

Power Transformations and Unit Mean and Constant Variance Assumptions of the Error Component of a Multiplicative Error Model: The Generalized Gamma Distribution

ABSTRACT

In this study, we investigated the implication of power transformations namely, inverse-square-root, inverse, inverse-square and square transformations on the unit-mean and constant variance assumptions of the error component of the multiplicative error model. The distributions of the error component studied were the various forms of the generalized gamma distribution namely Gamma (a, b, 1), Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributions. The purpose of the study is to investigate whether the unit-mean and constant variance assumptions necessary for modeling using the multiplicative error model are either violated or retained after the various power transformations. Firstly, the functions describing distributional characteristics of interest for the generalized power transformed error component were established and secondly the unit-mean conditions of the untransformed distributions were applied to the established functions with a view to studying their impacts on the transformed distribution. From the results of the study, the following were discovered; (i) For the inverse-square-root transformation, the unit-mean and constant variance assumptions are approximately maintained for all the distributions under study except the Chi-square distribution where the unit mean assumption was violated. (ii) For the inverse transformation, the unit-mean assumptions are violated after transformation except for the Rayleigh and Maxwell distributions. (iii) For the square transformation, the unit-mean assumption is violated for all the distributions under study. (iv) For the square transformation, it is only the Maxwell distribution that maintained the unit-mean assumption. (v) For all the studied transformations the variances of the transformed distributions were found to be greater than those of the untransformed distribution. Finally, based on the results of this study, the following recommendations were made; (i) Inverse-square-root transformation is appropriate for Gamma (a, b, 1), where $a = b = 1$, Exponential, Weibull, Rayleigh and Maxwell distributed error components. (ii) Inverse transformation is appropriate for a data set whose error component belongs to a Rayleigh and Maxwell distributed distribution. (iii) Inverse square transformation is not appropriate for a data set whose error component belong to the six studied distributions. (iv) Square transformation is only appropriate for a data set whose error component belongs to a Maxwell distribution.

Key words: Error Component; Mean; Multiplicative Error Model; Power Transformation; Variance

1. INTRODUCTION

Given that a multiplicative error model is given as

$$X_{t,t \in N} = \Psi(X_{t-1})\xi_t \quad (1)$$

where $X_{t,t \in N}$ is a real-valued stochastic process defined on $[0, +\infty)$, $\Psi(X_{t-1})$, the information available for forecasting $X_{t,t \in N}$ and ξ_t , a random variable defined over a $[0, +\infty)$ support with unit mean and unknown constant variance, σ_1^2 . That is

$$\xi_t \sim V_1^+(1, \sigma_1^2) \quad (2)$$

In principle the distribution of ξ_t in (1) can be specified by means of any probability density function (pdf) having the characteristics in (2). Examples are Gamma, Log-Normal, Weibull, and mixtures of them (Brownlees et al., (2011)). Engle and Gallo (2006) favor a Gamma (ϕ, ϕ) (which implies $\sigma^2 = 1/\phi$); Bauwens and Giot (2000), in ACD framework considered a Weibull $\Gamma((1+\phi)^{-1}, \phi)$ (in this case, $\sigma^2 = \Gamma(1+2\phi)/\Gamma((1+\phi)^2 - 1)$). As a result of the above suggested specifications, the error component ξ_t would be generally studied under the generalized gamma distributions (GGD) given by

$$f(\xi) = \frac{a c (a \xi)^{bc-1} e^{-(a \xi)^c}}{\Gamma(b)}, \quad \xi > 0 \quad (3)$$

where a (a shape parameter) and b are real numbers. c can in principle take any real value but normally we consider the case where $c \geq 0$. The reason of using the GGD as the study distribution is because the various distributional forms (The 3-parameter gamma, Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributions) of the GGD for various values of a , b and c , have the distributional characteristics given in (2). The distributional forms of (3) for various values of a , b and c are given in Table 1. For more details on the generalized gamma distribution, see Stacy (1962).

In statistical modeling, the familiar application of the normal linear model involves a response variable that is assumed normally distributed with constant variance. In other applications a response variable may occur in a form that suppresses an underlying normal linear structure (Fraser (1967)). Sometimes in these applications the context may suggest a logarithm or inverse or square root transformation and so on, which reveals the normal linear form. Transformation may also be necessary to either stabilize the variance component of a model or to normalize it. Details on the reasons for transformations are found in; Box and Cox (1964); Iwueze et al., (2011).

Many important results in statistical analysis follow from the assumption that the population being sampled or investigated is normally distributed with a common variance and additive error structure. For the multiplicative error model, the assumptions of interest are unit mean and constant variance. When the relevant theoretical assumptions relating to a selected method of analysis are approximately satisfied, the usual procedures can be applied in order to make inferences about unknown parameters of interest. In situations where the assumptions are seriously violated several options are available (Sakia (1992)): (i) Ignore the violation of the assumptions and proceed with the analysis as if all assumptions are satisfied. (ii) Decide what is the correct assumption in place of the one that is violated and use a valid procedure that takes into account the new assumption. (iii) Design a new model that has important aspects of the original model and satisfies all the assumptions, e.g. by applying a proper transformation to the data or

filtering out some suspect data point which may be considered outlying. (iv) Use a distribution-free procedure that is valid even if various assumptions are violated. For more details on the above listed options, see Graybill (1976, p. 213).

