

THE CORE ERROR PROBLEM IN VARIABLES FOR STATISTICAL LEAST SQUARES WITH INTERVAL DATA

Abstract: The paper discusses the core problem error in variables for statistical Least Squares data with intervals. Via Mobius transformation for complex disk in the sense of Petkovic, the exact inversion of a disk was described, wherein, Modified Certifylss relaxed refinement Cholesky decomposition in the sense of Rump comes in handy, as a useful tool .

As a measure of comparison on the performance of this approach, Oettli-Prager theorem was adopted to provide the solution to the resulting linear interval system of equations. We draw conclusion based on the outcome of our findings.

Keywords: least squares, interval relaxed iterative refinement, mobius transformation, modified Certifylss relaxed refinement Cholesky decomposition, Oettli-Prager theorem

1. Introduction

Consider statistical measurements without noise which is described by the equation

$$y = f(\beta) \quad (1.1)$$

Where, $y \in R^m$ is a measurement vector, and $\beta \in R^n$ is a parameter vector respectively. It is assumed that function f has at least $C^1 \in [a, b]$, $a \leq \beta \leq b$. Often in practice, the measured parameters are corrupted by noise, thus applying traditional Statistical methods to solve this will undoubtedly fail. There is thus, a need to overcome this setback, by applying the interval arithmetic.

The aim is to solve

$$\begin{aligned} y(\beta) &= \min_{\beta \in IR^n} \left\| f(\beta) - \hat{y} \right\|_2^2 = \min_{\beta \in IR^n} \left\| A\beta - \hat{y} \right\|_2^2 = \left(A\beta - \hat{y} \right)^T \left(A\beta - \hat{y} \right) \\ &= \beta^T A^T A \beta - 2\beta^T A^T \hat{y} + \hat{y}^T A^T A \hat{y} \end{aligned} \quad (1.2)$$

Where $f(\beta) = A\beta$ and A is a real $m \times n$ matrix with $m > n$.

Wherefrom, we minimized β in the form

$$\beta = (A^T A)^{-1} A^T \hat{y} \quad (1.3)$$

29 The statistical meaning [1] of the matrix $(A^T A)^{-1}$, is as follows: Consider the components
 30 $y_i : i = 1, 2, \dots, m$ which are independently normally distributed random variables having μ_i as a
 31 mean and, variance δ^2 :

$$32 \quad E(y_i) = \mu_i ; E(y_i - \mu_i)(y_k - \mu_k) = \begin{cases} \delta^2, & i = k \\ 0 & \text{otherwise} \end{cases} \quad (1.4)$$

33 Letting $\mu = (\mu_1, \mu_2, \dots, \mu_m)^T$ it can be proved [1] that

$$34 \quad E(y) = \mu, \quad E(y - \mu)(y - \mu)^T = \delta^2 I.$$

35 It follows that

$$36 \quad E[(\beta - E(\beta))(\beta - E(\beta))^T] = E\left[(A^T A)^{-1} A^T (y - \mu)(y - \mu)^T A (A^T A)^{-1}\right]$$

$$37 \quad = (A^T A)^{-1} A^T E[(y - \mu)(y - \mu)^T] A (A^T A)^{-1} = \delta^2 (A^T A)^{-1} = \delta^2 \sum_{j=1}^n \frac{1}{\lambda_j},$$

38 $\lambda_j : j = 1, 2, \dots, p$ are eigenvalues of $A^T A$.

39 The optimum $\beta = (A^T A)^{-1} A^T y$ is also a random vector with

$$40 \quad E[(A^T A)^{-1} A^T y] = (A^T A)^{-1} A^T E(y) = (A^T A)^{-1} A^T \mu.$$

