

A GENERAL INEQUALITY RELATED TO VARIATIONAL INEQUALITIES AND ITS CONSEQUENCES

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Let X be a Hausdorff topological vector space with dual X^* , K a nonempty subset of X , and $f: K \times K \rightarrow \mathbb{R}$ be any map. In this paper we study the following problem: “Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ ”. Some results on this problem have been studied by Takahasi, Park and Kim, and Yen. Behera and Panda recognized this problem as an independent problem and proved several results on the existence of solution of this problem in the setting of a Banach space. This problem includes as special cases, many problems on variational inequalities and generalized variational inequalities studied by many authors. In this paper we also introduce and study another problem, “Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ ” where $S: K \rightarrow 2^K$ is any point-to-set map. This problem includes as a particular case the classical quasi-variational inequality problem. A generalization of Minty’s Lemma is also studied.

Key words : Partition of unity, upper and lower semicontinuous point-to set map, monotone operator

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1. Introduction

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty subset of X . Let the value of $u \in X^*$ at $x \in X$ be denoted by (u, x) . Let $g: K \rightarrow \mathbb{R}$ be a map (possibly nonlinear). The classical minimization problem for the pair (g, K) is to find $x_0 \in K$ such that

$$g(x_0) = \min_{y \in K} g(y).$$

If we define a function $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = g(y) - g(x)$$

for all $x, y \in K$, then the above problem reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

If $T: K \rightarrow X^*$, then the nonlinear variational inequality problem is to find $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$. If we define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = (Tx, y - x)$$

then it reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

The above discussions lead to consider the following problem:

42 $\mathbf{P}_1$: Given $f : K \times K \rightarrow \mathbb{R}$, find $x_0 \in K$ such that
 43 $f(x_0, y) \geq 0$
 44 for all $y \in K$.
 45

46 **Special cases**

47 1. If $f(x, y) = (Tx, \theta(y, x))$ for some maps $T : K \rightarrow X^*$, and $\theta : K \times K \rightarrow X$, then
 48 $\mathbf{P}_1$ reduces to the problem of finding $x_0 \in K$ such that

$$49 (Tx_0, \theta(y, x_0)) \geq 0$$

50 for all $y \in K$, which is the *variational-like inequality problem* studied by Behera
 51 and Panda [2, 4], Dien[9], Noor[15] and Siddiqi et. al.[21–23]. Yang and Chen [27]
 52 also studied this problem calling it a *pre-variational inequality problem* because of
 53 the relation of this problem with the *pre-invex functions* (see [6, 27]).
 54

55 2. If $z \in K$ is fixed and $f(x, y) = (T((z+x)/2), y-x)$, then $\mathbf{P}_1$ reduces to the
 56 problem of finding $x_0 \in K$ such that

$$57 (T((z+x_0)/2), y-x_0) \geq 0$$

58 for all $y \in K$, which is the *variational-type inequality problem* originally introduced
 59 by Behera and Panda[3] in the setting of Banach spaces and subsequently studied
 60 by Panda and Dash [16] in the setting of topological vector spaces.
 61

62 3. If $f(x, y) = (Tx - Ax, y - x)$ for some maps $T, A : K \rightarrow X^*$, then $\mathbf{P}_1$ reduces
 63 to the problem of finding $x_0 \in K$ such that

$$64 (Tx_0, y - x_0) \geq (Ax_0, y - x_0)$$

65 for all $y \in K$, which is the *strongly nonlinear variational inequality problem*
 66 studied by Nanda[12] and Noor[13].

67 4. If $f(x, y) = (Tx - Ax, g(y) - g(x))$ for some maps $T, A : K \rightarrow X^*$, $g : K \rightarrow K$,
 68 then $\mathbf{P}_1$ reduces to the problem of finding $x_0 \in K$ such that

$$69 (Tx_0, g(y) - g(x_0)) \geq (Ax_0, g(y) - g(x_0))$$

70 for all $y \in K$, which is the *strongly nonlinear implicit variational inequality*
 71 *problem* introduced and studied by Noor[13] in connection with the solution of the
 72 differential equations of odd order.
 73

74 5. If $f(x, y) = (Tx - Ax, \theta(y, x))$ for some maps $T, A : K \rightarrow X^*$ and
 75 $\theta : K \times K \rightarrow X$, then $\mathbf{P}_1$ reduces to the problem of finding $x_0 \in K$ such that

$$76 (Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$$

77 for all $y \in K$, which is the *strongly nonlinear variational-like inequality problem*
 78 studied by Panda and Dash [17].
 79

80 Some results on Problem $\mathbf{P}_1$ are available in Takahasi[25], Park and Kim[18]
 81 and Yen[28]. Behera and Panda [5] recognized $\mathbf{P}_1$ as an independent problem and
 82 proved several results on the existence of solution of this problem in the setting of

83 Banach spaces. Many authors used results on Problem P_1 to prove the existence of
 84 solution of variational and generalized variational inequality problems.