Most researchers, however, have opted for (iii) which has attracted much attention as documented by Thoeni (1967) and Hoyle (1973) among others. In this study our interest would center on transformation as a remedy for situations where the assumptions for parametric data analysis are seriously violated.

Data transformations are the applications of mathematical modifications to the values of a variable. However caution should be exercised in the choice of the type of transformation to be adopted so that the fundamental structure of the series is not distorted and thereby rendering the interpretation very difficult or impossible. There are two major methods of data transformations namely Bartlett and Box and Cox methods of data transformation, however for ease of application we would only consider the Bartlett's techniques.

Bartlett(1947) used the simple relation between mean and standard deviation over several groups. Akpanta and Iwueze (2009) had shown how to apply Bartlett's transformation technique to time series data using the Buys-Ballot table. For details on Buys-Ballot table, see Iwueze et al., (2011). The relationship between variance and mean over several groups is what is needed. If we take random samples from a population, the means and standard deviations of these samples will be independent (and thus uncorrelated) if the population has a normal distribution (Hogg and Craig (1978)). Akpanta and Iwueze (2009) showed that Bartlett's transformation for time series data is to regress the natural logarithms of the group standard deviations $(\hat{\sigma}_i, i=1,2,...,m)$ against the natural logarithms of group means $(\bar{X}_i, i=1,2,...,m)$ and determine the slope, p of the relationship

$$\log_e \hat{\sigma}_i = \alpha + p \log_e \bar{X}_i + \text{error} \quad (4)$$

For non-seasonal data that require transformation, we split the observed time series $X_t, t=1,2,...,n$ chronologically into m fairly equal different parts and compute $(\bar{X}_i, i=1,2,...,m)$ and $(\hat{\sigma}_i, i=1,2,...,m)$ for the parts. For seasonal data with the length of the periodic intervals, s , the Buys-Ballot table naturally partitions the observed data into m periods or rows for easy application. Akpanta and Iwueze (2009) also showed that Bartlett's transformation may also be regarded as the power transformation

$$\begin{cases} \log_e X_t, & p=1 \\ X_t^{(1-p)}, & p \neq 1 \end{cases} \quad (5)$$

Summary of transformations for various values of p is given in Table 2.

Studies on the effects of transformation on the error component of the multiplicative error models whose distributional characteristics is given in (2) are not new in the statistical literature. The overall aim of such studies is to establish the conditions for successful transformation. A successful transformation is achieved when the desirable properties of a data set remains unchanged after transformation. These basic properties or assumptions of interest for this study are; (i) Unit mean and (ii) constant variance.

In this end, for the purely multiplicative time series model whose error component in addition to be normally distributed is classified under the characteristics given in (2), Iwueze (2007) investigated the effect of logarithmic transformation on the error component ξ_t where

$\xi_t \sim N(1, \sigma_1^2)$) and discovered that the logarithm transform; $Y = \text{Log } \xi_t$ can be assumed to be normally distributed with mean, zero and the same variance, σ^2 for $\sigma_1 < 0.1$. Similarly Nwosu et al., (2013) and Otuonye (2012) had studied the effects of inverse and square root transformation on the error component of the same model. Nwosu et al., (2013) discovered that the inverse transform $Y = \frac{1}{\xi_t}$ can be assumed to be normally distributed with mean, one and the same variance provided $\sigma \leq 0.10$. Similarly Otuonye (2012) discovered that the square root transform; $Y = \sqrt{\xi_t}$ can be assumed to be normally distributed with unit mean and variance, $4\sigma_1^2$ for $\sigma_1 \leq 0.3$, where σ_1^2 is the variance of the original error component before transformation.

The application of power transformation to model (1) gives

$$X_{t,t \in N}^* = \Psi^*(X_{t-1}) \xi_t^* \quad (6)$$

where $X_{t,t \in N}^* = (X_{t,t \in N})^p = X_{t,t \in N}^p$, $\Psi^*(X_{t-1}) = (\Psi(X_{t-1}))^p = \Psi^p(X_{t-1})$, $\xi_t^* = (\xi_t)^p = \xi_t^p$ and

$$\xi_t^p \sim V_2^+(1, \sigma_2^2), p = -1, \frac{1}{2}, 1, \frac{3}{2}, 2, 3 \quad (7)$$

where ξ_t^p is the generalized power transformed error component of model (1). The most popular and common power transformations are $\log_e X_t, \sqrt{X_t}, 1/X_t, X_t^2$ and $1/X_t^2$. The results of the transformations on model (1) are given in Table 3. The logarithm transformation converts the multiplicative error model (1) to the additive model while the other listed transformations leave the model still multiplicative. For the logarithm transformation

$$Y_t = \log_e X_t = \log_e \Psi(X_{t-1}) + \log_e \xi_t = \Psi^*(X_{t-1}) + \xi_t^* \quad (8)$$

In Table 3 the following notations were adopted

Y_t = Transformed observed series

$\Psi^*(X_{t-1})$ = The Transform of the information available for forecasting X_t

ξ_t^* = Transformed error component

It is clear from Table 3, that only the logarithm transformation alters the assumptions placed on error components of the multiplicative error model and as a result interest would be centered on the transformations that leaves model (1) still multiplicative.

