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Research Paper

THE BETA GENERALIZED INVERSE WEIBULL DISTRIBUTION

ABSTRACT

The beta generalized inverse Weibull distribution (BGIW) is suggested in this paper. The mathematical properties of the BGIW distribution are provided and the expression for the moment generating function is derived. Also, the analytical shapes of the corresponding probability density function, reliability function, hazard rate function, and mode are derived with graphical illustrations. Expressions for the r^{th} positive and negative moments are calculated and the variation of the skewness and kurtosis measures is investigated. Moment and likelihood estimators of the parameters are derived. The observed information matrix is obtained. Simulation study is carried out to investigate the performance of the maximum likelihood method of estimation. Moreover, analysis of real data set is conducted to demonstrate the usefulness of the proposed distribution.

Keywords: *Beta generalized inverse Weibull distribution, Fisher information matrix, moment generating function, moment estimation, maximum likelihood estimation.*

1 INTRODUCTION

The inverse Weibull distribution (IW) usually used in reliability and biological studies. The three-parameter generalized inverse Weibull (GIW) distribution, which extends to several distributions, and commonly used in the lifetime-literature, is more flexible than the inverse Weibull distribution. Mudholkar et al. [1], Jiang et al. [2] and De Gusmao et al. [3] have introduced and discussed the three-parameter GIW distribution.

Let T be a random variable from GIW distribution, so the cumulative distribution function (cdf) of T is defined as

$$G(t;\alpha,\beta,\theta) = e^{-\theta(\frac{\alpha}{t})^{\beta}}, \qquad t > 0. \tag{1}$$

Therefore, the probability density function (pdf) of the (GIW) distribution with three parameters ($\alpha>0$, $\beta > 0$, $\theta >0$) is given by

$$g(t;\alpha,\beta,\theta) = \theta\beta\alpha^{\beta}t^{-(\beta+1)}e^{-\theta(\frac{\alpha}{t})^{\beta}}, \qquad t > 0. \tag{2}$$

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Recent developments focus on new techniques to construct useful distributions. These include the approach explored by Eugene et al. [4] and Jones [5]. The new technique is motivated by the following general class:

If G(.) is the cdf of a random variable, then a generalized class of distributions can be defined by

$$F(x) = I_{G(x)}(a,b), \qquad x \in \mathfrak{T}, \tag{3}$$

where $I_{G(x)}(a,b) = \frac{B_{G(x)}(a,b)}{B(a,b)}$, $B_{G(x)}(a,b) = \int_0^{G(x)} z^{a-1}(1-z)^{b-1}\,dz$, $B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$, and $\Gamma(.)$ denotes the incomplete beta function ratio, the incomplete beta function, the beta function, and the gamma function, respectively. The beta density has different type of shapes. The beta density is symmetric if a = b = 1, is unimodal when a, b >1 and "U" shaped when a, b < 1. It has positively skewed when a < b and negatively skewed when a >b.

The pdf of this general class of distributions namely the Beta type I distribution is defined by

$$f(u;a,b) = B(a,b)^{-1}u^{a-1}(1-u)^{b-1}, \qquad 0 \le u \le 1. \tag{4}$$

Recently, this class of generalized distributions has gained more attention from many researches. Eugene et al. [4] have constructed the beta normal distribution by substituting G(.) in (3) to be the cdf of the normal distribution with parameters μ and σ. Jones [5] focused on cases where G(.) is symmetric about zero with location and scale parameters. Moreover, beta distributions have been studied by Nadarajah and Kotz [6,7]. They have proposed the beta Gumbel distribution by substituting G(.) in (3) to be the cdf of the Gumbel distribution with parameters μ and σ. Beta distributions have been studied also by Akinsete et al. [8], Zografos and Balakrishnan [9]. Beta-Weibull distribution was first introduced by Famoye et.al. [10]. Properties of the beta Weibull model were studied by Lee et al. [11]. Cordeiro et al. [12] has provided an expansion for the cumulative distribution function, the moments, and the moment generating function of the beta Weibull distribution. Moreover, maximum likelihood estimation and formulas for the elements of the Fisher information matrix were provided. Barreto-Souza et al. [13] introduced the beta generalized exponential distribution and provided comprehensive mathematical properties of this distribution. They obtained the density, moment generating function and rth moment of the order statistics. Cordeiro et al. [14] introduced the beta exponentiated Weibull distribution; they provided the moments and two expressions for the moment generating function of the new distribution.

