpha^5 + 7\alpha^6, \quad (2.29)$$

$$a'(\alpha)b(\alpha) = -2592 + 18144\alpha - 57456\alpha^2 + 115200\alpha^3 - 159354\alpha^4 + 157122\alpha^5$$

$$-103542\alpha^6 + 30120\alpha^7 + 24849\alpha^8 - 44007\alpha^9 + 36042\alpha^{10} - 19818\alpha^{11} + 7644\alpha^{12}$$

$$-2010\alpha^{13} + 333\alpha^{14} - 27\alpha^{15},$$

$$a(\alpha)b'(\alpha) = 5184\alpha - 28080\alpha^2 + 73440\alpha^3 - 124776\alpha^4 + 150228\alpha^5 - 130083\alpha^6 +$$

$$76920\alpha^7 - 22338\alpha^8 - 9603\alpha^9 + 16890\alpha^{10} - 11610\alpha^{11} + 5031\alpha^{12} - 1434\alpha^{13} + 252\alpha^{14}$$

$$-21\alpha^{15}. \quad (2.31)$$

Now, we show that $H'(\alpha)$ has only one root in $(0, 1)$ which is equivalent to $s(\alpha)$ has only one root in $(0, 1)$. Because $s(0) = -2304$ and $s(1) = 864$ it suffices to show that $s'(\alpha) > 0$

$$s'(\alpha) = 11232 - 53952\alpha + 139116\alpha^2 - 236320\alpha^3 + 298555\alpha^4 - 288774\alpha^5 +$$

$$220486\alpha^6 - 133072\alpha^7 + 62820\alpha^8 - 22500\alpha^9 + 5643\alpha^{10} - 900\alpha^{11} + 78\alpha^{12}. \quad (2.32)$$

Using $\alpha < 1$ we get

$$s'(\alpha) > \xi(\alpha) = 11232 - 53952\alpha + 139116\alpha^2 - 236320\alpha^3 + 298555\alpha^4 - 288774\alpha^5 +$$

$$220486\alpha^6 - 133072\alpha^7 + 62820\alpha^8 - 22500\alpha^9 + 4743\alpha^{10}. \quad (2.33)$$

$$\frac{\xi(\alpha)}{10^5} > p(\alpha) = 0.11 - 0.54\alpha + 1.39\alpha^2 - 2.364\alpha^3 + 2.985\alpha^4 - 2.888\alpha^5 + 2.204\alpha^6$$

$$-1.331\alpha^7 + 0.628\alpha^8 - 0.23\alpha^9 + 0.04\alpha^{10}. \quad (2.34)$$

Therefore, $s'(\alpha) > p(\alpha)$. Now we show $p(\alpha) > 0$. We distinguish three cases.

Case 1. $\alpha \in (0, 0.45]$.

Case 2. $\alpha \in [0.45, 0.57]$.

Case 3. $\alpha \in [0.57, 1)$.

Case 1. Denote

$$p_1(\alpha) = 2.204\alpha^6 - 1.331\alpha^7 + 0.628\alpha^8 - 0.23\alpha^9 + 0.04\alpha^{10}. \quad (2.35)$$

$$p_2(\alpha) = 0.11 - 0.54\alpha + 1.39\alpha^2 - 2.364\alpha^3 + 2.985\alpha^4 - 2.888\alpha^5. \quad (2.36)$$

We have

$$p_1(\alpha) = \alpha^6 q_1(\alpha) = \alpha^6(2.204 - 1.331\alpha + 0.628\alpha^2 - 0.23\alpha^3 + 0.04\alpha^4). \quad (2.37)$$

Some computations give

$$q_1'(\alpha) = -1.331 + 1.256\alpha - 0.69\alpha^2 + 0.16\alpha^3, \quad q_1''(\alpha) = 1.256 - 1.38\alpha + 0.48\alpha^2,$$

$$q_1'''(\alpha) = -1.38\alpha + 0.96\alpha.$$

From $q_1'''(\alpha) < 0$ and $q_1''(1) = 0.356$ we have $q_1''(\alpha) > 0$. From this and from $q_1'(1) = -0.605$ we obtain $q_1'(\alpha) < 0$. Because $q_1(1) = 1.311$ we have $q_1(\alpha) > 0$ and so $p_1(\alpha) > 0$ for $\alpha \in (0, 1)$.

$$p_2'(\alpha) = -0.54 + 2.78\alpha - 7.092\alpha^2 + 11.94\alpha^3 - 14.44\alpha^4,$$

$$p_2''(\alpha) = 2.78 - 14.184\alpha + 35.82\alpha^2 - 57.76\alpha^3.$$

Using the Cardano's formula we obtain that $p_2''(\alpha) = 0$ only for one real $\alpha^* = 0.3218153$. From $p_2'(0) = -0.54$, $p_2'(\alpha^*) = -0.1367703$, $p_2'(0.45) = -0.2292278$ we get $p_2'(\alpha) < 0$ for $\alpha \in (0, 0.45]$. From $p_2(0.45) = 0.0021674$ we have $p(\alpha) = p_1(\alpha) + p_2(\alpha) > 0$ for $\alpha \in (0, 0.45]$.