41 Basic properties of interval arithmetic are well documented in the standard text books by [2], [3] and
 42 [4].

43 The following expositions hold for the function F.

- 44 (i) Let $F : ID \subseteq IR^n \rightarrow IR^n$ be F-differentiable on the convex set $ID_0 \subseteq ID$, and
 45 $\|F'(\beta)\| \leq K < \infty$, for all $\beta \in ID_0$, then F is Lipschitz continuous on ID_0 .
- 46 (ii) If $F : ID \subseteq IR^n \rightarrow IR^n$ is F-differentiable on the convex set $ID_0 \subset ID$, then for any
 47 $\beta_0, \beta_1 \in ID_0$,
 48 $\|F(\beta_1) - F(\beta_0) - F'(\beta)(\beta_1 - \beta_0)\| \leq$
 49 $\sup \|F'(\beta_0 + t(\beta_1 - \beta_0)) - F'(\beta_0)\| \|\beta_1 - \beta_0\|, 0 \leq t \leq 1.$

50 The Lipschitz continuity of $F'(\beta)$ holds from the following convexity that, for arbitrary
 51 vectors, $v, z \in \bar{S}(\beta, \delta)$, and $t \in [0, 1]$, we have that

52

$$\|z + t(v - z) - v\| = \|t(v - \beta) + (1 - t)(z - \beta)\| < t\delta + (1 - t)\delta = \delta.$$

and the estimate

$$\|F(v) - F(z) - F'(\beta)(\beta - z)\| \leq \sup_{0 \leq t \leq 1} \|F'(z + t(\beta - z)) - F'(\beta)\| \|v - z\| \leq \varepsilon \|v - z\| \quad (1.4)$$

remain valid.

Hemi-continuity of F is implied by the fact that Jacobian matrix of F and Riemann integrability of F will be given in the form:

$$\begin{aligned} \|F(v) - F(\beta) - F'(\beta)(v - \beta)\| &= \left\| \int_0^1 (F'(\beta + t(v - \beta)) - F'(\beta))(v - \beta) dt \right\| \leq \\ &\int_0^1 \|F'(\beta + t(v - \beta)) - F'(\beta)\| \|v - \beta\| dt \leq \int_0^1 \|F'(\beta + t(v - \beta)) - F'(\beta)\| \|v - \beta\| dt \\ &\leq \|v - \beta\|^{p+1} \int_0^1 t^p dt \leq \left(\frac{\alpha}{p+1} \right) \|v - \beta\|^{p+1} \end{aligned}$$

Therefore, it follows from the reasoning given in [5] that F has a local homeomorphism at β_0 and $F'(\beta_0)$ is non singular too.

2. The Ball arithmetic

The ball arithmetic was incorporated into interval arithmetic computation for the first time by [6]) and had gained prominence in interval researchers amongst them are [4]) and [3]. Fundamental to this underlining work is [7].

The interval operations are defined as follows: Let $*$ represent an operation from the set $\{+, -, \cdot, /\}$

.Then assuming $[a] = [\underline{a}, \bar{a}]$ and $[b] = [\underline{b}, \bar{b}]$ are two real intervals as in [2], we have:

$[a].[b] = \{a * b \mid a \in [a], b \in [b]\}$: Such that the following four operations for intervals are defined.

$$\begin{aligned} [a] + [b] &= [\underline{a} + \underline{b}, \bar{a} + \bar{b}], \\ [a] - [b] &= [\underline{a} - \bar{b}, \bar{a} - \underline{b}], \\ [a] * [b] &= [\min(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}), \max(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b})] \quad . \quad (2.1) \end{aligned}$$

The extended multiplication for intervals $[a]$ and $[b]$ was further studied in [8] , see also, [9] in the form

$$\begin{aligned}
 & \left[\begin{array}{l} \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} \geq 0 \text{ and } \bar{b} \geq 0 \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} \geq 0 \text{ and } \bar{a} < 0 < \bar{b} \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} \geq 0 \text{ and } \bar{b} \leq 0 \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} < 0 < \bar{a} \text{ and } \bar{b} \geq 0 \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} < 0 < \bar{a} \text{ and } \bar{b} \leq 0 \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} \leq 0 \text{ and } \bar{b} \geq 0 \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} \leq 0 \text{ and } \bar{b} < 0 < \bar{b} \\ \left[\begin{array}{l} \bar{a} \bar{b}, \bar{a} \bar{b} \end{array} \right] \quad \text{if } \bar{a} \leq 0 \text{ and } \bar{b} \leq 0 \end{array} \right. \\
 & \left[\min \left(\bar{a} \bar{b}, \bar{a} \bar{b} \right), \max \left(\bar{a} \bar{b}, \bar{a} \bar{b} \right) \right] \quad \text{if } \bar{a} < 0 < \bar{a} \text{ and } \bar{b} < 0 < \bar{b}
 \end{aligned} \tag{2.2}$$