85
 86 In this paper, we prove some results on the existence of solution of Problem
 87 P_1 under different assumptions in the setting of Hausdorff topological vector
 88 spaces. We also use a result on the existence of solution of Problem P_1 for the study
 89 the following problem, which is a generalization of the quasi-variational inequality
 90 problem introduced by Benoussan and Lions [7] in connection with impulse control
 91 and subsequently studied by Baicchi and Capello [1] and Mosco [11].

92 **P_2 :** Find $x_0 \in K$ such that

93
$$f(x_0, y) \geq 0$$

94 for all $y \in S(x_0)$, where $S : K \rightarrow 2^K$ is any point-to-set map.

95 We also prove a theorem on the uniqueness of solution of Problem P_1 which would
 96 serve as a generalization of the well-known Minty's Lemma (see [8], p.6).

97 2. Preliminaries

98 In this section we recall some definitions and known results which will be
 99 needed in the sequel. Throughout this section X is a real Hausdorff topological
 100 vector space with dual X^* and K a nonempty convex subset of X .

101
 102 DEFINITION 2.1. A mapping $T : K \rightarrow X^*$ is said to be monotone if
 103 $(Tx - Ty, x - y) \geq 0$ for all $x, y \in K$. T is said to be strictly monotone if strict
 104 inequality holds whenever $x \neq y$. T is said to be pseudo-monotone if $x, y \in K$ and
 105 $(Tx, x - y) \geq 0$ then $(Ty, x - y) \geq 0$.

106
 107 DEFINITION 2.2. A function $h : K \rightarrow \mathbb{R}$ is said to be upper semi-continuous, if for each
 108 real number λ , the set $\{x \in K : h(x) < \lambda\}$ is open; function f is said to be lower
 109 semi-continuous if $-f$ is upper semi-continuous.

110
 111 DEFINITION 2.3. A function $h : K \rightarrow \mathbb{R}$ is said to be quasi-convex if for each real
 112 number λ , the set $\{x \in K : h(x) < \lambda\}$ is convex; function f is said to be quasi-
 113 concave if $-f$ is quasi-convex.

114
 115 DEFINITION 2.4. Let M and N be two topological spaces, and $h : M \rightarrow 2^N$, a point-
 116 to set map. Then h is said to be upper semi-continuous at $x_0 \in K$ if for each open
 117 set U of N with $h(x_0) \subset U$, there exists a neighborhood $n(x_0)$ of x_0 in K such that
 118 if $x \in n(x_0)$ then $h(x) \subset U$; h is said to be lower semi-continuous at $x_0 \in M$ if for
 119 each open set U of N with $h(x_0) \cap U \neq \emptyset$, there exists a neighborhood $n(x_0)$ of x_0
 120 in M such that if $x \in n(x_0)$ then $h(x) \cap U \neq \emptyset$; h is said to be upper semi-
 121 continuous (lower semi-continuous) on M if h is upper semi-continuous (lower
 122 semi-continuous) at each point of M .

124

125 THEOREM 2.5. (Isac [10]) Let K be a nonempty compact convex subset of a
 126 Hausdorff topological vector space. Let f and g be two real valued functions on
 127 $K \times K$ having the following properties :

- 128 (a) for all $x, y \in K$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq 0$,
 129 (b) for each fixed $y \in K$, $f(x, y)$ is a lower semicontinuous function of x
 130 on K ,
 131 (c) for each fixed $x \in K$, $g(x, y)$ is a quasi-concave function of x on K .

132 Then there exists a point $\hat{x} \in K$ such that $f(\hat{x}, y) \leq 0$ for all $y \in K$.

133 THEOREM 2.6. (Shih and Tan [24]). Let X be a Hausdorff topological vector space
 134 with dual X^* and K a nonempty convex subset of X . Let $S : K \rightarrow 2^K$ be an upper
 135 semicontinuous point-to-set map such that for each $x \in K$, $S(x)$ is nonempty and
 136 bounded. Then for each $p \in X^*$, the map $f_p : K \rightarrow \mathbb{R}$ defined by

137
$$f_p(x) = \sup_{y \in S(x)} (p, y)$$

138 is upper semicontinuous.

139

140 THEOREM 2.7. (Takahasi [25]) Let K be a nonempty compact convex subset of a
 141 locally convex Hausdorff topological vector space X and let $F : K \times K \rightarrow \mathbb{R}$ be a
 142 function such that

- 143 (a) for each $y \in K$, the map $x \mapsto F(x, y)$ is upper semicontinuous,
 144 (b) for each $x \in K$, the map $y \mapsto F(x, y)$ is convex,
 145 (c) $F(x, x) \geq \alpha$ for all $x \in K$ with some real number α .

