

COST COMPUTATIONS OF FUZZY M/M/1 AND M/G/1 QUEUEING MODELS

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Abstract

In this paper two models of planning queueing system and its effect on the cost of the each system by using two fuzzy queueing models of M/M/1 and M/G/1 are studied. These two fuzzy queueing models based on the cost of each model are compared and fuzzy ranking methods are used to select the optimal model due to the resulted complexity. This paper results in a better approach for computing different queueing models in the fuzzy environment that it can be more effective than deterministic queueing models. The basic idea is to transform a fuzzy queueing cost problem to a family of conventional crisp queue cost problem by applying the α -cut approach and Zadeh's extension principle. A set of parametric nonlinear programs are developed to calculate the lower and upper bound of the minimal expected total cost per unit time at α , through which the membership function of the total cost is constructed. Numerical example is illustrated to check the validity of the proposed method.

Key words: α -cut, membership function, total cost function, centroid ranking method.

1. Introduction

In this modern world, it is well known that manpower is inevitable in spite of existence of advanced technology. Manpower planning is a device with which an attempt is made to match the supply of people with the demand in the form of jobs available in an organisation so that the cost incurred is optimal.

Queueing decision problem play an important role in the queueing system design that involves one or many decision such as number of servers at a service facility, the efficiency of the servers. A queueing cost based decision model is to determine a suitable service rate such that the sum of the cost of offering the service and cost of delay in offering the service is minimized.

Crisp models of M/M/1 and M/G/1 are studied in the works of Hiler and Liberman in [5] and Taha [9]. Fuzzy queueing models have been described by such researchers like Li and Lee[7], Buckley[1,2], Negi and Lee [8]. Chen[3,4] analysed fuzzy queueing using Zadeh's extension principle. Kao et al[6] constructed membership function of the system characteristic for fuzzy queues using parametric linear programming.

In queuing theory it is usually assumed that the time between the two consecutive arrivals and the servicing time follows a special probability distribution. However, in the real world, this type of information is obtained using qualitative data and expressed by words like quick, medium and slow rather than the probability distribution. Hence fuzzy queuing models are more realistic and practical than classical ones. In this paper cost computations of fuzzy M/M/1 and M/G/1 queuing model is considered. Obviously when the cost coefficients, arrival rate service rate are fuzzy, the minimal expected total cost per unit time will be fuzzy. Therefore the minimal expected total cost should be described by the membership function rather than by a crisp value. Here a mathematical non-linear parametric programming approach for the queuing decision problem by the basic idea of Zadeh's extension principle and α cut representation is developed. A set of non linear programming problems are formulated to calculate the upper and lower bound of α cut of the minimal expected total cost and consequently membership function of the minimal expected total cost is derived.

2. Total cost function of Fuzzy M/M/1 and M/G/1 queuing decision problem

Consider an (FM/FM/1): (∞ /FCFS) and (FM/FG/1) : (∞ /FCFS)) queuing models in which customer arrive at the service facility at rate $\tilde{a}$ and at service rate $\tilde{\kappa}$, the cost of service per unit time is $\tilde{C}_1$, the cost waiting per customer per unit time is $\tilde{C}_2$ for both the models are fuzzy numbers. Let the service rate be $\tilde{\kappa}$ for the first model and $\tilde{\kappa}_1$ for the later model. Total expected cost for the first model can be computed as follows:

Let $\mu_{\tilde{a}}^{(x)}$, $\mu_{\tilde{\kappa}}^{(y)}$, $\mu_{\tilde{C}_1}^{(u)}$ and $\mu_{\tilde{C}_2}^{(v)}$ denoted the membership functions of $\tilde{a}$, $\tilde{\kappa}$, $\tilde{C}_1$, and $\tilde{C}_2$ respectively. We have the following fuzzy sets.