73 The operation for division is $\frac{1}{[b]} = \left[\frac{1}{\bar{b}}, \frac{1}{\bar{b}} \right], 0 \notin [b]$ and $\frac{[a]}{[b]} = [a].[b]^{-1}$.

74 More fundamentally, is the case when $[b]$ contains the origin, i.e., $0 \in [b]$.

75 Therefore, Kahan arithmetic in the sense of [10] could be defined in the form:

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$$\begin{aligned}
 \frac{[a]}{[b]} = & \begin{cases} \left[\frac{\bar{a}}{\bar{b}}, \infty \right] & \text{if } \bar{a} \leq 0 \text{ and } \bar{b} = 0 \\ \\ \left(-\infty, \frac{\bar{a}}{\bar{b}} \right] \cup \left[\frac{\bar{a}}{\bar{b}}, \infty \right) & \text{if } \bar{a} \leq 0 \text{ and } \bar{b} < 0 < \bar{b} \\ \\ \left(-\infty, \frac{\bar{a}}{\bar{b}} \right] & \text{if } \bar{a} \leq 0 \text{ and } \bar{b} = 0 \\ \\ (-\infty, \infty) & \text{if } \bar{a} < 0 < \bar{a} \\ \\ \left(-\infty, \frac{\bar{a}}{\bar{b}} \right) & \text{if } \bar{a} \geq 0 \text{ and } \bar{b} = 0 \\ \\ \left(-\infty, \frac{\bar{a}}{\bar{b}} \right] \cup \left[\frac{\bar{a}}{\bar{b}}, \infty \right) & \text{if } \bar{a} \geq 0 \text{ and } \bar{b} < 0 < \bar{b} \\ \\ \left[\frac{\bar{a}}{\bar{b}}, \infty \right) & \text{if } \bar{a} \geq 0 \text{ and } \bar{b} = 0 \end{cases} \quad (2.3)
 \end{aligned}$$

81 Besides the interval operation due to [2] mentioned above, we also have the circular complex
 82 interval arithmetic with centre $a_c = \frac{1}{2} \left(\bar{a} + \underline{a} \right)$ and radius $\Delta a = \frac{1}{2} [\bar{a} - \underline{a}]$ and is by the parametric
 83 notation defined in the form

$$84 \quad [a] = \{a : |a - a_c| < \Delta a\} \quad (2.4)$$

85 We need to stress salient concepts in circular interval computation- the negation and relative extent
 86 of a disk. The negation of a disk $[a] = -[a] = (-a_c, \Delta a)$, and the relative extent is defined:

$$87 \quad R\mathbf{x}(a) = \frac{\Delta a}{a_c}, \text{ for } a_c \neq 0.$$

88 Via Mobius transformation in the sense of [11], the exact inversion of a disk could be derived. First
 89 observe that:

$$90 \quad [a]^{-1} = \{a_c, \Delta a\}^{-1} = \left\{ \frac{\bar{a}_c}{|a_c|^2 - \Delta a^2}, \frac{\Delta a}{|a_c|^2 - \Delta a^2} \right\} = \left\{ \frac{1}{a} : a \in \{a_c, \Delta a\}, 0 \notin \{a_c, \Delta a\} \right\}. \quad (2.5)$$