146 Then there exists $x_0 \in K$ such that $F(x_0, y) \geq \alpha$ for all $y \in K$.

147

148 THEOREM 2.8. (Rudin [19]) Let K be a compact subset of a topological space and let
 149 $\{V_1, V_2, \dots, V_n\}$ be a finite open covering of K . Then there exists a family
 150 $\{\beta_1, \beta_2, \dots, \beta_n\}$ of continuous real valued function on K such that $\beta_i(x) = 0$ outside
 151 V_i , $0 \leq \beta_i(x) \leq 1$ for all $i \in \{1, 2, \dots, n\}$ and for all $x \in K$, and

152
$$\sum_{i=1}^n \beta_i(x) = 1$$

153 for all $x \in K$.

154

155 THEOREM 2.9. (Tarafdar [26]) Let K be a nonempty subset of a Hausdorff topological
 156 vector space X and let $F : K \rightarrow 2^K$ be a point-to-set map such that

- 157 (a) for each $x \in K$, $F(x)$ is nonempty,
 158 (b) for each $y \in K$, $F^{-1}(y) = \{x \in K : y \in F(x)\}$ contains a relatively open
 159 subset A_y of K (A_y may be empty for some y),
 160 (c) $\bigcup_{x \in K} A_x = K$,
 161 (d) K contains a nonempty set K_0 contained in a compact convex subset
 162 K_1 of K such that $D = \bigcap_{x \in K_0} A_x^c$ is compact (D may be empty).

163 Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.

164

165 THEOREM 2.10 ([24], p. 34). Let X be a Hausdorff topological vector space with
166 dual X^* and K a nonempty convex subset of X . Let $S : K \rightarrow 2^X$ be a point-to-set
167 upper semi-continuous map such that for each $x \in K$, $S(x)$ is nonempty and
168 bounded. Then for each $p \in X^*$ the map $f_p : K \rightarrow \mathbb{R}$ defined by

169 $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semi-continuous.

170

171 THEOREM 2.11 ([20], p.201). Let X be a locally convex Hausdorff topological vector
172 space with dual X^* and K a nonempty closed and convex subset of X . Let x^* be a
173 point of X not in K . Then there exists $p \in X^*$ such that $(p, x^*) > \sup_{x \in K} (p, x)$.

174

175 3. Existence of solution

176

177 The following theorems on the existence of solution of Problem P_1 are the
178 main results of this paper.

179

180 THEOREM 3.1. Let K be a nonempty compact convex subset of a Hausdorff
181 topological vector space X and let $f : K \times K \rightarrow \mathbb{R}$ be a map such that

182

(a) $f(x, x) \geq 0$ for each $x \in K$,

183

(b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,

184

(c) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K .

185 Then there exists $x_0 \in K$ such that

186

$$f(x_0, y) \geq 0$$

187

for all $y \in K$.

188

189 *Proof.* Same as the proof of Theorem 2.7. Indeed Takahasi assumed that for each y
190 $\in K$, the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is upper semi-continuous and for each $x \in K$,
191 the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex which are stronger than hypotheses (b)
192 and (c) of this theorem.

193

194 The following theorem generalizes Theorem 3.1 to an arbitrary nonempty
195 convex subset K of X .

196

197 THEOREM 3.2. Let K be a nonempty convex subset of a Hausdorff topological vector
198 space X and let $f : K \times K \rightarrow \mathbb{R}$ be a map such that

199

(a) $f(x, x) \geq 0$ for each $x \in K$,

200

(b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,

201 (c) for each $y \in K$, the set $\{x \in C : f(x, y) < 0\}$ is open in K for every
 202 compact subset C of K ,

203 (d) there exists a nonempty compact subset L of K such that for each
 204 $x \in K - L$, there exists $u \in L$ for which $f(x, u) < 0$.

205 Then there exists $x_0 \in L$ such that

$$206 \quad f(x_0, y) \geq 0$$

207 for all $y \in K$.

208

209 *Proof.* For each $y \in K$, let

$$210 \quad A(y) = \{x \in L : f(x, y) \geq 0\}.$$

211 From hypothesis (c) it follows that $A(y)$ is a nonempty compact subset of K . The
 212 conclusion of the theorem follows if

$$213 \quad \bigcap_{y \in K} A(y) \neq \emptyset.$$

214 For this, it is sufficient to show that the family $\{A(y) : y \in K\}$ has the finite inter-
 215 section property.

216 Let $y_1, y_2, \dots, y_n$ be arbitrary elements of K and K_0 , the convex hull of
 217 $L \cup \{y_1, y_2, \dots, y_n\}$. Then K_0 is a nonempty compact convex subset of K . By
 218 Theorem 3.1 there exists $\tilde{x}_0 \in K_0$ such that

$$219 \quad f(\tilde{x}_0, y) \geq 0 \quad \dots(1)$$

220 for all $y \in K_0$. We claim that $\tilde{x}_0 \in L$; for if $\tilde{x}_0 \notin L$, that is, $\tilde{x}_0 \in K_0 - L \subset K - L$.

