

Research Paper

Modeling and parameter sensitivity analysis of population dynamics in Uganda

ABSTRACT

Keeping track of the human population is essential for various reasons. In this work mathematical modeling and simulations are used to study the population dynamics of Uganda. The models are used for prediction of the country's population dynamics. Parameter sensitivity analysis shows huge influence of variations of the model parameters. The results show that the difference between the per capita birth and death rate parameters is crucial for changes in the country's population. Such findings can also be analogously applied to countries with a similar population structure and economy.

Keywords: population growth, parameter sensitivity analysis, modeling, birth rate, death rate.

1. INTRODUCTION

Currently Uganda has one of the highest population growth rates in the world. Already the provision of essential social (and basic and needs) services such as water, housing, education, hospitals is a challenge to not only the developed countries but even more so to the developing countries such as Uganda. It is often poorly understood how changes in certain crucial parameters that influence the total population of people in a country. Gaining insight into how small and large variations in parameters impact on the national population is useful not only for the government planning but also for ensuring that in the event of sudden outbreak of epidemics, the communities are well prepared to cope.

One of the tools that have been effectively used in tackling such challenges is mathematical and statistical modeling and simulation. However, this process requires good quality data that is collected from population census. Such data acquisition processes are very expensive and time-consuming. Many third world countries are not equipped both financially and technologically to cope with such challenges alongside the already existing ones. Therefore, by using existing data from previous studies or information available from government archives, mathematical simulations can be used to make predictions into the future. We live in a universe of limited resources. With the ever-increasing demand for food and fuels, it has become very crucial for every community to plan for its population.

In this paper mathematical simulations are performed using prior knowledge and existing knowledge on parameters that are considered crucial for understanding the population dynamics of a given population. In literature, little attention has been devoted to unraveling the complexity in the dynamics of population growth models with for a third world country such as Uganda, e.g. [1]. First, an analytic assessment of the model is given. This is then followed by an assessment of the implications of the results from the computer simulation. Already, the country's population has nearly doubled in just about two decades. Such a rapid population growth can be argued to be essential for the country's economic growth. Others might argue it is a big challenge for the country to cope with its high birth rate which by 2006 was reported to be at about 3.1% and the total birth rate of about 6 to 7 children per woman.

2. METHODOLOGY

2.1 Population growth models

The logistic population growth model has for long been used for making prediction and studying the population dynamics of different species of animals (see e.g. [2][3]. In some cases they are applied to a specific population. Consider the model for the population growth as described by Wali et al. [1] which is given as:

$$\frac{d}{dt}P(t) = \alpha P(t) \left(1 - \frac{\beta}{\alpha} P(t)\right); P(0) > 0 \quad (1)$$

where $P(t)$ is the population in the country at any given time t . Here α and β are referred to as the vital parameters of the population; α is often referred to as the Malthusian parameter, and it is basically the difference between the per capita birth and death rate in a given population. The parameter β is the amount by which the difference in per capita birth and death rate changes in response to the addition of one individual to the population. It is important to note that the logistic population growth model is a modification of the Malthusian model – which is in principle a simple exponential growth model.

Let the population at some arbitrary defined starting time zero be $P(0)$. Let the parameters be denoted as $\vartheta_r \in \{\alpha, \beta\}$. The differential equation given in eq. (1) has a solution:

$$P(t) = \frac{\beta}{\alpha} \left(P(0) / \left(P(0) + \left(\frac{\beta}{\alpha} - P(0) \right) e^{-\alpha t} \right) \right) \quad (2)$$

In the limiting time ($t \rightarrow \infty$), the population carrying capacity is obtained as $P_{\max} = P(\infty) = \alpha/\beta$ for $\beta \neq 0$. At steady state $d\tilde{P}/dt = 0$, where $\tilde{P}$ is the steady state population value. Clearly, $\tilde{P}(0) \gg 0$ (i.e. $\tilde{P}(0) \neq 0$), which implies that $\tilde{P} = \alpha/\beta$. Clearly, if $\alpha \gg \beta$ then the maximum population attained in a far future time becomes extremely large. This would imply too many people sharing limited resources. In the above model formulation the (per capita) birth and death parameters are coalesced together into a single parameter. In principle the value of this parameter can vary depending on changes in demographics and social economic factors in any given population.

