

On the Existence and Nonexistence of Positive Solutions for Singular Quasilinear Elliptic Equations with Gradient Terms¹

Abstract

In this paper, we consider the positive solutions of quasilinear elliptic equations of the form

$$\begin{cases} -\Delta_p u = a(\delta(x))g(u) + f(x, u) + \lambda|\nabla u|^q & \text{in } \Omega \subset \mathbf{R}^N, N \geq 2, \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where Ω is a smooth and bounded domain, g is an unbounded decreasing function, a is positive and continuous, f grows at most linearly at infinity, $\delta(x) = \text{dist}(x, \partial\Omega)$, $p \geq 2, p - 2 < q \leq p$, $\lambda \in R$. We emphasis the effect of all these terms in study of existence.

Keywords: Gradient terms; Quasilinear elliptic equation; Singular; Existence and Nonexistence

Mathematics Subject Classification 35J15, 35J75.

1 Introduction

In this paper, we are concerned with quasilinear elliptic problems in the from

$$\begin{cases} -\Delta_p u = a(\delta(x))g(u) + f(x, u) + \lambda|\nabla u|^q, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

Where $\Omega \subset \mathbf{R}^N, N \geq 2, \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, where Ω is a smooth and bounded domain, $\delta(x) = \text{dist}(x, \partial\Omega)$ which is assumed to be nonincreasing, nonnegative and Hölder continuous, $p \geq 2, p - 2 < q \leq p$, $\lambda \in R$. For the convenience, we denote $h(x, u, \nabla u) = a(\delta(x))g(u) + f(x, u) + \lambda|\nabla u|^q$, and $h(x, u, \nabla u)$ is nonincreasing respect to u .

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We assume that $g \in C^1(0, \infty)$ is a positive decreasing function and

$$(g1) \lim_{t \rightarrow 0^+} g(t) = \infty$$

The function $f : \bar{B}_R \times [0, \infty) \rightarrow [0, \infty)$ is *Hölder* continuous, nondecreasing with respect to the second variable and f fulfills the hypotheses:

$$(f1) \text{ the mapping } (0, \infty) \ni t \mapsto \frac{f(x, t)}{t^{p-1}} \text{ is nonincreasing for all } x \in \bar{B}_R;$$

$$(f2) \lim_{t \rightarrow 0^+} \frac{f(x, t)}{t^{p-1}} = \infty \text{ and } \lim_{t \rightarrow \infty} \frac{f(x, t)}{t^{p-1}} = 0, \text{ uniformly for } x \in \bar{B}_R.$$

Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluid theory([21]), non-Newtonian filtration([22]) and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, then they are Newtonian fluids.

The main features of the paper are the presence of the convection term $|\nabla u|^{p-1+q}$ combined with the singular weight $a : (0, \infty) \rightarrow (0, \infty)$ which is assumed to be nonincreasing and *Hölder* continuous.

Many papers have been devoted to the case $a \equiv 1$ or $\lambda = 0$ (see [1-7, 9, 10, 14-17, 19, 20] and the references therein). In [1], the author considered the following problem

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) + q(x)u^{-\gamma} = 0, \quad x \in \mathbf{R}^N, \quad (1.2)$$

has a positive entire solution if $1 < p < N$, $0 \leq \gamma < p - 1$, and $q(x) \in C(\mathbf{R}_+)$ satisfy some suitable conditions.

In [2], the author studied the existence of the positive solutions of the equation

$$\begin{cases} -\Delta_p u = \lambda f(x, u), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbf{R}^N$ is a $C^{1,\omega}$ bounded domain, for some $0 < \omega < 1$, $f : \Omega \times (0, \infty) \rightarrow [0, \infty)$ is a suitable function and allowed to be singular, $\lambda > 0$.