221 Now, by hypothesis (d), there exists $u \in L$ such that $f(\tilde{x}_0, u) < 0$ which contradicts
 222 (1) when $y = u$. Thus $\tilde{x}_0 \in L$, and in particular $\tilde{x}_0 \in A(y_i)$ for $i = 1, 2, \dots, n$; that is

$$223 \quad \tilde{x}_0 \in \bigcap_{i=1}^n A(y_i).$$

224 Hence,

$$225 \quad \bigcap_{i=1}^n A(y_i) \neq \emptyset,$$

226 which proves that the family $\{A(y) : y \in K\}$ has the finite intersection property.
 227 This completes the proof.

228

229

230 If X is a normed linear space and K is locally compact, we have the following
 231 result.

232

233 **THEOREM 3.3.** Let K be a nonempty locally compact convex subset of a real normed
 234 linear space X with $0 \in K$, and let $f : K \times K \rightarrow \mathbb{R}$ be a map such that

235 (a) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,

236 (b) for all $y \in K$ and $r > 0$, the set $\{x \in K_r : f(x, y) < 0\}$ is open in K ,

237 where $K_r = \{x \in K : \|x\| \leq r\}$,

- 238 (c) $f(x, x) = 0$ for each $x \in K_r$ and for each $r > 0$,
 239 (d) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

240 Then there exists $x_0 \in K$ such that

241
$$f(x_0, y) \geq 0$$

242 for all $y \in K$.

243

244 *Proof.* For each $r > 0$, the set K_r is a nonempty compact and convex subset of K .
 245 Hence by Theorem 3.1 there exists $x_r \in K_r$ such that

246
$$f(x_r, y) \geq 0 \quad \dots(2)$$

247 for all $y \in K_r$. We claim that $\|x_r\| < r$; for if $\|x_r\| = r$, then by hypothesis (d),
 248 $f(x_r, 0) < 0$, which contradicts (2) when $y = 0$. Now let $x \in K$ be arbitrary and
 249 choose $\lambda > 0$ sufficiently small so that

250
$$y_r = \lambda x + (1-\lambda)x_r \in K_r.$$

251 Now by hypotheses (a) and (b)

252
$$\begin{aligned} 0 &\leq f(x_r, y_r) \\ 253 &= f(x_r, \lambda x + (1-\lambda)x_r) \\ 254 &\leq \lambda f(x_r, x) + (1-\lambda)f(x_r, x_r) \\ 255 &= \lambda f(x_r, x). \end{aligned}$$

256 Since $\lambda > 0$, it follows that $f(x_r, x) \geq 0$. This completes the proof.

257

258 If X is a reflexive Banach space then local compactness of K can be relaxed
 259 from the above theorem.

260

261 **THEOREM 3.4.** Let K be a nonempty closed and convex subset of a reflexive real
 262 Banach space X with $0 \in K$ and $f: K \times K \rightarrow \mathbb{R}$, a map such that

- 263 (a) $f(x, x) = 0$ for each $x \in K$,
 264 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex,
 265 (c) the set $\{x \in C : f(x, y) < 0\}$ is weakly open in K for each $y \in K$, and
 266 for each weakly compact subset C of K .

267 Then there exists $x_0 \in K$ such that

268
$$f(x_0, y) \geq 0$$

269 for all $y \in K$, under each of the following conditions :

- 270 (i) K is bounded,
 271 (ii) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

272

273 *Proof.* Equip K with the weak topology. If K is bounded, then it becomes weakly
 274 compact and in this case, the proof follows from Theorem 3.1.

275

276 If K is not bounded let

277
$$K_r = \{x \in K : \|x\| \leq r\}.$$

278 Then K_r is a nonempty closed, convex and bounded subset of K . By the first part of
 279 the theorem, there exists $x_r \in K_r$ such that

$$280 \quad f(x_r, y) \geq 0$$

281 for all $y \in K_r$. The remaining part of the proof follows from the proof of Theorem
 282 3.3 and hence it is omitted.

283

284 The following theorem generalizes Theorem 3.4 to a Hausdorff topological
 285 vector space.

286

287 **THEOREM 3.5.** *Let K be a nonempty convex subset of a Hausdorff topological vector*
 288 *space X and L , a nonempty compact and convex subset of K . Let $f : K \times K \rightarrow \mathbb{R}$ be*
 289 *a map such that*

290 (a) $f(x, x) = 0$ for each $x \in L$,

291 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,

292 (c) for each $y \in K$, the set $\{x \in L : f(x, y) < 0\}$ is open in L ,

293 (d) for every x in the boundary of L , there exists $u \in L$ such that
 294 $f(x, u) < 0$.

295 Then there exists $x_0 \in L$ such that

$$296 \quad f(x_0, y) \geq 0$$

297 for all $y \in K$.