Case 2. Put

$$q_1(\alpha) = p_1(\alpha) - 0.77\alpha^5, \quad q_2(\alpha) = p_2(\alpha) + 0.77\alpha^5 \quad (2.38)$$

We have

$$q_1(\alpha) = \alpha^5 r_1(\alpha) = \alpha^5 (-0.77 + 2.204\alpha - 1.331\alpha^2 + 0.628\alpha^3 - 0.23\alpha^4 + 0.04\alpha^5). \quad (2.39)$$

Some computations give

$$r_1'(\alpha) = 2.204 - 2.662\alpha + 1.884\alpha^2 - 0.92\alpha^3 + 0.2\alpha^4,$$

$$r_1''(\alpha) = -2.662 + 3.768\alpha - 2.76\alpha^2 + 0.8\alpha^3,$$

$$r_1'''(\alpha) = 3.768 - 5.52\alpha + 2.4\alpha^2, \quad r_1''''(\alpha) = -5.52 + 4.8\alpha < 0.$$

From $r_1''''(\alpha) < 0$ and $r_1'''(0.57) = 1.40136$ we have $r_1'''(\alpha) > 0$. From this and from $r_1''(0.57) = -1.2628096$ we obtain $r_1''(\alpha) < 0$. From $r_1'(0.57) = 1.149506$ we have $r_1'(\alpha) > 0$ for $\alpha \in (0.45, 0.57]$. From $r_1(0.45) = 0.0008057$ we have $q_1(\alpha) > 0$ for $\alpha \in (0.45, 0.57]$.

$$q_2'(\alpha) = -0.54 + 2.78\alpha - 7.092\alpha^2 + 11.94\alpha^3 - 10.59\alpha^4,$$

$$q_2''(\alpha) = 2.78 - 14.184\alpha + 35.82\alpha^2 - 42.36\alpha^3,$$

$$q_2'''(\alpha) = -14.184 + 71.64\alpha - 127.08\alpha^2, \quad q_2''''(\alpha) = 71.64 - 254.16\alpha < 0.$$

From $q_2'''(0.45) = -7.6797$ we have $q_2'''(\alpha) < 0$. From $q_2''(0.45) = -0.209305$ we have $q_2''(\alpha) < 0$. From $q_2'(0.45) = -0.0713537$ we get $q_2'(\alpha) < 0$. From $q_2(0.57) = 0.003673$ we have $p(\alpha) = q_1(\alpha) + q_2(\alpha) > 0$ for $\alpha \in [0.45, 0.57]$.

Case 3. Using elementary computations we obtain

$$p(1) = 0.004, \quad p'(1) = -0.091, \quad p''(1) = -0.918, \quad p'''(1) = -6.966,$$

$$p^{(4)}(1) = -38.4, \quad p^{(5)}(1) = -161.64, \quad p^{(6)}(1) = -323.28,$$

$$p^{(7)}(1) = 1073.52, \quad p^{(8)}(1) = 14434.56, \quad p^{(9)}(1) = 61689.6, \quad p^{(10)}(1) = 145152.$$

From this we obtain

$$p(\alpha) = \zeta(1 - \alpha) = 0.004 + 0.091(1 - \alpha) - 0.459(1 - \alpha)^2 + 1.161(1 - \alpha)^3 - 1.6(1 - \alpha)^4 + \quad (2.40)$$

$$1.347(1 - \alpha)^5 - 0.449(1 - \alpha)^6 - 0.213(1 - \alpha)^7 + 0.358(1 - \alpha)^8 - 0.17(1 - \alpha)^9 + 0.04(1 - \alpha)^{10}.$$

Denote $s_\alpha = 1 - \alpha$. If we show $\zeta(s_\alpha) > 0$ for $0 < s_\alpha < 0.44$ then the proof will be completed. Denote

$$\zeta_1(s_\alpha) = 0.004 - 0.449s_\alpha^6 - 0.213s_\alpha^7 + 0.358s_\alpha^8 - 0.17s_\alpha^9 + 0.04s_\alpha^{10}. \quad (2.41)$$

$$\zeta_2(s_\alpha) = 0.091s_\alpha - 0.459s_\alpha^2 + 1.161s_\alpha^3 - 1.6s_\alpha^4 + 1.347s_\alpha^5. \quad (2.42)$$