In [10], the existence and uniqueness of positive solutions of the following quasilinear singular equations

$$\begin{cases} -\Delta_p u = \rho(x)f(u), & \text{in } \mathbf{R}^N \\ u > 0, & \text{in } \mathbf{R}^N \\ \lim_{|x| \rightarrow \infty} u = 0. \end{cases} \quad (1.4)$$

and

$$\begin{cases} -\Delta_p u = \rho(x)f(u), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

are considered

One of the works in the literature dealing with singular weights in connection with singular nonlinearities is due to [3], [5]. In [3], [5], the following problem has been considered

$$\begin{cases} -(\varphi_p(u'(t)))' = a(t)g(u(t)), & \text{in } (0, 1) \\ u > 0, & \text{in } (0, 1) \\ u(0) = u(1) = 0 \end{cases} \quad (1.6)$$

where g satisfies (g1) and a is positive and continuous on $(0, 1)$. It is shown that if

$$\int_0^\delta \varphi_p^{-1}\left(\int_s^\delta a(\tau)d\tau\right)ds + \int_\delta^1 \varphi_p^{-1}\left(\int_\delta^s a(\tau)d\tau\right)ds < \infty, \quad (1.7)$$

where $0 < \delta < 1$, then (1.6) may be a positive one classical solution. In our framework, g is continuous at $t = 1$, so (1.7) reads as

$$\int_0^1 \varphi_p^{-1}\left(\int_s^1 a(\tau)d\tau\right)ds < \infty, \quad (1.8)$$

where $\varphi_p^{-1}(t) = |t|^{\frac{1}{p-1}}$. In this paper we prove that the assumption (1.8) is also necessary for (1.1) to have solutions.

When $p = 2$, there is a vast literature on stable solutions to the equation (1.1), we refer to [11, 18, 23-27] and references therein. In particular, Marius Ghergu and Taliaferro S.D. [11, 23] dealing with singular weights in connection with singular nonlinearities. The present paper is an part extension of [11, 23] to the p -Laplacian equation. Our main ideas com from the paper [11, 23].

Theorem 1.(Nonexistence) Assume $\int_0^1 \varphi_p^{-1}\left(\int_s^1 a(\tau)d\tau\right)ds = \infty$. Then (1.1) has no solutions.

Theorem 2. Assume (f1), (f2), $q = p - 1$, and

$$\int_0^1 \varphi_p^{-1}\left(\int_s^1 a(\tau)d\tau\right)ds < \infty, \quad \Omega = B_R(0)$$

for some $R > 0$, if g satisfies

(g2): there exist $C > 0$, such that

$$a(\delta(x))g(u) > C > 0, \quad (1.9)$$

has a solution $\omega \in C_{loc}^{1+\alpha}(B_R(0))$.

Then the problem (1.1) has at least one solution for all $\lambda < 0$.

2 Preliminary

Now we introduce a sub-supersolution method with the boundary restriction.

Definition 2.1. A function $\underline{u} \in C^{1+\alpha}(\Omega) \cap C(\bar{\Omega})$ is called a subsolution of problem (2.3) if

$$-\operatorname{div}(|\nabla \underline{u}|^{p-2} \nabla \underline{u}) \leq h(x, \underline{u}, \nabla \underline{u}), \quad \underline{u} > 0, x \in \Omega, \underline{u} = 0, x \in \partial\Omega.$$

A function $\bar{u} \in C^{1+\alpha}(\Omega) \cap C(\bar{\Omega})$ is called a supersolution of problem (2.3) if

$$-\operatorname{div}(|\nabla \bar{u}|^{p-2} \nabla \bar{u}) \geq h(x, \bar{u}, \nabla \bar{u}), \quad \bar{u} > 0, x \in \Omega, \bar{u} = 0, x \in \partial\Omega.$$

Lemma 2.1. ([8, Theorem 9.5.])

$$Q(u, \phi) = \int_{\Omega} (A(x, u, \nabla u)) \cdot \nabla \phi - B(x, u, \nabla u) \phi dx.$$

for all non-negative $\phi \in C_0^1(\Omega)$. Let $u, v \in C^1(\bar{\Omega})$ satisfy $Qu \geq 0$ in Ω , $Qv \leq 0$ in Ω and $u \leq v$ on $\partial\Omega$, where the functions A, B are continuously differentiable with respect to the z, p variables in $\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N$, the operator Q is elliptic in Ω , and the function B is non-increasing in z for fixed $(x, p) \in \Omega \times \mathbf{R}^N$. Then, if either

- (i) the vector function A is independent of z ; or
- (ii) the function B is independent of p .