91 Introducing thus, the mapping $a \rightarrow \frac{1}{a}$, equation 2.5 can be written in polynomial form:

$$\left\{ \frac{\bar{a}_c}{|a_c|^2 - \Delta a}, \frac{\Delta r}{|a_c|^2 - \Delta a^2} \right\} = \left\{ \frac{1}{a_c} + \frac{(\Delta a / |a_c|)^2}{a_c} \left(1 + (\Delta a / |a_c|)^2 - (\Delta a / |a_c|)^4 + \dots \right), \frac{\Delta a}{|a_c|^2 - \Delta a^2} \right\} \quad (2.6)$$

The circumference of exact inversion of a disk D_c would be defined as

$$D_c = \left\{ a : \left| a - \frac{\bar{a}_c}{|a_c|^2 - \Delta a^2} \right| = \frac{\Delta a}{|a|^2 - \Delta a^2} \right\}. \quad (2.6)$$

Where from,

$$\{a_c, \Delta a\}^{I_c} = D_c \bigcup \text{int } D_c \supseteq \{a_c, \Delta a\}^{-1}. \quad (2.7)$$

By writing an inversion of a disk $\{a_c, \Delta a\}^{I_c} = \left\{ \frac{1}{a_c}, \rho \right\}$, it can be derived [11] that

$$\begin{aligned} \rho = \text{rad}(a_c, \Delta a)^{I_c} &= \max_{\tau \in \{a_c, \Delta a\}^{-1}} \left| \frac{1}{a_c} - \tau \right| = \max_{\theta \in [0, 2\pi]} \left| \frac{1}{a_c} - \frac{a_c^{-1} \Delta a \exp(i\theta)}{|a_c|^2 - \Delta a^2} \right| \\ &= \frac{\Delta a}{|a_c|} \max_{\theta \in [0, 2\pi]} \left| \frac{\Delta a + a_c \exp(i\theta)}{|a_c|^2 - \Delta a^2} \right| = \frac{\Delta a}{|a_c|} \max_{\tau \in [0, 2\pi]} \left| \frac{\Delta a + |a_c| \exp(i\tau)}{|a_c|^2 - \Delta a^2} \right| = \frac{\Delta a}{|a_c|(|a_c| - \Delta a)}, \quad (2.8) \end{aligned}$$

The centred inversion of a disk in our work would be adopted to be in the form:

$$\{a_c, \Delta a\}^{I_c} = \left\{ \frac{1}{a_c}, \frac{\Delta a}{|a_c|(|a_c| - \Delta a)} \right\} \supseteq \{a_c, \Delta a\}^{-1}, 0 \notin \{a_c, \Delta a\}. \quad (2.9)$$

In what follows we introduce multiplication of two disks $a = \{a_c, \Delta a\}$, $b = \{b_c, \Delta b\}$ where

$|a_c| > \Delta a$, and $|b_c| > \Delta b$ in the form:

$$a.b = \{a_c.b_c, |a_c|\Delta b + |b_c|\Delta a + \Delta a\Delta b\}, \quad (2.10)$$

Further applications of equation 2.10 were used in [12, 13] in the form:

$$a.b \subseteq \{a_c.b_c + \eta a.\eta b, |a_c|\Delta b + \Delta a(|b_c| + \Delta b) - |\eta a| |\eta b|\} \quad (2.11)$$

Where,

$$\eta a = \text{sign}(a_c) * \min(|a_c|, \Delta a). \quad (2.12)$$

In what follows, we denote R as approximate inverse of midpoint interval matrix A, as a result, [14] proved that the bound

110 $\left| \mathbf{A}^{-1} - \mathbf{R} \right| \leq \left(\left(\mathbf{I} - \left| \mathbf{R} \Delta \mathbf{A} \right| \right)^{-1} - \mathbf{I} \right) \left| \mathbf{R} \right|$. Available information we shall need for certifying the regularity
111 of interval matrices are well documented in [15] and [16] in the form:

- 112 (i) If $\mathbf{A}_c$ is regular and $\max_j \left(\Delta \mathbf{A} \left| \mathbf{R} \right| \right)_{jj} \geq 1$ then $\mathbf{A}$ is regular.
113 (ii) If $\lambda_{\max} \left(\left(\Delta \mathbf{A} \right)^T \Delta \mathbf{A} \right) < \lambda_{\min} \left(\mathbf{A}_c^T \mathbf{A}_c \right)$ holds, then $\mathbf{A}$ is regular whenever λ is the eigenvalue of
114 the matrix.
115 (iii) If there exists a matrix $\mathbf{R}$ such that $\left(\mathbf{I} + \left| \mathbf{I} - \mathbf{A}_c \mathbf{R} \right| \right)_j \leq \left(\Delta \mathbf{A} \left| \mathbf{R} \right| \right)_j$ for some $j \in (1, 2, \dots, n)$ then
116 $\mathbf{A}$ is singular.
117 (iv) If $\mathbf{A}_c$ is regular and $\rho \left(\left| \mathbf{R} \Delta \mathbf{A} \right| \right) < 1$ then $\mathbf{A}$ is regular.

118 Let us take note that verification for positive definiteness of an interval matrix $\mathbf{A}$ can be obtained if

119 $\mathbf{A}_c^T \mathbf{A}_c - \left\| \Delta \mathbf{A}^T \Delta \mathbf{A} \right\| > 0.$

120 The above named properties of interval arithmetic will be used in the solution to system (1.1)

121

122 3. Iterative Methods for the problem

123 Following [7], we give Cholesky decomposition in the form:

124 **Cholesky algorithm**

125 Input a symmetric matrix $\mathbf{A} = (a_{ij})$:

126 For j=1:n

127 For i=1:j-1

128
$$r_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj} \right) / r_{ii}$$

129 end

130
$$r_{jj} = \left(a_{jj} - \sum_{k=1}^{j-1} r_{kj}^2 \right)^{\frac{1}{2}}$$

131 end

132

133 Computation of Cholesky decomposition when executed in finite precision runs to completion

134 with approximate $\bar{\mathbf{R}}$ such that, $\bar{\mathbf{R}}^T \bar{\mathbf{R}} = \mathbf{A} + \Delta \mathbf{A}$ where the residual satisfies the error estimates,
135 [7]:

$$136 \quad \left| a_{ij} - \sum_{k=1}^i \bar{r}_{ki} \bar{r}_{kj} \right| \leq \eta_i \sum_{k=1}^i \left| \bar{r}_{ki} \right| \left| \bar{r}_{kj} \right| \text{ for } i < j \text{ and}$$

$$137 \quad \left| a_{jj} - \sum_{k=1}^j \bar{r}_{kj}^2 \right| \leq \eta_{j+1} \sum_{k=1}^j \bar{r}_{kj}^2$$

$$138 \quad \text{Where } \eta_k = \frac{ku}{(1-ku)}; u = 2^{-53} \text{ and } \eta^{-1074}$$

139 The operator under consideration is the modified Certifylss relaxed Cholesky decomposition which,
140 takes into consideration the reach ability theory in a manner likened to [17] in the form:

141

142 Algorithm (Modified Certifylss relaxed via Cholesky decomposition)

143 Input: matrix $A \in IR^{n \times n}$, $b \in IR^n$.

144 Compute $\bar{\beta} = A_c \setminus b_c$ (through Cholesky decomposition)

145 While (not converged)

$$146 \quad r = (b_c - A \bar{\beta})$$

147 $[e] = A \setminus r$ (Through method 2.9)

$$148 \quad \beta = \bar{\beta} + [e]$$

149

150 In [18], Oettli-Prager formula for the Linear Least Squares was used in the form

$$151 \quad \beta = \begin{pmatrix} \beta \\ - \end{pmatrix}_i = \sum_{j=1}^n \min \left\{ (Q)_{ij} \begin{pmatrix} y \\ - \end{pmatrix}_j, (Q)_{ij} \begin{pmatrix} - \\ y \end{pmatrix}_j \right\} \leq (\beta)_i \leq \sum_{j=1}^n \max \left\{ (Q)_{ij} \begin{pmatrix} y \\ - \end{pmatrix}_j, (Q)_{ij} \begin{pmatrix} - \\ y \end{pmatrix}_j \right\} = \begin{pmatrix} - \\ \beta \end{pmatrix}_i \quad (3.4)$$