Since model (6) is still a multiplicative error model and therefore ξ_t^* must also be characterized with unit mean and some constant variance, σ_2^2 which may or may not be equal to σ_1^2 . It is on these premises that we want to study the effect of power transformations on a non-normal distributed error component of a multiplicative error model whose distributional characteristics belong to the Generalized Gamma distribution. The purpose is to determine if the assumed fundamental structure (unit mean and constant variance) is maintained after the power transformation and also to investigate what happens to σ_1^2 and σ_2^2 in terms of equality or non-equality. The overall reason for concentrating on the error component of model (6) is as plane as

the nose on the face: the reason is that the assumptions for model analysis are always placed on the error component, ξ_t .

Ohakwe et al. (2012) had studied the implication of square root transformation on the unit mean and constant variance assumptions of the error component of model (1) whose distributional characteristics belong to the Generalized Gamma Distribution under the various forms; Chi-square, Exponential, Gamma (a, b, 1), Weibull, Maxwell and Rayleigh distributions. From the results of the study, the unit mean assumption is approximately maintained for all the given distributional forms of the GGD. However there were reductions in the variances of the distributions except those of the Gamma (a, b, 1), for $a > 1$, Rayleigh and Maxwell that increased, hence they concluded that square-root transformation is not appropriate for multiplicative error model with a Gamma (a, b, 1) for $a > 1$ or Rayleigh or Maxwell distributed error component. Finally, Ohakwe et al., (2012) recommended that square-root transformation, where applicable for a multiplicative error model are successful for the studied distributions if the variance of the transformed error component < 0.5 .

Out of the six popular power transformations namely logarithm, square root, inverse, inverse square root, square and inverse square, only the effect of square root transformation on the error component of the multiplicative error model with regard to unit mean and constant variance had been studied (see Ohakwe et al., (2012)) leaving the others yet to be studied. Whereas the logarithm transformation converts the multiplicative error model to an additive model, the others still leave it multiplicative therefore in this paper we study the effect of the power transformations namely; inverse square root, inverse, inverse square and square on the error component of the multiplicative error model with the overall aim of investigating on what happens to the unit mean and constant variance assumptions after the transformations. This study would be carried out under the generalized gamma distribution considering that all the suggested distributions of the error component of the multiplicative error model (Brownlees et al., (2011); Engle and Gallo(2006);Bauwens and Giot(2000)) are the various forms of the generalized gamma distribution.

This paper is organized as follows; Section One contains the introduction while the distributional characteristics of the generalized power transformed error are contained in Section Two. While the results of the study are in Section Three, the conclusion, references and appendix are respectively contained in Sections Four, Five and Six.

For simplicity, the notation $\xi_t = \xi$ and $Y_t = Y$ would be adopted.

2 DISTRIBUTIONAL CHARACTERISTICS OF THE GENERALIZED POWER TRANSFORMED ERROR COMPONENT

Suppose the distributional characteristics of ξ belongs to the generalized gamma distribution whose probability density function denoted as $f(\xi)$ is (3), let

$$y = \xi^p \quad (9)$$

where p is a power transformation, therefore

$$\xi = y^{\frac{1}{p}}$$

and

$$\frac{d\xi}{dy} = \frac{1}{p} y^{\frac{1}{p}-1}$$

hence the probability density function of y as given in Hogg and Craig (1978) is given by

$$\begin{aligned} f(y) &= \frac{a^{bc} c y^{\frac{1}{p}(bc-1)} e^{-\left(a y^{\frac{1}{p}}\right)^c}}{\Gamma(b)} \frac{1}{p} y^{\frac{1}{p}-1} \\ &= \frac{a^{bc} c y^{\left(\frac{bc}{p}-1\right)} e^{-(a^p y)^{\frac{c}{p}}}}{p \Gamma(b)}, y > 0 \end{aligned} \quad (10)$$

(10) is a probability density function (pdf) and in what follows, we have to show that its integral is unity (That is, $\int_0^\infty f(y) dy = 1$). We proceed as follows;

$$\int_0^\infty f(y) dy = \frac{a^{bc} c}{p \Gamma(b)} \int_0^\infty y^{\left(\frac{bc}{p}-1\right)} e^{-(a^p y)^{\frac{c}{p}}} dy \quad (11)$$