It follows that $u \leq v$ in Ω .

lemma 2.2. ([12, Lemma 2.2]) Let $h(x, u, \xi)$ satisfy the following two basic conditions:

(A) $h(x, u, \xi)$ is locally Holder continuous function in $\Omega \times \mathbf{R}^+ \times \mathbf{R}^N$ and continuously differentiable with respect to the variables u and ξ ;

(B) For every bounded domain $\Omega_1 \subset \subset \Omega$, for any $M > 0$, $\exists \rho(\Omega, M) > 0$, such that

$$|h(x, u, \xi)| < \rho(\Omega, M)(1 + |\xi|^p), x \in \Omega, 0 \leq u \leq M, \xi \in \mathbf{R}^N.$$

If problem (1.1) has a supersolution $\bar{u} \in C^1(\Omega)$ and a subsolution $\underline{u} \in C^1(\Omega)$ such that $\underline{u} \leq \bar{u}$ in Ω , the problem (1.1) has at least one solution $u(x) \in C^1(\Omega)$ with $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$.

3 Proof of Theorem 1.

The proof of Theorem 1 follows from the following more general result.

Proposition 1. Assume $\int_0^1 \varphi_p^{-1}(\int_s^1 a(\tau) d\tau) ds = \infty$. Then the inequality boundary problem

$$\begin{cases} -\Delta_p u + \lambda(p-1)|\nabla u|^p \geq a(\delta(x))g(u), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

(3.1) has no classical solutions.

Proof. Let (λ_1, φ_1) be the first eigenvalue and eigenfunction of $-\Delta_p$ in Ω subject to a homogenous Dirichlet boundary condition. It is known that λ_1 and by normalization, one can assume φ_1 in Ω . It suffices to prove the result only for $\lambda > 0$. We argue by contradiction and assume that there exists $u \in C^{1+\alpha}(\Omega) \cap C(\bar{\Omega})$ a solution of (3.1). Using (g1), we can find $C_1 > 0$ such $\underline{u} := c_1 \varphi_1^{p-1}$ verifies

$$-\Delta_p u + \lambda(p-1)|\nabla u|^p \leq a(\delta(x))g(u), \text{ in } \Omega$$

. Since comparison principle, we easily obtain

$$u \geq \underline{u}, \text{ in } \Omega \quad (3.2)$$

We make in (3.1) the change of variable $v = 1 - e^{-\lambda u}$. Therefore

$$\begin{cases} -\Delta_p v = \lambda^{(p-1)}(1-v)^{p-1}(\lambda(p-1)|\nabla u|^p - \Delta_p u) \geq \lambda^{(p-1)}(1-v)^{p-1}a(\delta(x))g\left(\frac{\ln(1-v)}{-\lambda}\right), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.3)$$

In order to avoid the singularities in (3.3), let us consider the approximated problem

$$\begin{cases} -\Delta_p v = \lambda^{(p-1)}(1-v)^{p-1}a(\delta(x))g\left(\epsilon - \frac{\ln(1-v)}{\lambda}\right), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.4)$$

with $0 < \epsilon < 1$, clearly v is a super-solution of (3.4). By (3.2) and the fact that $\lim_{t \rightarrow 0^+} \frac{1-e^{-\lambda t}}{t} = \lambda > 0$, there exist $c_2 > 0$ such that $v \geq c_2 \varphi^{p-1}$ in Ω . On the other hand, there exist $0 < c < c_2$ such that $c\varphi^{p-1}$ is a sub-solution of (3.4) and obviously $c\varphi^{p-1} \leq v$ in Ω . Then, by the standard sub- and sup-solution method, the problem (3.4) has a solution $v_\epsilon \in C^{1+\alpha}(\bar{\Omega})$ such that

$$c\varphi^{p-1} \leq v_\epsilon \leq v, \text{ in } \Omega \quad (3.5)$$

Then

$$v_\epsilon = G_p^1(\lambda^{(p-1)}(1-v_\epsilon)^{p-1}a(\delta(x))g\left(\epsilon - \frac{\ln(1-v_\epsilon)}{\lambda}\right))$$

where G_p^1 is the inverse operator of $A_p^1 = -\Delta_p$ under the Dirichlet boundary condition. So,

$$\int_{\Omega} v_\epsilon dx = \int_{\Omega} G_p^1(\lambda^{(p-1)}(1-v_\epsilon)^{p-1}a(\delta(x))g\left(-\frac{\ln(1-v_\epsilon)}{\lambda}\right)) dx$$