We have

$$\zeta_1'(s_\alpha) = s_\alpha^5 \varrho_1(s_\alpha) = s_\alpha^5 (-2.694 - 1.491s_\alpha + 2.864s_\alpha^2 - 1.53s_\alpha^3 + 0.4s_\alpha^4). \quad (2.43)$$

Some computations give

$$\varrho_1'(s_\alpha) = -1.491 + 5.728s_\alpha - 4.59s_\alpha^2 + 1.6s_\alpha^3.$$

Using the Cardano's formula we obtain that $\varrho_1''(s_\alpha) = 0$ only for one real $s_\alpha^* = 0.3435535$. From $\varrho_1(0) = -2.694$, $\varrho_1(0.44) = -2.9109087$, $\varrho_1(s_\alpha^*) = -2.9246712$ we get $\zeta_1'(s_\alpha) < 0$. From $\zeta_1(0.44) = 0.0004706$ we have $\zeta_1(s_\alpha) > 0$ for $s_\alpha \in (0, 0.44]$. Next we have

$$\zeta_2(s_\alpha) = s_\alpha z(s_\alpha) = s_\alpha (0.091 - 0.459s_\alpha + 1.161s_\alpha^2 - 1.6s_\alpha^3 + 1.347s_\alpha^4),$$

$$z'(s_\alpha) = -0.459 + 2.322s_\alpha - 4.8s_\alpha^2 + 5.388s_\alpha^3.$$

Using the Cardano's formula we obtain that $z'(s_\alpha) = 0$ only for one real $s_\alpha^{**} = 0.3534804$. From $z(0) = 0.091$, $z(0.44) = 0.280021$, $z(s_\alpha^{**}) = 0.0241802$ we have $\zeta_2(s_\alpha) > 0$ for $s_\alpha \in (0, 0.44]$. The proof is complete.

Competing interests

The author declare that he has no competing interests.

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References

- Qiu, Y., Wang, M., Chu, Y. (2011). The sharp combination bounds of arithmetic and logarithmic means for Seiffert's mean. *Int. J. Pure Appl. Math.* 72, no. 1, 11-18.
- Qiu, Y-F., Wang, M-K., Chu, Y-M. (2011). The optimal generalized Heronian mean bounds for the identric mean. *Int. J. Pure Appl. Math.*, 72(1): 19-26.
- Chu, Y-M., Hou, S-W., Gong, V-M. (2011). Inequalities between logarithmic, harmonic, arithmetic and centroidal means. *J. Math. Anal.*, 2(2): 1-5.
- Chu, Y-M., Hou, S-W., Shen, Z-H. (2012). Sharp bounds for Seiffert mean in terms of root mean square. *J. Inequal. Appl.* 2012:11, 6 pp.
- Chu, Y-M., Qiu, Y-F., Wang, M-K. (2010). Sharp power mean bounds for the combination of Seiffert and geometric means. *Abstr. Appl. Anal.*, Art. ID 108920, 12 pages.
- Chu, Y-M., Qiu, Y-F., Wang, M-K., Wang, G-D. (2010). The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean. *J. Inequal. Art.* ID 436457, 7 pp.
- Chu, Y-M., Wang, M-K. (2011). Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means, *J. Appl. Math.*, 2011, Art. ID 618929, 9 pages.
- Chu, Y-M., Wang, M-K., Gong, W-M. (2011). Two sharp double inequalities for Seiffert mean. *J. Inequal. Appl.* 2011:44, 7 pp.
- Chu, Y-M., Wang, M-K., Qiu, Y-F. (2010). An optimal double inequality between power - type Heron and Seiffert means. *J. Inequal. Art.* ID 146945, 11 pp.