By adopting the substitution

$$w = (a^p y)^{\frac{c}{p}} \quad (12)$$

in (10) we have the following results

$$y = \frac{w^{\frac{p}{c}}}{a^p}; \quad \frac{dy}{dw} = \frac{p w^{\frac{p}{c}-1}}{c a^p}; \quad dy = \frac{p w^{\frac{p}{c}-1}}{c a^p} dw \quad (13)$$

Now substituting the results in (12) and (13) into (11) we obtain

$$\int_0^\infty f(y) dy = \frac{a^{bc} c}{p \Gamma(b)} \int_0^\infty \left(\frac{w^{\frac{p}{c}}}{a^p} \right)^{\left(\frac{bc}{p}-1\right)} e^{-w} \frac{p w^{\frac{p}{c}-1}}{c a^p} dw \quad (14)$$

After a series of algebraic evaluation in (14) we have that

$$\int_0^\infty f(y) dy = \frac{a^{bc} (a^{-p})^{\frac{bc}{p}} \Gamma(b)}{\Gamma(b)} = 1.$$

Having shown that (10) is a proper pdf by showing that its integral is unity, we now proceed to obtain its generalized k th moment, where k is a positive integer.

By definition $E(Y^k) = \int_0^\infty y^k f(y) dy$, hence

$$\begin{aligned} E(Y^k) &= \int_0^\infty y^k f(y) dy = \frac{a^{bc} c}{p \Gamma(b)} \int_0^\infty y^k y^{\left(\frac{bc}{p}-1\right)} e^{-(a^p y)^{\frac{c}{p}}} dy \\ &= \frac{a^{bc} c}{p \Gamma(b)} \int_0^\infty y^{\left(\frac{bc}{p}+k\right)-1} e^{-(a^p y)^{\frac{c}{p}}} dy \end{aligned} \quad (12)$$

By applying the substitution in (12) and its results in (13) into (15), we obtain

$$\begin{aligned}
 E(Y^k) &= \frac{a^{bc} c}{p \Gamma(b)} \int_0^\infty \left(\frac{w^c}{a^p} \right)^{\left(\frac{bc}{p} + k - 1 \right)} e^{-w} \frac{p w^{\frac{p}{c}-1}}{c a^p} dw \\
 &= \frac{a^{bc} (a^{-p})^{\left(\frac{bc}{p} + k \right)}}{\Gamma(b)} \int_0^\infty w^{\left(\frac{bc}{p} + k \right) - 1} e^{-w} dw \\
 &= \frac{\Gamma\left(b + \frac{pk}{c}\right)}{a^{pk} \Gamma(b)}
 \end{aligned} \tag{16}$$

For $k = 1$ and 2 , we obtain

$$E(Y) = \frac{\Gamma\left(b + \frac{p}{c}\right)}{a^p \Gamma(b)} \tag{17}$$

$$E(Y^2) = \frac{\Gamma\left(b + \frac{2p}{c}\right)}{a^{2p} \Gamma(b)} \tag{18}$$

Hence the variance of Y denoted as σ_y is given by

$$\sigma_y = \frac{\Gamma\left(b + \frac{2p}{c}\right)}{a^{2p} \Gamma(b)} - \left[\frac{\Gamma\left(b + \frac{p}{c}\right)}{a^p \Gamma(b)} \right]^2 \tag{19}$$

The mean and variance of the untransformed distribution given in (3) had been obtained by Ohakwe et al., (2012) as

$$E(\xi) = \frac{1}{a \Gamma(b)} \Gamma\left(b + \frac{1}{c}\right) \tag{20}$$

and

$$\sigma_\xi = Var(\xi) = E(\xi^2) - (E(\xi))^2 = \frac{1}{a^2 \Gamma(b)} \Gamma\left(b + \frac{2}{c}\right) - \left[\frac{1}{a \Gamma(b)} \Gamma\left(b + \frac{1}{c}\right) \right]^2 \tag{21}$$

The means and the variances of the various forms are given in Ohakwe et al., (2012)

For the unit mean condition and its impact on the variance for the various forms of the distributions under study as obtained by Ohakwe et al., (2012), see Table 4.

Similarly for the various power transformations under study namely, inverse ($p=2$), inverse square root ($p=\frac{3}{2}$), inverse square ($p=3$) and square ($p=-1$), the corresponding means and variances of the various distributions under study are obtained and the results are given in Table 5 through 8 while the means and variances resulting from the applications of the unit mean conditions of the untransformed distributions to those of the transformed distributions are given in Tables 9 through 12.