Using (3.5), we obtain

$$\begin{aligned} M &= \int_{\Omega} v dx \\ &\geq \int_{\Omega} G_p^1(\lambda^{(p-1)}(1-v)^{p-1}a(\delta(x))g(-\frac{\ln(1-v)}{\lambda}))dx \\ &\geq C \int_{\Omega_{\delta}} G_p^1(a(\delta(x)))dx \end{aligned}$$

where $\Omega_{\delta} \supset \{x \in \Omega; \delta(x) < \delta\}$, for some $\delta > 0$ sufficient small. Therefore

$$\int_0^1 \varphi_p^{-1}(\int_s^1 a(\tau)d\tau)ds = \infty,$$

by the above inequality, we find a contradiction. Hence, (3.1) has no classical solutions and the proof is now complete.

4 Proof of Theorem 2.

Let us note first that in our setting problem (1.1) reads

$$\begin{cases} -\Delta_p u = a(R - |x|)g(u) + f(x, u) + \lambda|\nabla u|^{p-1}, & \text{in } |x| < R \\ u > 0, & \text{in } |x| < R \\ u = 0, & \text{on } |x| = R \end{cases} \quad (4.1)$$

In order to provide a super-solution of (4.1), we consider the problem

$$\begin{cases} -\Delta_p u = a(R - |x|)g(u) + 1 + \lambda|\nabla u|^{p-1}, & \text{in } |x| < R \\ u > 0, & \text{in } |x| < R \\ u = 0, & \text{on } |x| = R \end{cases} \quad (4.2)$$

Lemma 4.1. Assume (g1), problem (4.2) has at least one solution.

Proof. We are looking for radially decreasingly symmetric solution of (4.2), that is $u = u(r)$, $0 \leq r = |x| \leq R$. In this case, problem (4.2) becomes

$$\begin{cases} -[(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u'] = a(R - |x|)g(u(r)) + 1 + \lambda|u'|^{p-1}, & \text{in } |x| < R \\ u(r) > 0, & \text{in } |x| < R \\ u(r) = 0, & \text{on } |x| = R \end{cases} \quad (4.4)$$

Since $u(r)$ is radial decreasingly, that is $u'(r) \leq 0$, then (4.4) gives

$$-[(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' - \lambda|u'|^{p-2}u'] = a(R - |x|)g(u(r)) + 1, 0 \leq r < R.$$

We obtain

$$-(e^{-\lambda r} r^{N-1} |u'|^{p-2} u')' = e^{-\lambda r} r^{N-1} \psi(r, u(r)), 0 \leq r < R, \quad (4.5)$$

where

$$\psi(r, u(r)) = a(R - |x|)g(u(r)) + 1, (r, t) \in [0, R) \times (0, \infty)$$

From (4.5) we obtain

$$u(r) = u(0) - \int_0^r [e^{\lambda t} t^{1-N} \int_0^t e^{-\lambda s} s^{N-1} \psi(s, u(s)) ds]^{\frac{1}{p-1}} dt, 0 \leq r < R. \quad (4.6)$$

On the other hand, due to [6] and the symmetry of the domain, there exist a solution $\omega = \omega(r) \in C^{1+\alpha}(B_R(0)) \cap C(\bar{B}_R(0))$ of the problem

$$\begin{cases} -\Delta_p \omega = a(R - |x|)g(\omega) + 1, & \text{in } |x| < R \\ \omega > 0, & \text{in } |x| < R \\ \omega = 0, & \text{on } |x| = R \end{cases} \quad (4.7)$$

Clearly, ω is a sub-solution of (4.2). As above we get

$$\omega(r) = \omega(0) - \int_0^r [t^{1-N} \int_0^t s^{N-1} \psi(s, \omega(s)) ds]^{\frac{1}{p-1}} dt, 0 \leq r < R. \quad (4.8)$$

We claim that there exists a solution $v \in C^{1+\alpha}(B_R(0)) \cap C(\bar{B}_R(0))$ of (4.6) such that $v > 0$ in $[0, R)$.