$$\tilde{a} = \{(x, \mu_{\tilde{a}}^{(x)}) / x \in X\} \dots\dots\dots (1a)$$

$$\tilde{\kappa} = \{(y, \mu_{\tilde{\kappa}}^{(y)}) / y \in Y\} \dots\dots\dots (1b)$$

$$\tilde{C}_1 = \{(u, \mu_{\tilde{C}_1}^{(u)}) / u \in U\} \dots\dots\dots (1c)$$

$$\tilde{C}_2 = \{(v, \mu_{\tilde{C}_2}^{(v)}) / v \in V\} \dots\dots\dots (1d)$$

Where X, Y, C₁ and C₂ are the crisp universal sets of arrival service and cost coefficient. Let f(x, y, u, v) denote the system characteristics of interest. Since $\tilde{a}$, $\tilde{\kappa}$, $\tilde{C}_1$, $\tilde{C}_2$ are fuzzy numbers, f($\tilde{a}$, $\tilde{\kappa}$, $\tilde{C}_1$, $\tilde{C}_2$) is also a fuzzy number.

Following Zadeh's extension principle [11] the membership function of expected total cost is defined as

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$$f(\tilde{a}, \tilde{\alpha}, C_1, C_2)(z) = \sup \min \{ \mu_{\tilde{a}}(x), \mu_{\tilde{\alpha}}(y), \mu_{\tilde{C}_1}(u), \mu_{\tilde{C}_2}(v) /$$

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$$z = f(x, y, u, v) \dots\dots\dots (2)$$

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The minimal expected total cost of a crisp queuing system is given by

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$$E(c) = C_1 + LC_2 \dots\dots\dots (3)$$

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The membership function of the minimal cost is

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$$\mu_{E(\tilde{C})}(z) = \sup \min \{ \mu_{\tilde{a}}(x), \mu_{\tilde{\alpha}}(y), \mu_{\tilde{C}_1}(u), \mu_{\tilde{C}_2}(v) / z = u + Lv \} \dots\dots\dots (4)$$

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In this paper we approach the representation problem using a mathematical programming technique parametric NLPs are developed to find α cut $f(\tilde{a}, \tilde{\alpha}, \tilde{C}_1, \tilde{C}_2)$ based on the extension principle.

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3. Solution procedure

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Definitions for the α cuts of $\tilde{a}$, $\tilde{\alpha}$, $\tilde{C}_1$, and $\tilde{C}_2$ as crisp intervals are as follows.

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$$a(\sim) = [x_{\sim}^L, x_{\sim}^U] = [\min \{x / \mu_{\tilde{a}}(x) \geq \sim\}, \max \{x / \mu_{\tilde{a}}(x) \geq \sim\}] \dots\dots\dots (5a)$$

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$$\alpha(\sim) = [y_{\sim}^L, y_{\sim}^U] = [\min \{y / \mu_{\tilde{\alpha}}(y) \geq \sim\}, \max \{y / \mu_{\tilde{\alpha}}(y) \geq \sim\}] \dots\dots\dots (5b)$$

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$$u(\sim) = [u_{\sim}^L, u_{\sim}^U] = [\min \{u / \mu_{\tilde{C}_1}(u) \geq \sim\}, \max \{u / \mu_{\tilde{C}_1}(u) \geq \sim\}] \dots\dots\dots (5c)$$

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$$v(\sim) = [v_{\sim}^L, v_{\sim}^U] = [\min \{v / \mu_{\tilde{C}_2}(v) \geq \sim\}, \max \{v / \mu_{\tilde{C}_2}(v) \geq \sim\}] \dots\dots\dots (5d)$$

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As a result, the bound of these intervals can be described as functions of α and can be obtained as

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$$x_{\sim}^L = \min \mu_{\tilde{a}}^{-1}(\sim) \quad x_{\sim}^U = \max \mu_{\tilde{a}}^{-1}(\sim)$$

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$$y_{\sim}^L = \min \mu_{\tilde{\alpha}}^{-1}(\sim) \quad y_{\sim}^U = \max \mu_{\tilde{\alpha}}^{-1}(\sim)$$

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$$u_{\sim}^L = \min \mu_{\tilde{C}_1}^{-1}(\sim) \quad u_{\sim}^U = \max \mu_{\tilde{C}_1}^{-1}(\sim)$$

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$$v_{\sim}^L = \min \mu_{\tilde{C}_2}^{-1}(\sim) \quad v_{\sim}^U = \max \mu_{\tilde{C}_2}^{-1}(\sim)$$

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Therefore the α cuts to construct its membership function is used. Since the membership function in (4) is parameterized by α .