3 RESULTS

Tables 5 through 8 give the theoretical expressions for the mean and variances of the various forms after inverse square root, inverse, inverse square and square transformations respectively. Based on the methodology of this study, it is important to note that the mean and variance of the exponential distribution as well as the variance of the Rayleigh distribution are undefined (Table 8) for inverse transformation. In Tables 9 through 12, the means and variances of the transformed distributions resulting from the applications of the unit-mean-conditions for the various forms of the untransformed GGD are respectively given for the various transformations: (Table 9 for inverse-square-root; Table 10 for inverse; Table 11 for inverse-square; Table 12 for square transformations). It is seen in Table 12 that by the methodology, the means and variances resulting from the application of the unit-mean-conditions of the untransformed distribution to the transformed distribution are undefined for the Gamma(a, b, 1) where $a = b = 1$, Chi-square, Exponential and Weibull distributions as well as the variance of the Rayleigh distribution.

For the inverse-square-root transformation (Table 9), except the Chi-square distribution that has mean ≈ 2.0 , the means of all the other forms under study are approximately unity, however, the variances increased. That is $\sigma_y > \sigma_\xi$ for all the distributions. Here the unit-mean and constant variance assumptions are approximately maintained for all the distributions under study except the Chi-square distribution where the unit mean assumption is violated.

For the inverse transformation (Table 10), only the means of the Rayleigh and Maxwell distributions are approximately 1.0, others have mean ≥ 2.0 , also $\sigma_y > \sigma_\xi$ for all the distributions. Here the unit-mean assumptions are violated except for the Rayleigh and Maxwell distributions.

Furthermore, for the square transformation (Table 11), they were increases in the variances, $\sigma_y > \sigma_\xi$ for all the distributions however the means are all ≥ 2.0 . Under this transformation, the unit-mean assumption is violated for all the distributions under study.

Finally, for the square transformation (Table 12), either the mean or the variance or both are undefined for the various distributions except the Maxwell distribution that maintained the unit mean even though its increased after the transformation. Here it is only the Maxwell distribution that maintained the unit-mean assumption.

4.0 CONCLUSION

In this study, we investigated the implication of power transformations namely, inverse-square-root, inverse, inverse-square and square transformations on the unit-mean and constant variance assumptions of the error component of the multiplicative error model. The distributions

of the error component studied were the various forms of the generalized gamma distribution namely Gamma (a, b, 1), Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributions. The purpose of the study is to investigate on whether the unit-mean and constant variance assumptions necessary for modeling using the multiplicative error model are either violated or retained after the various power transformations. Firstly, the functions describing distributional characteristics of interest for the generalized power transformed error component were established and secondly the unit-mean conditions of the untransformed distributions were applied to the established functions with a view to studying their impacts on the transformed distribution. From the results of the study, the following were discovered;

- (i) For the inverse-square-root transformation, except the Chi-square distribution that has mean ≈ 2.0 , the means of all the other forms under study are approximately unity. However, the variances increased. That is $\sigma_y > \sigma_\xi$ for all the distributions. Here the unit-mean and constant variance assumptions are approximately maintained for all the distributions under study except the Chi-square distribution where it is violated.
- (ii) For the inverse transformation, only the means of the Rayleigh and Maxwell distributions are approximately 1.0, others have mean ≥ 2.0 , also $\sigma_y > \sigma_\xi$ for all the distributions. Here the unit-mean assumptions are violated except for the Rayleigh and Maxwell distributions.
- (iii) For the square transformation, they were increases in the variances, $\sigma_y > \sigma_\xi$ for all the distributions however the means are all ≥ 2.0 . Under this transformation, the unit-mean assumption is violated for all the distributions under study.
- (iv) For the square transformation, either the mean or the variance or both are undefined for the various distributions except the Maxwell distribution that maintained the unit mean even though its variance increased after the transformation. Here it is only the Maxwell distribution that maintained the unit-mean assumption.

Furthermore, based on the results of this study, we make the following recommendations for the various transformations where absolutely necessary for the data with error component having the various distributions under study (Gamma (a, b, 1), where a = b =1, Chi-square, Exponential, Weibull, Rayleigh and Maxwell);

- (i) Inverse-square-root transformation is appropriate for Gamma (a, b, 1), where a = b =1, Exponential, Weibull, Rayleigh and Maxwell distributed error components.
- (ii) Inverse transformation is appropriate for a data set whose error component belongs to Rayleigh or Maxwell distributions.
- (iii) Inverse square transformation is not appropriate for a data set whose error component belong to the six studied distributions.

- (iv) Square transformation is only appropriate for a data set whose error component belongs to a Maxwell distribution.