- Chu, Y.-M., Wang, M.-K., Wang, G.-D. (2012). The optimal generalized logarithmic mean bounds for Seiffert's mean. *Acta Math. Sci. Ser. B Engl. Ed.* 32 , no. 4, 1619-1626.
- Chu, Y.-M., Wang, M.-K., Wang, Z.-K. (2012). Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means. *Math. Inequal. Appl.* 15 , no. 2, 415-422.
- Chu, Y.-M., Wang, M.-K., Qiu, S.-L., Qiu, Y.-F. (2011). Sharp generalized Seiffert mean bounds for Toader mean, *Abstr. Appl. Anal.*, Art. ID 605259, 8 pages.
- Chu, Y.-M., Wang, M.-K., Wang, Z.-K., Qiu, Y.-F. (2011). An optimal double inequality between Seiffert and geometric means, *J. Appl. Math.*, Art. ID 261237, 6 pages.
- Chu, Y.-M., Wang, M.-K., Wang, Z.-K., Qiu, Y.-F. (2011). A best-possible double inequality between Seiffert and harmonic means, *J. Inequal. Appl.*, 2011: 94.
- Chu, Y.-M., Wang, S.-S., Zong, C. (2011). Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, *Abstr. Appl. Anal.*, 2011, Art. ID 520648, 9 pages.
- Chu, Y.-M., Li, Y.-M., Long, B.-Y. (2012). Sharp bounds by power mean for generalized Heronian mean, *J. Inequal. Appl.*, 2012: 129.
- Chu, Y.-M., Xia, W.-F. (2010). Two optimal double inequalities between power mean and logarithmic mean, *Comput. Math. Appl.*, 60: 83-89.
- Fu, L.-L., Xi B.-Y., Srivastava H.-M. (2011). Schur-convexity of the generalized Heronian means involving two positive numbers, *Taiwanese Journal of Mathematics* 15, 2721-2731.
- Gao S. (2011). Inequalities for the Seiffert's means in terms of the identric mean. *J. Math. Sci. Adv. Appl.* 10, no. 1-2, 23-31.
- Gao, H., Guo, J., Li, M. (2012). Sharp Bounds for the First Seiffert and Logarithmic Means in Terms of Generalized Heronian Mean. *Acta Mathematica Scientia*, accepted 2012.
- Gong, W.-M., Song, Y.-Q., Wang, M.-K., Chu, Y.-M. (2012). A sharp double inequality between Seiffert, arithmetic, and geometric means, *Abstr. Appl. Anal.*, Art. ID 684834, 7 pages.
- Hastö, P. (2004). Optimal inequalities between Seiffert's mean and power means. *Math. Inequal. Appl.* 7, no. 1, 47-53.
- Hou, S., Chu, Y. (2011). Optimal convex combination bounds of root-square and harmonic root-square means for Seiffert mean. *Int. J. Math. Anal. (Ruse)* 5, no. 37-40, 1897-1904.
- Hu, H., Hou, S., Xu, Y., Chu, Y. (2011). Optimal convex combination bounds of root-square and harmonic root-square means for Seiffert mean. *Int. Math. Forum* 6 (2011), no. 57-60, 2823-2831.
- Janous W. (2001). A Note on Generalised Heronian Mean. *Math. Ineq. Appl.* 3, 369-375.
- Liu, H., Meng, X.-J. (2011). The optimal convex combination bounds for Seiffert's mean. *J. Inequal. Appl.* ID 686834, 9 pp.
- Long, B.-Y., Li, Y.-M., Chu, Y.-M. (2012). Optimal inequalities between generalized logarithmic, identric and power means, *Int. J. Pure Appl. Math.*, 80(1): 41-51.
- Matejíčka, L. (2010). Proof of One Optimal Inequalities for Generalized Logarithmic, Arithmetic and Geometric Means, *J. Inequal. Appl.*, Article ID 902432, 5 pages.

- Seiffert W. (1993). Problem 887, Nieuw Archief voor Wiskunde vol. 11, no.2, pp. 176-176.
- Shi, H-X., Long, B-Y., Chu, Y-M. (2012). Optimal generalized Heronian mean bounds for the logarithmic mean, J. Inequal. Appl., 2012: 63.
- Shi, H-Y., Chu, Y-M., Jiang, Y-P. (2009). Optimal inequalities among various means of two arguments, Abstr. Appl. Anal., Art. ID 694394, 10 pages.
- Wang, M-K., Qiu, Y-F., Chu, Y-M. (2010). Sharp bounds for Seiffert means in terms of Lehmer means. J. Math. Inequal. 4, no. 4, 581-586.
- Wang, S., Chu, Y-M. (2010). The best bounds of the combination of arithmetic and harmonic means for the Seiffert's mean. Int. J. Math. Anal. (Ruse) 4, no. 21-24, 1079-1084.
- Zong, C., Chu, Y. (2010). An inequality among identric, geometric and Seiffert's means. Int. Math. Forum 5, no. 25-28, 1297-1302. 26E60
- Xia, W-F., Chu, Y-M., Wang, G-D. (2010). The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means, Abstr. Appl. Anal., Art. ID 604804

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