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92 Using Zadeh's extension principle, $\mu_{E(\tilde{C})}$ is minimum of $\mu_{\tilde{a}}^{(x)}$, $\mu_{\tilde{\kappa}}^{(y)}$, $\mu_{\tilde{C}_1}^{(y)}$, and $\mu_{\tilde{C}_2}^{(y)}$.

93 To derive the $\mu_{E(\tilde{C})}(z) = \sim$ we need at least one of the following cases to hold such that

94 satisfies $\mu_{E(\tilde{C})}(z) = \sim$

95 Case 1 : $\mu_{\tilde{a}}^{(x)} = \sim, \mu_{\tilde{\kappa}}^{(y)} \geq \sim, \mu_{\tilde{C}_1}^{(u)} \geq \sim, \mu_{\tilde{C}_2}^{(v)} \geq \sim$

96 Case 2 : $\mu_{\tilde{a}}^{(x)} \geq \sim, \mu_{\tilde{\kappa}}^{(y)} = \sim, \mu_{\tilde{C}_1}^{(u)} \geq \sim, \mu_{\tilde{C}_2}^{(v)} \geq \sim$

97 Case 3 : $\mu_{\tilde{a}}^{(x)} \geq \sim, \mu_{\tilde{\kappa}}^{(y)} \geq \sim, \mu_{\tilde{C}_1}^{(u)} = \sim, \mu_{\tilde{C}_2}^{(v)} \geq \sim$

98 Case 4 : $\mu_{\tilde{a}}^{(x)} \geq \sim, \mu_{\tilde{\kappa}}^{(y)} \geq \sim, \mu_{\tilde{C}_1}^{(u)} \geq \sim, \mu_{\tilde{C}_2}^{(v)} = \sim$

99 This can be accomplished using parametric NLP techniques. The NLP to find the lower and
100 upper bounds of the cut of $\mu_{E(\tilde{C})}$ for case 1 are

$$101 [E(C)]_{\sim}^L = \min[u + Lv] \quad \dots\dots\dots (6a)$$

$$102 [E(C)]_{\sim}^U = \max[u + Lv] \quad \dots\dots\dots (6b)$$

103 For case 2 are

$$104 [E(C)]_{\sim}^L = \min[u + Lv] \quad \dots\dots\dots (6c)$$

$$105 [E(C)]_{\sim}^U = \max[u + Lv] \quad \dots\dots\dots (6d)$$

106 For case 3 are

$$107 [E(C)]_{\sim}^L = \min[u + Lv] \quad \dots\dots\dots (6e)$$

$$108 [E(C)]_{\sim}^U = \max[u + Lv] \quad \dots\dots\dots (6f)$$

109 For case 4 are

$$110 [E(C)]_{\sim}^L = \min[u + Lv] \quad \dots\dots\dots (6g)$$

$$111 [E(C)]_{\sim}^U = \max[u + Lv] \quad \dots\dots\dots (6h)$$

112 From the definitions of $\alpha(\sim), \kappa(\sim), u(\sim)$ and $v(\sim)$ in (5a, 5b, 5c, 5d)

113 $x \in \alpha(\sim), y \in \kappa(\sim), u \in C_1(\sim)$ and $v \in C_2(\sim)$ can be replaced by

$$114 x \in [x_{\sim}^L, x_{\sim}^U], y \in [y_{\sim}^L, y_{\sim}^U], u \in [u_{\sim}^L, u_{\sim}^U], v \in [v_{\sim}^L, v_{\sim}^U].$$