5.0 REFERENCES

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6.0 APPENDIX

Table 1: Relation of the GGD to other Distributions

S/N	Generalized Gamma Distribution (GG(a, b, c))	a	b	c
1	Gamma ($\text{Gamma}(a, b, 1)$)	a	B	1
2	Chi- square $\left(\chi^2\left(\frac{1}{2}, \frac{n}{2}, 1\right)\right)$	$\frac{1}{2}$	$\frac{n}{2}$	1
3	Exponential $\left(\text{Exp}\left(\frac{1}{\alpha}, 1, 1\right)\right)$	$\frac{1}{\alpha}$	1	1
4	Weibull $\left(\text{Wb}\left(\frac{1}{\sigma}, 1, \alpha\right)\right)$	$\frac{1}{\sigma}$	1	α
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}}, 1, 2\right)\right)$	$\frac{1}{\sigma\sqrt{2}}$	1	2
6	Maxwell $\left(\text{Max}\left(\frac{1}{\sigma\sqrt{2}}, \frac{3}{2}, 2\right)\right)$	$\frac{1}{\sigma\sqrt{2}}$	$\frac{3}{2}$	2

Table 2: Bartlett's Transformation for some values of p .

S/N	p	Required Transformation
1	0	No transformation
2	$\frac{1}{2}$	$\sqrt{X_t}$
3	1	$\log_e X_t$
4	$\frac{3}{2}$	$\frac{1}{\sqrt{X_t}}$
5	2	$\frac{1}{X_t}$
6	3	$\frac{1}{X_t^2}$
7	-1	X_t^2

Table 3: Transformations of the multiplicative Error Model

Y_t	$\Psi(X_{t-1})$	ξ_t^*	Model for Y_t	Assumption on e_t^*
$\log_e X_t$	$\log_e \Psi(X_{t-1})$	$\log_e \xi_t$	Additive	$\xi_t^* \sim V_2^+(0, \sigma_2^2)$
$\sqrt{X_t}$	$\sqrt{\Psi(X_{t-1})}$	$\sqrt{\xi_t}$	Multiplicative	$\xi_t^* \sim V_2^+(1, \sigma_2^2)$
$1/X_t$	$1/\Psi(X_{t-1})$	$1/\xi_t$	Multiplicative	$\xi_t^* \sim V_2^+(1, \sigma_2^2)$
X_t^2	$\Psi(X_{t-1})^2$	ξ_t^2	Multiplicative	$\xi_t^* \sim V_2^+(1, \sigma_2^2)$
$1/\sqrt{X_t}$	$1/\sqrt{\Psi(X_{t-1})}$	$1/\sqrt{\xi_t}$	Multiplicative	$\xi_t^* \sim V_2^+(1, \sigma_2^2)$
$1/X_t^2$	$1/\Psi(X_{t-1})^2$	$1/\xi_t^2$	Multiplicative	$\xi_t^* \sim V_2^+(1, \sigma_2^2)$

Table 4: Condition for Unit Mean and its implication on the Variance of the special cases of the original GGD

S/n	Distribution	Mean	Condition for Unit Mean	Variance
1	Gamma $(Gamma(a, b, 1))$	$\frac{b}{a}$	$a = b$	$\frac{1}{a}$
2	Chi- square $\left(\chi^2\left(\frac{1}{2}, \frac{n}{2}, 1\right)\right)$	n	$n = 1$	2
3	Exponential $\left(Exp\left(\frac{1}{\alpha}, 1, 1\right)\right)$	α	$\alpha = 1$	1
4	Weibull $\left(Wb\left(\frac{1}{\sigma}, 1, \alpha\right)\right)$	$\frac{\sigma}{\alpha} \Gamma\left(\frac{1}{\alpha}\right)$	$\alpha = 1 = \sigma$	1
5	Rayleigh $\left(R\left(\frac{1}{\sigma\sqrt{2}}, 1, 2\right)\right)$	$\sigma\sqrt{\frac{\pi}{2}}$	$\sigma = \sqrt{\frac{2}{\pi}}$	$\frac{1}{\pi}(4 - \pi) = 0.2732$
6	Maxwell $\left(Max\left(\frac{1}{\sigma\sqrt{2}}, \frac{3}{2}, 2\right)\right)$	$2\sigma\sqrt{\frac{2}{\pi}}$	$\sigma = \frac{1}{2}\sqrt{\frac{\pi}{2}}$	$\frac{3\pi}{8} - 1 = 0.1781$

Table 5: Mean and Variance of the special cases of the original GGD under inverse square root transformation $p = \frac{3}{2}$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{\Gamma\left(b + \frac{3}{2}\right)}{a^{\frac{3}{2}} \Gamma(b)}$	$\frac{b(b+1)(b+2)}{a^3} - \left[\frac{\Gamma\left(\frac{2b+3}{2}\right)}{a^{\frac{3}{2}} \Gamma(b)} \right]^2$
2	Chi- square	$\frac{2^{\frac{3}{2}} \Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$	$n(n+2)(n+4) - 8 \left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \right]^2$
3	Exponential	$\frac{3}{4} \alpha^{\frac{3}{2}} \Gamma\left(\frac{1}{2}\right)$	$3\alpha^3 \left[8 - \frac{3}{16} \left[\Gamma\left(\frac{1}{2}\right) \right]^2 \right]$
4	Weibull	$\sigma^{\frac{3}{2}} \frac{3}{2\alpha} \Gamma\left(\frac{3}{2\alpha}\right)$	$\sigma^3 \frac{3}{\alpha} \left[\Gamma\left(\frac{3}{\alpha}\right) - \frac{3}{4\alpha} \left(\Gamma\left(\frac{3}{2\alpha}\right) \right)^2 \right]$
5	Rayleigh	$\sigma^{\frac{3}{2}} 2^{\frac{3}{4}} \Gamma\left(\frac{7}{4}\right)$	$\sigma^3 2^{\frac{3}{2}} \left[\Gamma\left(\frac{5}{2}\right) - \left(\Gamma\left(\frac{7}{4}\right) \right)^2 \right]$
6	Maxwell	$\sigma^{\frac{3}{2}} 2^{\frac{3}{4}} \frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}$	$\sigma^3 2^{\frac{3}{2}} \left[\frac{4}{\Gamma\left(\frac{1}{2}\right)} - \left(\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} \right)^2 \right]$