In this paper, a new beta distribution is introduced by taking G(.) to be the generalized inverse Weibull distribution and will refer to it as beta generalized inverse Weibull distribution (BGIW). The inverse Weibull distribution is a special case of (2) when θ = 1 and represents a particular case of the BGIW distribution when θ = a = b =1.

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The rest of the paper is organized as follows: Section 2 provides the cumulative distribution, probability density, reliability, and hazard functions of the BGIW distribution with graphical illustrations. Some special models of BGIW are studied in Section 3. Theoretical properties are discussed in Section 4. In Section 5, the parameters estimation are obtained by the method of moment and maximum likelihood method in addition to presenting the Fisher information matrix. The performance of the estimators is investigated through simulation study in Section 6. Section 7 presents the analyses of real data set to illustrate the usefulness of the proposed distribution. Finally, some concluding remarks are presented in Section 8.

2. THE BETA GENERALIZED INVERSE WEIBULL DISTRIBUTION

The cdf of the BGIW distribution can be obtained as follows:

$$F(t) = I_{G(t)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(t)} z^{a-1}(1-z)^{b-1}\, dz, \quad 0 < z < 1, a, b > 0. \tag{5}$$

Substituting (1) into (5) yields

$$F(t) = \frac{1}{B(a,b)} \int_0^{\exp\left[-\theta\left(\frac{\alpha}{t}\right)^{\beta}\right]} z^{a-1}(1-z)^{b-1}\, dz, \quad 0 < z < 1,\ a, b > 0. \tag{6}$$

Property 2.1 Probability density function of BGIW distribution

Differentiating both sides of (6) with respect to t, the explicit form of the pdf of the BGIW distribution can be obtained as:

$$f(t) = \frac{\beta\theta}{\alpha B(a,b)} \left(\frac{\alpha}{t}\right)^{\beta+1} e^{-a\theta\left(\frac{\alpha}{t}\right)^{\beta}} \left(1 - e^{-\theta\left(\frac{\alpha}{t}\right)^{\beta}}\right)^{b-1}, \alpha, \beta, \theta, a, b > 0, t > 0. \tag{7}$$

For non-integer positive value b > 0, (7) can be rewritten as an infinite power series in the form

$$f(t) = \frac{\beta\theta}{\alpha\, b\, B(a,b)} \left(\frac{\alpha}{t}\right)^{\beta+1} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j, j+1)} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}, \quad \alpha, \beta, \theta, a, b > 0, t > 0,\ b > j. \tag{8}$$

Throughout the paper, this distribution will be referred to as the beta generalized inverse Weibull distribution BGIW (α, β, θ, a, b). The density given by (8) is a proper density and when a=b=1, it reduces to the generalized inverse Weibull density with scale parameter α and the two shape parameters, β and θ. This model is constructed based on the beta and the generalized inverse Weibull distributions.