298

299 *Proof.* By theorem 3.1 there exists $x_0 \in L$ such that

$$300 \quad f(x_0, y) \geq 0 \quad \dots(3)$$

301 for all $y \in L$. We claim that $x_0 \in \text{int } L$ (where $\text{int } L$ stands for the interior of L), for
 302 if $x_0 \notin \text{int } L$, then by hypothesis (d) there exists $u \in L$ such that $f(x_0, u) < 0$ which
 303 contradicts (3) when $y = u$.

304

305 Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently small so that

$$306 \quad y_\lambda = \lambda x + (1 - \lambda)x_0 \in L.$$

307 Now by hypotheses (a) and (b)

$$\begin{aligned} 308 \quad 0 &\leq f(x_0, y_\lambda) \\ &\leq \lambda f(x_0, x) + (1 - \lambda)f(x_0, x_0) \\ &= \lambda f(x_0, x). \end{aligned}$$

309 Since $\lambda > 0$, it follows that $f(x_0, x) \geq 0$. This completes the proof.

310

311 The following theorem is a variant of Theorem 3.1 in which hypotheses (a)
 312 and (b) of Theorem 3.1 have been replaced by a single generalized condition.

313

314 **THEOREM 3.6.** *Let K be a nonempty compact convex subset of a Hausdorff*
 315 *topological vector space X and let $f : K \times K \rightarrow \mathbb{R}$ be a map such that*

- 316 (a) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K ,
 317 (b) for any finite set $\{y_1, y_2, \dots, y_n\} \subset K$, and $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$,
 318
$$\max_{1 \leq i \leq n} f(u, y_i) \geq 0.$$

319 Then there exists $x_0 \in K$ such that

320
$$f(x_0, y) \geq 0$$

321 for all $y \in K$.

322

323 *Proof.* Assume that the assertion is false. Then for each $x \in K$, there exists $y \in K$
 324 such that $f(x, y) < 0$. For each $y \in K$, let

325
$$A(y) = \{x \in K : f(x, y) < 0\}.$$

326 Then by hypothesis (a) $A(y)$ is open for every $y \in K$. Further

327
$$K = \bigcup_{y \in K} A(y).$$

328 Since K is compact, there exists $\{y_1, y_2, \dots, y_n\} \subset K$ such that

329
$$K = \bigcup_{i=1}^n A(y_i).$$

330 Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be a partition of unity subordinate to the covering $\{A(y_i) : i = 1,$
 331 $2, \dots, n\}$ and let $S = \text{conv}(\{y_1, y_2, \dots, y_n\})$. Then S is a compact convex subset of K .

332 Define a continuous map $p : K \rightarrow K$ as

333
$$p(x) = \sum_{i=1}^n \beta_i(x) y_i.$$

334 Since $p(x)$ is a convex linear combination of points of the set S , $p(x) \in S$ for each
 335 $x \in K$. In particular, p maps S into S . Now by Browuer's fixed point theorem p has
 336 a fixed point; that is, there exists $\hat{x} \in K$ such that $\hat{x} = p(\hat{x})$.

337

338 Since $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a partition of unity, $\beta_i(\hat{x}) > 0$ for at least one i .

339

340 If $\beta_i(\hat{x}) > 0$ then $\hat{x} \in A(y_i)$ so that

341
$$f(\hat{x}, y_i) < 0. \quad \dots(4)$$

342 If

343
$$J = \{i : 1 \leq i \leq n \text{ and } \beta_i(\hat{x}) > 0\}$$

344 then

345
$$\hat{x} = p(\hat{x}) = \sum_{i=1}^n \beta_i(\hat{x}) y_i = \sum_{i \in J} \beta_i(\hat{x}) y_i$$

346 so that $\hat{x} \in \text{conv}\{y_i : i \in J\}$. Now by hypothesis (b) there exists $i_0 \in J$ such that

347 $f(\hat{x}, y_{i_0}) \geq 0$ which contradicts (4). This contradiction proves the theorem.

348

349 The following theorem generalizes Theorem 3.6 to an arbitrary nonempty
 350 convex subset K of X .

351

THEOREM 3.7. Let X be a locally convex Hausdorff topological vector space, K a nonempty convex subset of X , K_0 a nonempty compact convex subset of K , and L a nonempty compact subset of K . Let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- (a) for each $y \in K$, the set $\{x \in L : f(x, y) \geq 0\}$ is closed in K ,
- (b) for any finite set $\{y_1, y_2, \dots, y_n\} \subset K$, and $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$,

$$\max_{1 \leq i \leq n} f(u, y_i) \geq 0.$$
- (c) for each $x \in K - L$, there exists $u \in \text{conv}(K_0 \cup \{x\})$ such that $f(x, u) < 0$.

Then there exists $x_0 \in L$ such that

$$f(x_0, y) \geq 0$$

for all $y \in K$.

Proof. For each $y \in K$, let

$$A(y) = \{x \in L : f(x, y) \geq 0\}.$$

From (b) it follows that $A(y)$ is closed and consequently compact for each $y \in K$.

The assertion of the theorem follows if

$$\bigcap_{y \in K} A(y) \neq \emptyset.$$

For this, it is sufficient to prove that the family $\{A(y) : y \in K\}$ has the finite intersection property.