152 In equation 3.4, [19] gave hints of how to compute the solution in the following way.

153 It is assumed that there exists an orthant

154

155 $IR_z^n = \{\beta; T_z \beta \geq 0\}$, T_z is a diagonal matrix with sign vector z on the diagonal, $z_i \in \{-1, 1\}$ where
156 from,

$$157 \quad \beta \cap IR^n = \{\beta, (A_c + \Delta T_z)z \geq b_c - \delta, (A_c - \Delta T_z)z \leq b_c + \delta, T_z \beta \geq 0\} \quad (3.5)$$

158 The matrices Q, Q' appearing in the inequality 3.4 solves the system of linear equation

$$QA_c T_z + |Q|\Delta = E \quad (3.6)$$

$$Q' A_c T_z - |Q'|\Delta = E \quad (3.7)$$

with unique solution .

The solution bounds provided by the numerical methods to the given problem should fulfil in the sense of [20] the estimate

$$\left\| \hat{\beta} - A^{-1}b \right\|_{\infty} = \left\| (1 - (I - RA))^{-1} R \left(A \hat{\beta} - b \right) \right\|_{\infty} \leq \frac{\left\| R \left(A \hat{\beta} - b \right) \right\|_{\infty}}{1 - \|I - RA\|_{\infty}}$$

4. NUMERICAL EXAMPLE

Consider fitting of polynomial of degree 3: $\hat{y}(\hat{\beta}) = \hat{\beta}_0 + \hat{\beta}_1 t + \hat{\beta}_2 t^2 + \hat{\beta}_3 t^3$ for the sample data taken from [18] given in Table 1

Table 1: the set of Statistical data for experiments

T	y
2	[1.5,2.5]
-1	[-1.5,-0.5]
0	[-0.5,0.5]
1	[3.5,4.5]
2	[3.5,4.5]
3	[5.5,6.5]
4	[8.5,9.5]
5	[6.5,7.5]
6	[10.5,11.5]
7	[10.5,11.5]
8	[9.5,10.5]

In floating point arithmetic we have the result using MATLAB 2007

$\hat{\beta} = (0.4465, 1.5357, 0.1595, -0.0246)^T$. If we use the Oettli-Prager formula given in equation 3.4, as well as Modified Certylss Cholesky method inclusion intervals were obtained as showed in Tables 2-3.

Table 2: Results for original set of data from Table 1

Modified certylss relaxed Cholesky method	Oettli-Prager formula r
Values of interval $[\hat{\beta}]$	Values of interval $[\hat{\beta}]$

[0.4465, 0.4465]	[-0.0535, 0.9465]
[1.5357, 1.5357]	[1.5357, 1.5357]
[0.1595, 0.1595]	[0.1595, 0.1595]
[-0.0246, -0.0246]	[-0.0246, -0.0246]

176

177 Table 3: Results for the variable t with 0.5 tolerance from Table 1

Modified Certylss relaxed Cholesky method in midpoint radius vector	Oetlli-Prager formula
[0.4465, 0.3977×10^{-14}]	[0.1416, 0.7033]
[1.5357, -0.0859×10^{-14}]	[1.4623, 1.6311]
[0.1595, -0.0161×10^{-14}]	[0.1053, 0.2271]
[-0.0246, 0.0028×10^{-14}]	[-0.0285, -0.0196]

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180 4. Conclusion

181 We have discussed the core problem error in variables for the Statistical Least squares using Oetlli-
 182 Prager formula and the modified Certylss Cholesky decomposition via Mobius transformation of ball
 183 arithmetic. The procedure was tested using polynomial of degree 3 for the interval data set where
 184 the parameters appearing as coefficients were obtained with high degree accuracy, as can be seen in
 185 Tables 2 and 3. The two methods performed optimally fast to the solution but with modified Certylss
 186 Cholesky decomposition preferred to Oetlli-Prager formula. This will provide further insight in the
 187 study of Least squares using circular interval arithmetic.

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