Property 2.2 Cumulative distribution function of BGIW distribution

From (8) the corresponding cdf can be written as:

$$F(t) = \frac{1}{b\, B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j, j+1)\,(a+j)} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}, t > 0, \alpha, \beta, \theta, a, b > 0. \tag{9}$$

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Property 2.3 Reliability function of BGIW distribution

The reliability function of the BGIW (α, β, θ, a, b) distribution can be written as

$$R(t) = 1 - \frac{1}{b\,B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j,j+1)\,(a+j)} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}, t > 0, \alpha, \beta, \theta, a, b > 0. \quad (10)$$

Property 2.4 Hazard function of BGIW distribution

The hazard function of the BGIW (α, β, θ, a, b) distribution can be written in the form:

$$h(t) = \frac{\frac{\beta\theta}{\alpha\, b\, B(a,b)} \left(\frac{\alpha}{t}\right)^{\beta+1} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j,j+1)} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}}{1 - \frac{1}{b\,B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j,j+1)\,(a+j)} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}}, \; t > 0, \alpha, \beta, \theta, a, b > 0. \quad (11)$$

Remark 2.1 The index j in (8) – (11) stops at (b-1), when b > 0 is a positive integer.

Graphical representation of the pdf and reliability function for selected values of α, β, θ, a, and b are given in Figures 1 and 2, respectively. It indicates the stability of the characteristics of the initial distribution (GIW) after adding the new parameters a and b.

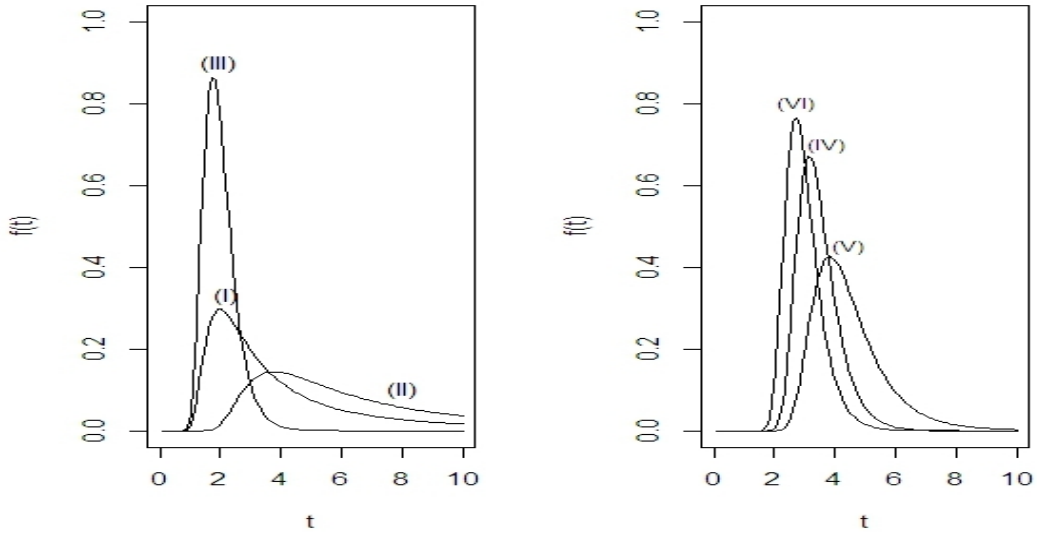


Fig. 1. Plots of the BGIW density function. α=2, β =2.5, θ =2.5, (I): a =0.5, b= 0.5. (II): a = 2, b=0.5, (III): a =0.5, b=4, (IV): a =5, b=5. (V):a =6, b= 3. (VI): a =3, b=5