Let $\{y_1, y_2, \dots, y_n\} \subset K$ and $S = \text{conv}(K_0 \cup \{y_1, y_2, \dots, y_n\})$. Clearly S is a non-empty compact convex subset of K . By Theorem 3.6 there exists $\hat{x} \in S$ such that

$$f(\hat{x}, y) \geq 0 \quad \dots(5)$$

for all $y \in S$. We claim that $\hat{x} \in L$; for if $\hat{x} \notin L$, that is, $\hat{x} \in S - L \subset K - L$, then there exists $u \in \text{conv}(K_0 \cup \{\hat{x}\})$ such that $f(\hat{x}, u) < 0$ which contradicts (5) when $y = u$. Thus $\hat{x} \in L$, and in particular, $\hat{x} \in A(y_i)$ for each $i \in \{1, 2, \dots, n\}$, that is,

$$\hat{x} \in \bigcap_{i=1}^n A(y_i). \text{ Hence } \bigcap_{i=1}^n A(y_i) \neq \emptyset, \text{ showing that the family } \{A(y) : y \in K\} \text{ has the}$$

finite intersection property. This completes the proof.

The following theorem also deals with the existence of solution of Problem P_1 under pseudo-monotonicity type assumptions.

THEOREM 3.8. Let K be a nonempty closed and convex subset of a Hausdorff topological vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- (a) $f(x, x) \geq 0$ for each $x \in K$,
- (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex and lower semicontinuous for each $x \in K$,

390 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line
 391 segments of K to the usual topology of $\mathbb{R}$,
 392 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$,
 393 (e) there exists a nonempty set K_0 contained in a compact convex subset
 394 K_1 of K such that
 395
$$D = \bigcap_{y \in K_0} \{y \in K : f(x, y) \leq 0\}$$

 396 is either compact or empty.

397 Then there exists $x_0 \in K$ such that
 398
$$f(x_0, y) \geq 0 \quad \dots(6)$$

 399 for all $y \in K$.

400
 401 Note: If we set $f(x, y) = (Tx, y - x)$ for some operator $T : K \rightarrow X^*$ then Condition
 402 (d) of the above theorem is equivalent to the fact that T is pseudo-monotone.

403
 404 To prove the above theorem, we need the following lemma.

405
 406 LEMMA 3.9. Let K be a nonempty closed and convex subset of a Hausdorff
 407 topological vector space X and let $f : K \times K \rightarrow \mathbb{R}$ be a map such that

- 408 (a) $f(x, x) \geq 0$ for each $x \in K$,
 409 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 410 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line
 411 segments of K to the usual topology of $\mathbb{R}$,
 412 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$.

413 Then the following are equivalent :

- 414 (A) $x_0 \in K$ and $f(x_0, y) \geq 0$ for all $y \in K$.
 415 (B) $x_0 \in K$ and $f(y, x_0) \leq 0$ for all $y \in K$.

416
 417 Note: If we set $f(x, y) = (Tx, y - x)$ for some operator $T : K \rightarrow X^*$ then Lemma
 418 3.9 reduces to Minty's Lemma ([8], p. 6).

419
 420 Proof. Assume that (A) holds. Then for some $x_0 \in K$ and for all $y \in K, f(x_0, y) \geq 0$.
 421 By hypothesis (d) $f(y, x_0) \leq 0$ for all $y \in K$ from which (B) follows.

422
 423 Now assume that (B) holds. Then for some $x_0 \in K, f(y, x_0) \leq 0$ for all
 424 $y \in K$. Fix $y \in K, t \in (0, 1)$ and set $y_t = (1 - t)x_0 + ty$. Then $y_t \in K$, and by (B),
 425 $f(y_t, x_0) \leq 0$ for each $t \in (0, 1)$. Now it follows from hypotheses (a) and (b) that

$$\begin{aligned}
& 0 \leq f(y_t, y_t) \\
& \leq (1-t)f(y_t, x_0) + tf(y_t, y).
\end{aligned}$$

Thus

$$f(y_t, y) \geq -\frac{1-t}{t} f(y_t, x_0) \geq 0$$

for each $t \in (0,1)$. Since the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line segments of K to the usual topology of $\mathbb{R}$, taking limit $t \rightarrow 0^+$ in the above inequality we get $f(x_0, y) \geq 0$. Since $y \in K$ is arbitrary (B) follows. This completes the proof of the lemma.