Table 6: Mean and Variance of the special cases of the original GGD under inverse transformation $p = 2$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{b(b+1)}{a^2}$	$\frac{b(b+1)}{a^4} [(b+2)(b+3) - b(b+1)]$
2	Chi- square	$n(n+2)$	$n(n+2) [(n+4)(n+6) - n(n+2)]$
3	Exponential	$2\alpha^3$	$4\alpha^4 (6 - \alpha^2)$
4	Weibull	$\frac{2\sigma^2}{\alpha} \Gamma\left(\frac{2}{\alpha}\right)$	$\frac{4\sigma^4}{\alpha} \left[\Gamma\left(\frac{4}{\alpha}\right) - \frac{1}{\alpha} \left(\Gamma\left(\frac{2}{\alpha}\right) \right)^2 \right]$

5	Rayleigh	$2\sigma^2$	$4\sigma^4$
6	Maxwell	$3\sigma^2$	$6\sigma^4$

Table 7: Mean and Variance of the special cases of the original GGD under inverse square transformation $p = 3$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{b(b+1)(b+2)}{a^3}$	$\frac{b(b+1)(b+2)}{a^6} [(b+3)(b+4)(b+5) - b(b+1)(b+2)]$
2	Chi- square	$n(n+2)(n+4)$	$n(n+2)(n+4) [(n+6)(n+8)(n+10) - n(n+2)(n+4)]$
3	Exponential	$6\alpha^3$	$684\alpha^6$
4	Weibull	$\frac{3\sigma^3}{\alpha} \Gamma\left(\frac{3}{\alpha}\right)$	$\frac{3\sigma^6}{\alpha} \left[2\Gamma\left(\frac{6}{\varepsilon}\right) - \frac{3}{\alpha} \left(\Gamma\left(\frac{3}{\alpha}\right) \right)^2 \right]$
5	Rayleigh	$\frac{3\sigma^3}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right)$	$3\sigma^6 \left[16 - \frac{3}{2} \left(\Gamma\left(\frac{1}{2}\right) \right)^2 \right]$
6	Maxwell	$\frac{8\sigma^3\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}$	$\sigma^6 \left[105 - \frac{128}{\left[\Gamma\left(\frac{1}{2}\right) \right]^2} \right]$

Table 8: Mean and Variance of the special cases of the original GGD under square transformation $p = -1$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{a}{b-1}$	$\frac{a^2}{b-1} \left[\frac{1}{(b-1)} - \frac{1}{(b-2)} \right]$
2	Chi- square	$\frac{1}{n-2}$	$\frac{1}{n-2} \left[\frac{1}{(n-4)} - \frac{1}{(n-2)} \right]$
3	Exponential	Undefined	Undefined
4	Weibull	$\frac{1}{\sigma} \Gamma\left(1 - \frac{1}{\alpha}\right), \alpha > 1$	$\frac{1}{\sigma^2} \left[\Gamma\left(1 - \frac{2}{\alpha}\right) - \left(\Gamma\left(1 - \frac{1}{\alpha}\right) \right)^2 \right], \alpha > 2$

5	Rayleigh	$\frac{1}{\sigma\sqrt{2}}\Gamma\left(\frac{1}{2}\right)$	Undefined
6	Maxwell	$\frac{\sqrt{2}}{\sigma\Gamma\left(\frac{1}{2}\right)}$	$\frac{1}{\sigma^2}\left[1-\frac{2}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\right]$

Table 9: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under inverse square root transformation $p = \frac{3}{2}$