Let $A = \omega(0)$ and define the sequence $(v_k)_{k \geq 0}$ by

$$\begin{cases} v_k(r) = A - \int_0^r [e^{\lambda t} t^{1-N} \int_0^t e^{-\lambda s} s^{N-1} \psi(s, v_{k-1}(s)) ds]^{\frac{1}{p-1}} dt, & 0 \leq r < R \\ v_0(r) = \omega \end{cases} \quad (4.9)$$

Note that $v_k(r)$ is decreasing in $[0, R)$ for all $k \geq 0$ and by induction we reduce $v_k(r) \geq v_{k-1}(r)$ for all $k \geq 1$. Hence

$$\omega = v_0 \leq v_1 \leq v_2 \leq \dots \leq v_k \leq \dots \leq A, \text{ in } B_R(0).$$

Thus, there exists $v(r) := \lim_{k \rightarrow \infty} v_k(r)$, for all $0 \leq r < R$ and $v > 0$ in $[0, R)$. We now can pass to the limit in (4.9). By classical regularity results we also obtain $v \in C^{1+\alpha}([0, R)) \cap C([0, R])$. This proves the claim and the proof of Lemma 4.1 is completed.

Proof of Theorem 2. In order to provide a subsolution of (4.1), we consider the following problem

$$\begin{cases} -\Delta_p u = a(R - |x|)g(u) + \lambda |\nabla u|^{p-1}, & \text{in } |x| < R \\ u > 0, & \text{in } |x| < R \\ u = 0, & \text{on } |x| = R \end{cases} \quad (4.10)$$

It is easy to see that the solution of (4.10) is the subsolution of (4.1), so next we are looking for the solution of (4.10).

In [9], it is easy to see that there exist $\bar{u}_1 \in C^{1+\alpha}(B_R) \cap C(\bar{B}_R)$ such that

$$\begin{cases} -\Delta_p u = a(R - |x|)g(u), & \text{in } |x| < R \\ u > 0, & \text{in } |x| < R \\ u = 0, & \text{on } |x| = R \end{cases} \quad (4.11)$$

It is obvious that $\bar{u}_1$ is a supsolution of (4.10) for all $\lambda < 0$

Since (g2), there exists $\underline{u}_1$ such that

$$\begin{cases} -\Delta_p u = C + \lambda|\nabla u|^{p-1}, & \text{in } B_R(0) \\ u > 0, & \text{in } B_R(0) \\ u = 0, & \text{on } \partial B_R(0) \end{cases} \quad (4.12)$$

It is obvious that $\underline{u}_1 > 0$ is a sub-solution of (4.1) for all $\lambda < 0$. Since comparison principle, we easily obtain

$$\bar{u}_1 > \underline{u}_1$$

Then, by the standard sub- and sup-solution method, the problem (4.10) has a solution $\underline{u}$ and $\bar{u}_1 > \underline{u} > \underline{u}_1$. It follows that $\underline{u}$ is a super-solution of (4.1).

Let u be a solution of (4.4). For $M > 1$ we have

$$\begin{aligned} -\Delta_p(Mu) &= M^{p-1}a(R - |x|)g(u) + M^{p-1} + \lambda|\nabla(Mu)|^{p-1} \\ &\geq a(R - |x|)g(Mu) + M^{p-1} + \lambda|\nabla(Mu)|^{p-1} \end{aligned}$$

Since (f_1) , we can choose $M > 1$ such that

$$M^{p-1} \geq f(x, M|u|_\infty), \text{ in } B_R(0)$$

Then $\bar{u}_\lambda := Mu$ satisfies

$$-\Delta_p(\bar{u}_\lambda) \geq a(R - |x|)g(\bar{u}_\lambda) + f(x, \bar{u}_\lambda) + \lambda|\nabla(\bar{u}_\lambda)|^{p-1}$$

It follows that $\bar{u}_\lambda$ is a super-solution of (4.1). Since Lemma 2.1, we know $\underline{u} \leq \bar{u}_\lambda$ in $B_R(0)$. So (4.1) has at least one solution. Hence the proof of Theorem 2 is completed.

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