PROOF OF THEOREM 3.8. Assume that inequality (6) has no solution. Then, for each $x \in K$, there exists $u \in K$ such that

$$f(x, u) < 0 \quad \dots(7)$$

and by Lemma 3.9, there exists $v \in K$ such that

$$f(v, x) > 0. \quad \dots(8)$$

Thus for each $x \in K$, it follows from (7) that the set

$$F(x) = \{y \in K : f(x, y) < 0\}$$

is nonempty, which is also convex by hypothesis (b) of the theorem. Again for each $y \in K$, since

$$F^{-1}(y) = \{x \in K : f(x, y) < 0\},$$

it follows that

$$\begin{aligned}
[F^{-1}(y)]^c &= \{x \in K : f(x, y) \geq 0\} \\
&\subset \{x \in K : f(y, x) \leq 0\} \quad (\text{by (d)}) \\
&= G(y) \quad (\text{say}).
\end{aligned}$$

Thus

$$F^{-1}(y) \supset [G(y)]^c = A(y) \quad (\text{say}).$$

Since for each $x \in K$, the map $y \mapsto f(x, y)$ lower semicontinuous, it follows that $G(y)$ is closed in K . Thus for each $y \in K$, $A(y)$ is open in K .

Now, let $x \in K$ be arbitrary. By (8) there exists $v \in K$ such that $f(v, x) > 0$, so that $x \in A(v)$. Thus

$$\bigcup_{y \in K} A(y) = K.$$

Finally,

$$D = \bigcap_{y \in K_0} G(y) = \bigcap_{y \in K_0} [A(y)]^c$$

is either empty or compact by hypothesis (e) so that all conditions of Theorem 2.9 are fulfilled by F . Thus there exists $x_0 \in K$ such that $x_0 \in F(x_0)$, which is tantamount to $f(x_0, x_0) < 0$, which is a contradiction to hypothesis (a) of the theorem. Hence the conclusion of the theorem follows.

4. Uniqueness of solution

The following theorem characterizes the uniqueness of solution of the Problem P_1 under certain conditions.

THEOREM 4.1. Let K be a nonempty subset of a topological vector space X and $f: K \times K \rightarrow \mathbb{R}$ a map such that

(a) for all $x, y \in K$,

$$f(x, y) + f(y, x) \leq 0,$$

(b) equality does not hold in (a) unless $x = y$.

Then, if the problem P_1 is solvable, it has a unique solution.

Proof. If possible let x_1, x_2 be two solutions of P_1 . Then

$$f(x_1, y) \geq 0$$

and

$$f(x_2, y) \geq 0$$

for all $y \in K$. Putting $y = x_1$ in the former inequality and $y = x_2$ in the latter, we get

$$f(x_1, x_2) \geq 0$$

and

$$f(x_2, x_1) \geq 0.$$

Adding the last two inequalities we get,

$$f(x_1, x_2) + f(x_2, x_1) \geq 0.$$

This inequality combined with hypothesis (a) gives

$$f(x_1, x_2) + f(x_2, x_1) = 0.$$

Now an application of hypothesis (b) yields $x_1 = x_2$. This completes the proof.

Note: If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then Condition (a) of the above theorem is equivalent to the fact that T is monotone.

The following examples illustrate Theorem 4.1. Example 1 shows that the fulfillment of conditions (a) and (b) does not guarantee the existence of a solution to Problem P_1 . Examples (2) and (3) show that if conditions (a) and (b) are fulfilled and the Problem P_1 is solvable, then it has a unique solution.

Example 1. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = -|x - y|.$$

Then

$$f(x, y) + f(y, x) = -2|x - y|,$$

and

$$f(x, y) + f(y, x) = 0$$

if $|x - y| = 0$, that is $x = y$. Thus hypotheses (a) and (b) of Theorem 4.1 are fulfilled by f . But $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $|x_0 - y| \leq 0$ for all $y \geq 0$. Thus in this case no solution exists.

507

508 *Example 2.* Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \rightarrow \mathbb{R}$ as

509
$$f(x, y) = x(y - x).$$

510 Then

511
$$f(x, y) + f(y, x) = -(x - y)^2 \leq 0$$

512 and

513
$$f(x, y) + f(y, x) = 0$$

514 if and only if $(x - y)^2 = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is
 515 equivalent to $x_0(y - x_0) \geq 0$ for all $y \geq 0$, so that $x_0 = 0$ is the only solution in this
 516 case.

517

518 *Example 3.* Let $X = K = \mathbb{R}$, and define $f: K \times K \rightarrow \mathbb{R}$ as

519
$$f(x, y) = -x|x^2 - y^2|.$$

520 Then

521
$$f(x, y) + f(y, x) = -(x + y)^2|x - y| \leq 0$$

522 and

523
$$f(x, y) + f(y, x) = 0$$

524 if and only if $|x - y| = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is
 525 equivalent to $x_0|x_0^2 - y^2| \leq 0$ for all $y \in \mathbb{R}$ so that $x_0 = 0$ is the only solution in this
 526 case also.