115 The α cut form a nested structure with respect to $\alpha \{6a, 6b, 6c, 6d, 6e, 6f, 6g, 6h\}$. For given

116 $0 < \alpha_2 < \alpha_1 < 1$ we have

$$117 [y_{\sim_1}^L, y_{\sim_1}^U] \subseteq [y_{\sim_2}^L, y_{\sim_2}^U]$$

$$[u_{\sim 1}^L, u_{\sim 1}^U] \subseteq [u_{\sim 2}^L, u_{\sim 2}^U]$$

$$[v_{\sim 1}^L, v_{\sim 1}^U] \subseteq [v_{\sim 2}^L, v_{\sim 2}^U]$$

Therefore ((6a),(6c),(6e),(6g)) have the same smallest element and ((6b),(6d),(6f),(6h)) have the same largest element.

To find the lower and upper bounds of $E(C)$,

$$[E(C)]^L = \min[u + Lv] \text{ such that } \dots\dots\dots (7a)$$

$$x_{\sim}^L \leq x \leq x_{\sim}^U, y_{\sim}^L \leq y \leq y_{\sim}^U, u_{\sim}^L \leq u \leq u_{\sim}^U, v_{\sim}^L \leq v \leq v_{\sim}^U$$

$$[E(C)]^U = \max[u + Lv] \text{ such that } \dots\dots\dots (7b)$$

$$x_{\sim}^L \leq x \leq x_{\sim}^U, y_{\sim}^L \leq y \leq y_{\sim}^U, u_{\sim}^L \leq u \leq u_{\sim}^U, v_{\sim}^L \leq v \leq v_{\sim}^U$$

At least any one of x, y, u, v must hit the boundaries of their α cut satisfying $\mu_{E(\tilde{C})}(z) =$

α . Applying the results of Zimmerman [12] and convexity properties, we have

$$[E(C)]_{\sim 1}^L \geq [E(C)]_{\sim 2}^L \text{ and } [E(C)]_{\sim 1}^U \geq [E(C)]_{\sim 2}^U$$

Where $0 < \alpha_2 < \alpha_1 < 1$

In both $[E(C)]_{\sim}^L$ and $[E(C)]_{\sim}^U$ are invertible with respect to, then a left shape function

$L(z) = [E(C)_{\sim}^L]^{-1}$ and right shape function $R(z) = [E(C)_{\sim}^U]^{-1}$ can be derived, such that

$$\mu_{E(\tilde{C})}(z) = \begin{cases} L(z), [E(c)]_{\alpha=0}^L \leq z \leq [E(c)]_{\alpha=1}^L \\ 1, [E(c)]_{\alpha=1}^L \leq z \leq [E(c)]_{\alpha=1}^U \\ R(z), [E(c)]_{\alpha=1}^U \leq z \leq [E(c)]_{\alpha=0}^U \end{cases} \dots\dots\dots (8)$$

In most cases the values $[E(C)]_{\sim}^L$ and $[E(C)]_{\sim}^U$ cannot be solved analytically.

Consequently, a closed form membership function for $E(\tilde{C})$ cannot be obtained.

However, the numerical solution for $[E(C)]_{\sim}^L$ and $[E(C)]_{\sim}^U$ at different possibility levels can be collected to approximate the shape of $L(z)$ and $R(z)$. Similarly the total expected cost for the (FM/FG/1) : (∞ /FCFS) model can be computed.

Since the superiority of each model cannot be studied intuitively, to decide in an uncertain environment and to compare the different decisions in the fuzzy environment, ranking methods have been proposed. Centroid ranking method [10] is used in this paper.