2.2 Parameter sensitivity analysis

To study the sensitivity of the parameter in the population growth model, the expression in eq. (1), is differentiated partially with respect to each parameter in the model. This leads to a local sensitivity function that generally takes the form:

$$dP|_{\vartheta_r}(t)/dt = (\partial\phi/\partial P)P|_{\vartheta_r}(t) + (\partial\phi/\partial\vartheta_r) \quad (3)$$

where ϕ represents the right hand side of eq. (1) and $\partial\phi/\partial P$ is a diagonal system Jacobian matrix. The theory and details of applications of sensitivity analysis can be found in the work of Gunawan et al. [4]. The sensitivity functions are obtained by computing the partial derivatives with respect to each of the parameters in a given system, in this case eq. (1). The time-dependent parameter sensitivities are computed using the expressions given in eqs. (1) and (3) for each parameter,. The resulting sensitivities are given by the expressions:

$$dS_{\alpha}(t)/dt = (\alpha - 2\beta P)S_{\alpha}(t) + P \quad (4)$$

$$dS_{\beta}(t)/dt = (\alpha - 2\beta P)S_{\beta}(t) - P^2 \quad (5)$$

with the initial conditions satisfying the condition that $S_{\alpha}(0) = S_{\beta}(0) = 0$. The eqs. (4) and (5) are then solved to obtain the sensitivity functions, i.e. $S_{\alpha}(t)$ and $S_{\beta}(t)$.

2.3 Evaluation of parameter sensitivities

Since the parameters and state variables of a given population model may take on a wide range of values, it is important to assess how these parameters influence the population. The ranking metric is computed from the area under the parameter sensitivity function using:

$$\Phi_{\theta_r} = \int_{t_0}^{t_f} \|S_{\theta_r}(t)\|_1 dt \quad (6)$$

where $t(0)$ and $t(f)$ are the start and final time instants. This integral sensitivity metric is based on time integrals that are analogous to the criterion described in [5]. The metric specified in eq. (6) was used to compute the parameter sensitivities from the expressions given in eq. (4) and (5). All the analysis and simulation in this paper was performed in the MATLAB® Version 7.12.0 (R2011a) software under the Windows®7 environment.

3. RESULTS AND DISCUSSION

In the simulation, we used nominal parameter values of $\hat{\alpha} = \bar{\alpha} \approx 0.0356$ and $\hat{\beta} = \bar{\beta} \approx 1.20569e - 10$, respectively. In this case the nominal values are considered to be equal to the estimated parameter values from [1]. The simulations also yielded the value $\Phi_{\alpha} \approx 8.061e + 12$ and $\Phi_{\beta} \approx 0$, respectively. By considering the population data provided in Table 1 (Appendix), simulations were performed and the results are provided in Figure 1. In the simulation the point $t(0)$ is considered to be at 1980. It is important to note that the point of inflection, which was computed to be at $t^* = 88$ years (at the year 2068, see [1]; with a small change of notation). This means that in less than a century half of the resources that can support the population will have been consumed. The population is sensitive to a change in α (Figure 1). This implies that a drastic or sudden variation in the difference between the birth and death rates is likely to result in a rapid population change (Figure 1: C).

As indicated in the results, assuming the population dynamics and demography continues in the projected path, then it is very likely that around in about 120 years into the future (counting from 1980, the time for the $P(0)$), there is likely to be a significant influence on the demography. Off course these predictions are all based on the assumption that little will change in the rates of parameters such as immigration and emigration. This calls for perhaps even better models to that with the aid of demographic data in the future can help improve the predictions. This will aid improved planning for social and economic services for the population. Compared to decades ago, today Ugandan has a longer life expectancy and lower death rates (death as a result of all atrocities combined) – improved health care and education. This simply ensures that α increases and so does the overall population.