527

528 **5. Some consequences**

529

530 In this section, we discuss some consequences of the theorems proved in
 531 Section 3.

532

533 The following theorem deals with the existence of solution of Problem P_2 .

534

535 **THEOREM 5.1.** *Let X be a locally convex Hausdorff topological vector space with*
 536 *dual X^* , K a nonempty compact convex subset of X , $f: K \times K \rightarrow \mathbb{R}$ a point-to-point*
 537 *map, and $S: K \rightarrow 2^K$, a point-to-set upper semicontinuous map such that*

538 (a) $f(x, x) \geq 0$ for each $x \in K$,

539 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi convex for each $x \in K$,

540 (c) the set

541
$$\left\{ x \in K : \sup_{y \in S(x)} f(x, y) < 0 \right\}$$

542 is empty or open in K .

543 Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $f(x_0, y) \geq 0$ for all $y \in S(x_0)$.

544

545 *Proof.* Assume that the assertion is false. Then for each $x \in K$ either
 546 $x \notin S(x)$, or there exists $y \in K$ such that $f(x, y) < 0$. If $x \notin S(x)$, then by Theorem
 547 2.11, there exists $p \in X^*$ such that

$$(p, x) - \sup_{y \in S(x)} (p, y) > 0.$$

Let

$$V_0 = \left\{ x \in K : \sup_{y \in S(x)} f(x, y) < 0 \right\},$$

and for each $p \in X^*$ let

$$V(p) = \left\{ x \in K : (p, x) - \sup_{y \in S(x)} (p, y) > 0 \right\}.$$

Then by hypothesis (c), V_0 is open in K . By Theorem 2.6 since the map $x \mapsto \sup_{y \in S(x)} (p, y)$ is upper semicontinuous for each $p \in X^*$, it follows that the map

$$x \mapsto (p, x) - \sup_{y \in S(x)} (p, y)$$

is lower semicontinuous. Hence $V(p)$ is open in K for each $p \in X^*$. Further,

$$K = V_0 \cup \left[\bigcup_{p \in X^*} V(p) \right].$$

Since K is compact there exists $\{p_1, p_2, \dots, p_n\} \subset X^*$ such that

$$K = V_0 \cup \left[\bigcup_{i=1}^n V(p_i) \right].$$

Let $\{\beta_0, \beta_1, \dots, \beta_n\}$ be a partition of unity subordinate to the covering $\{V_0, V(p_1), \dots, V(p_n)\}$, and define $g : K \times K \rightarrow \mathbb{R}$ as

$$g(x, y) = \beta_0(x) f(x, y) + \sum_{i=1}^n \beta_i(x) (p_i, y - x).$$

Then it is clear that for each $x \in K$, $g(x, x) \geq 0$, the map $y \mapsto g(x, y)$ is quasi-convex, for each $y \in K$, the map $x \mapsto g(x, y)$ is upper semicontinuous so that the set $\{x \in K : f(x, y) < 0\}$ is open in K . Thus all conditions of Theorem 3.1 are fulfilled. Hence there exists $\hat{x} \in K$ such that

$$g(\hat{x}, y) \geq 0 \quad \dots(9)$$

for all $y \in K$, which is equivalent to

$$\beta_0(\hat{x}) f(\hat{x}, y) + \sum_{i=1}^n \beta_i(\hat{x}) (p_i, y - \hat{x}) \geq 0.$$

Since $\{\beta_0, \beta_1, \dots, \beta_n\}$ is a partition of unity $\beta_i(\hat{x}) > 0$ for at least one i . If $\beta_0(\hat{x}) > 0$ then $\hat{x} \in V_0$ so that

$$\sup_{y \in S(\hat{x})} f(\hat{x}, y) < 0.$$

Let $\hat{y} \in S(\hat{x})$ such that $f(\hat{x}, \hat{y}) < 0$. If $\beta_i(\hat{x}) > 0$ then $\hat{x} \in V(p_i)$ so that

$$(p_i, \hat{x}) - \sup_{y \in S(\hat{x})} (p_i, y) > 0,$$

from which it follows that

$$(p_i, \hat{x}) > \sup_{y \in S(\hat{x})} (p_i, y) > (p_i, \hat{y})$$

which implies that $(p_i, \hat{y} - \hat{x}) < 0$. Thus

$$g(\hat{x}, \hat{y}) = \beta_0(\hat{x})f(\hat{x}, \hat{y}) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, \hat{y} - \hat{x}) < 0$$

which contradicts (9) when $y = \hat{y}$. This contradiction proves the theorem.

The following result on the existence of solution of the quasi-variational inequality problem is a direct consequence of the above theorem.

COROLLARY 5.2. *Let X be a locally convex Hausdorff topological vector space with dual X^* , K a nonempty compact convex subset of X , $f : K \times K \rightarrow \mathbb{R}$ a point-to-point map and $S : K \rightarrow 2^K$ a point-to-set upper semi-continuous map such that the set*

$$\left\{ x \in K : \sup_{y \in S(x)} (Tx, y - x) < 0 \right\}$$

is empty or open in K . Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and

$$(Tx_0, y - x_0) \geq 0$$

for all $y \in S(x_0)$.

Proof. Follows directly from Theorem 5.1 with $f(x, y) = (Tx, y - x)$.

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