4. Numerical example.

Consider a single server Poisson input queue with mean arrival rate $\tilde{a} = [1, 2, 3, 4]$ customers per unit time, currently the server works according to an exponential distribution with mean service rate $\tilde{\kappa} = [6, 7, 8, 9]$ customers per unit time. Management has a training course which will result in an improvement in the service rate. After the completion of the course, it is estimated that mean service rate will increased. The increased service rate is $\tilde{\kappa}_1 = [11, 12, 13, 14]$. It is assumed that now the service rate follows the geometric distribution. The cost of the server is a trapezoidal fuzzy number $\tilde{C}_1 = [1200, 1500, 1800, 2100]$ per unit time, waiting time cost of the customer is $\tilde{C}_2 = [50, 60, 70, 80]$ per unit time. Let $\tilde{C}_3 = [10, 20, 30, 40]$ be the training cost of the server. The management wants to know whether the training is useful.

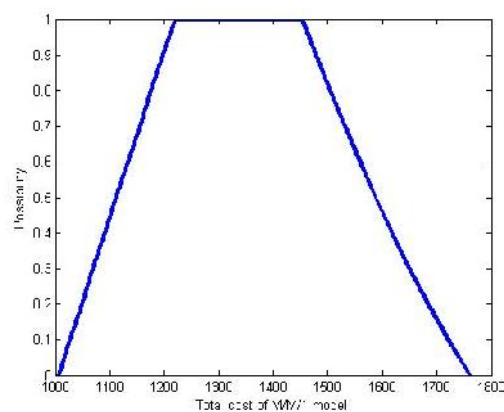
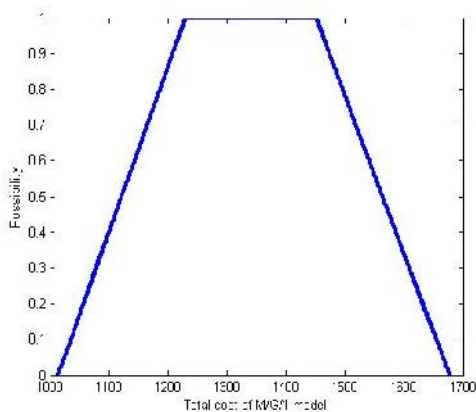
It is easy to find that

$$[x_{\alpha}^L, x_{\alpha}^U] = [1 + \alpha, 4 - \alpha], \quad [y_{\alpha}^L, y_{\alpha}^U] = [6 + \alpha, 9 - \alpha],$$

$$[y_{1\alpha}^L, y_{1\alpha}^U] = [11 + \alpha, 14 + \alpha], \quad [u_{\alpha}^L, u_{\alpha}^U] = [1200 + 300\alpha, 2100 + 300\alpha]$$

$$[v_{\alpha}^L, v_{\alpha}^U] = [50 + 10\alpha, 80 - 10\alpha]$$

It is obvious that $x = x_{\alpha}^L, y = y_{\alpha}^U, u = u_{\alpha}^L, v = v_{\alpha}^L$ the expected total cost attains its minimum value and when $x = x_{\alpha}^U, y = y_{\alpha}^L, u = u_{\alpha}^U, v = v_{\alpha}^U$ the expected total cost attains its maximum value.



Now, by getting all the parameters and according to formula (7a) and (7b), the cost function of first model is $E(C_1) = [1006.25, 1220, 1452.5, 1760]$ and the cost function of second model is $E(C_2) = [1013.69, 1229.97, 1450.48, 1677.44]$.

By comparing the total cost for the both models, it is observed that the result of cost analysis leads to the fuzzy values for costs of each model that has overlapped with each other. To decide about the case which provides the minimum cost, the decision making techniques in uncertain environments have been applied. To compare these costs the fuzzy centroid ranking method is used. By applying the centroid ranking methods it is found that expected total cost of the system after training is minimum, hence it can be concluded that training is useful.

5. Conclusion.

The fuzzy queuing model has more applicability in the real environments than the crisp systems. In this paper, two practical systems are compared in real environment. To analyse the conditions more precise and more practical, it is assumed that the rate of arrivals and servicing rate are fuzzy numbers also it is considered that the system costs are fuzzy numbers to express the uncertainty in the system completely. The results are more useful for system designers and practitioners.

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