S/ n	Distribution	Unit Mean of the Untransfor med Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$b = a$	$\frac{\Gamma\left(a + \frac{3}{2}\right)}{a^{\frac{3}{2}}\Gamma(a)}$	$\frac{(a+1)(a+2)}{a^2} - \left[\frac{\Gamma\left(a + \frac{3}{2}\right)}{a^{\frac{3}{2}}\Gamma(a)}\right]^2$
2	Chi- square	$n = 1$	$\frac{2^{\frac{3}{2}}\Gamma(2)}{\Gamma\left(\frac{1}{2}\right)} = 1.5958$	$n(n+2)(n+4) - 8\left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2 = 1.5958$
3	Exponential	$\alpha = 1$	$\frac{3}{4}\Gamma\left(\frac{1}{2}\right) = 1.3293$	$3\alpha^3\left[8 - \frac{3}{16}\left[\Gamma\left(\frac{1}{2}\right)\right]^2\right] = 22.2329$
4	Weibull	$\alpha = \sigma = 1$	$\frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = 1.3293$	$\sigma^3 \frac{3}{\alpha}\left[\Gamma\left(\frac{3}{\alpha}\right) - \frac{3}{4\alpha}\left(\Gamma\left(\frac{3}{2\alpha}\right)\right)^2\right] = 22.2329$
5	Rayleigh	$\sigma = \sqrt{\frac{2}{\pi}}$	$\left(\sqrt{\frac{2}{\pi}}\right)^{\frac{3}{2}} 2^{\frac{3}{4}}\Gamma\left(\frac{7}{4}\right) = 1.2331$	$\sigma^3 2^{\frac{3}{2}}\left[\Gamma\left(\frac{5}{2}\right) - \left(\Gamma\left(\frac{7}{4}\right)\right)^2\right] = 0.6963$

6	Maxwell	$\sigma = \frac{1}{2}\sqrt{\frac{2}{\pi}}$	$\left(\frac{1}{2}\sqrt{\frac{2}{\pi}}\right)^{\frac{3}{2}} 2^{\frac{3}{4}} \frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} = 0.5$	$\sigma^3 2^{\frac{3}{2}} \left[\frac{4}{\Gamma\left(\frac{1}{2}\right)} - \left(\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} \right)^2 \right] = 0.2121$
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Table 10: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under inverse transformation $p = 2$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{(a+1)}{a}$	$\frac{(a+1)}{a^3} [(a+2)(a+3) - a(a+1)]$
2	Chi- square	3	$n(n+2)[(n+4)(n+6) - n(n+2)] = 96.0$
3	Exponential	$2\alpha^3 = 2.0$	$4\alpha^4(6 - \alpha^2) = 1.0$
4	Weibull	$\frac{2\sigma^2}{\alpha} \Gamma\left(\frac{2}{\alpha}\right) = 2.0$	$\frac{4\sigma^4}{\alpha} \left[\Gamma\left(\frac{4}{\alpha}\right) - \frac{1}{\alpha} \left(\Gamma\left(\frac{2}{\alpha}\right) \right)^2 \right] = 20.0$
5	Rayleigh	$2\sigma^2 = 1.2732$	$4\sigma^4 = 1.6211$
6	Maxwell	$3\sigma^2 = 1.1781$	$6\sigma^4 = 0.9253$

Table 11: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under inverse square transformation $p = 3$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{(a+1)(a+2)}{a^2}$	$\frac{(a+1)(a+2)}{a^5} [(a+3)(a+4)(a+5) - a(a+1)(a+2)]$
2	Chi- square	$n(n+2)(n+4) = 15$	$n(n+2)(n+4)[(n+6)(n+8)(n+10) - n(n+2)(n+4)]$
3	Exponential	$6\alpha^3 = 6.0$	$\frac{3\sigma^6}{\alpha} \left[2\Gamma\left(\frac{6}{\alpha}\right) - \frac{3}{\alpha} \left(\Gamma\left(\frac{3}{\alpha}\right) \right)^2 \right] = 684.0$
4	Weibull	$\frac{3\sigma^3}{\alpha} \Gamma\left(\frac{3}{\alpha}\right) = 6.0$	$684\alpha^6 = 684.0$

5	Rayleigh	$\frac{3\sigma^3}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right) = 3.759$	$3\sigma^6 \left[16 - \frac{3}{2} \left(\Gamma\left(\frac{1}{2}\right) \right)^2 \right] = 0.9254$
6	Maxwell	$\frac{8\sigma^3\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)} = 1.5708$	$\sigma^6 \left[105 - \frac{128}{\left[\Gamma\left(\frac{1}{2}\right) \right]^2} \right] = 3.8913$

Table 12: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under square transformation $p = -1$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a, b, 1)$	$\frac{a}{a-1}$	$\frac{a^2}{a-1} \left[\frac{1}{(a-1)} - \frac{1}{(a-2)} \right]$
2	Chi- square	Undefined	Undefined
3	Exponential	Undefined	Undefined
4	Weibull	Undefined	Undefined
5	Rayleigh	$\frac{1}{\sigma\sqrt{2}} \Gamma\left(\frac{1}{2}\right) = 1.0$	Undefined
6	Maxwell	$\frac{\sqrt{2}}{\sigma\Gamma\left(\frac{1}{2}\right)} = 1.2732$	$\frac{1}{\sigma^2} \left[1 - \frac{2}{\left(\Gamma\left(\frac{1}{2}\right) \right)^2} \right] = 0.9253$