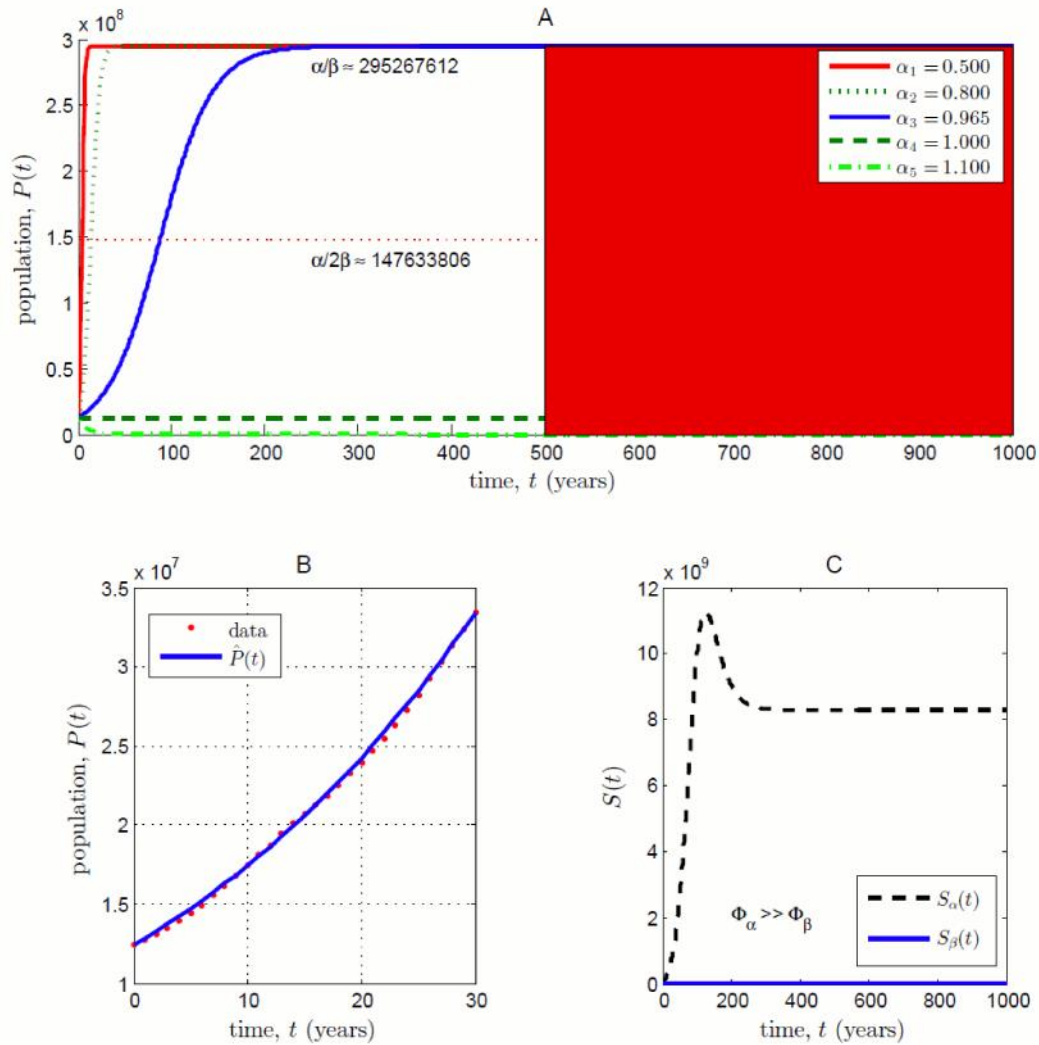


Figure 1. Population growth and results of parameter sensitivity analysis. A: projected population growth. The shaded region indicates the time frame at which the population carrying capacity is reached (assuming the population keeps growing). B: Plot showing the normalized parameter sensitivities. C: The change in sensitivities varies for α and β .

It is recommendable that for more accurate predictions the parameters α and β have to be estimated perhaps every decade. This is because of variations that can arise from, e.g. inaccurate accountability of the number of people in the country Uganda, porous borders, non-registered and non-counted individuals and those entering the country illegally. All these are factors that can largely influence mathematical predictions, thereby, hugely biasing the resulting inferences and recommendations that arise from such analysis.

4. CONCLUSION

The results in this work have demonstrated the impact of changes in parameters on the overall population of a community – as illustrated for the case of Uganda. Changes in such parameters need to be closely monitored by ensuring that quality demographic data is collected and used for better predictions and planning to improve the livelihoods of the

citizens. Although not considered or discussed here, in real life there are surely other factors that influence on the population dynamics. Results from the mathematical modeling and simulations based on prior knowledge and demographic data is useful especially in cases when population growth needs to be accurately predicted for proper planning purposes.

ACKNOWLEDGEMENTS

The study had no other sponsors, i.e. it was self-sponsored.

COMPETING INTERESTS

The author declares no competing interests.

AUTHORS' CONTRIBUTIONS

The author conceived the idea, modeling and analysis, read and approved the manuscript.

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APPENDIX

Table 1. Actual values of population as specified in the manuscript by Wali et al. [1].

Year	Actual Population	Year	Actual Population	Year	Actual Population
1980	12414719	1991	18082137	2002	25469579
1981	12725252	1992	18729453	2003	26321962
1982	13078930	1993	19424376	2004	27233661
1983	13470393	1994	20127590	2005	28199390
1984	13919514	1995	20689516	2006	29206503
1985	14391743	1996	21248718	2007	30262610
1986	14910724	1997	21861011	2008	31367972
1987	15520093	1998	22502140	2009	32369558
1988	16176418	1999	23227669	2010	33398682
1989	16832384	2000	23955822		
1990	17455758	2001	24690002		