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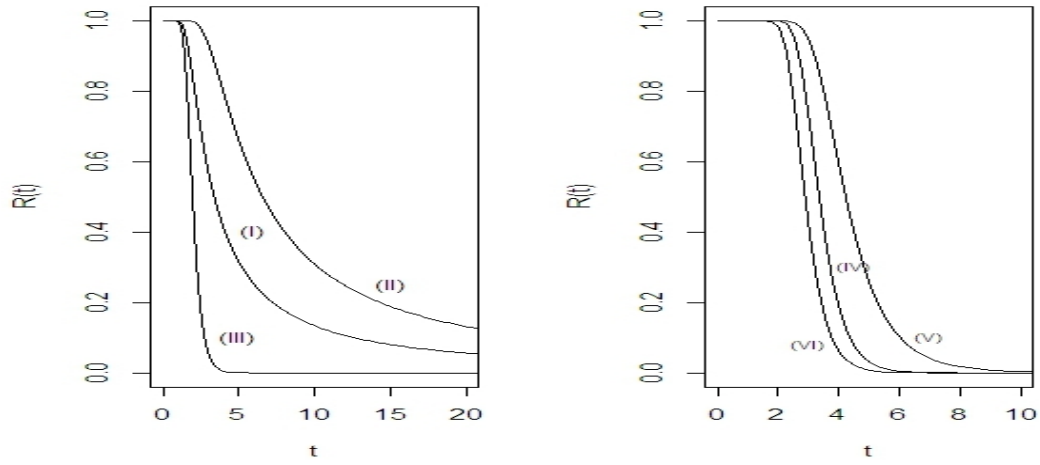


Fig. 2. Plots of the BGIW reliability function. α=2, β =2.5, θ =2.5, (I): a =0.5, b=0.5, (II): a = 2, b=0.5. (III): a =0.5, b=4. (IV): a =5 , b=5 (V) :a =6 , b= 3 (VI): a =3, b=5

Furthermore, Figure 3 illustrates different shapes of the failure rate function of the BGIW distribution which take monotonically decreasing case (III) and upside down bathtub shapes cases (I) , (II) and (IV).

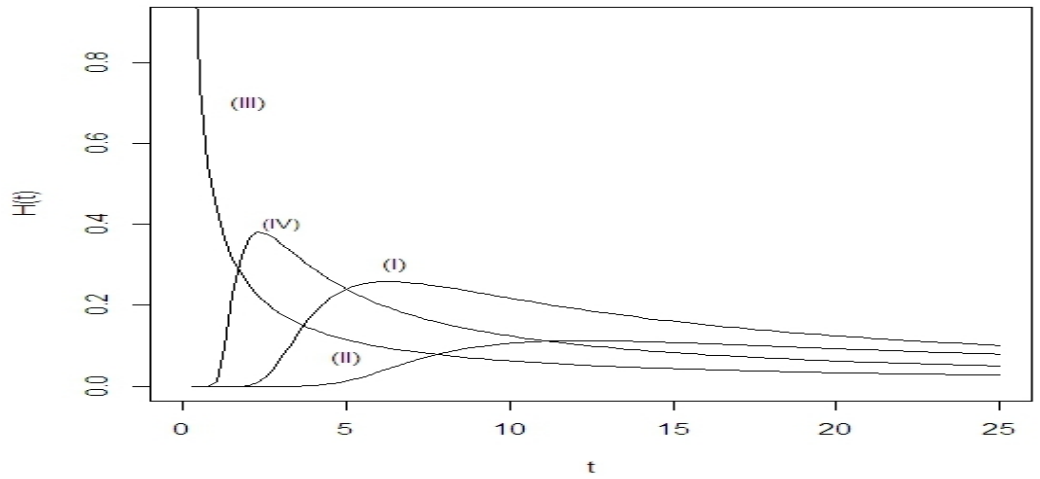


Fig. 3. Plots of the BGIW hazard rate functions. (I) α = 4, β = 2, θ =1.5, a = 1.5, b = 1.3, (II) α = 10, β= 2, θ =1.2, a = 1.1, b = 1.1, (III) α = 5, β = 0.2, θ = 0.2, a = 12, b = 5, (IV) α = 2, β = 2.5 , θ = 2.5, a=0.5, b = 0.5

3. SPECIAL MODELS

The BGIW distribution in (7) represents a generalization of several distributions that have been considered in the literature. In particular, BGIW distribution becomes generalized inverse Weibull (GIW(α, β, θ)) distribution when a = b =1. The beta inverse Weibull (BIW(α, β, a, b))

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distribution is clearly a special case of BGIW distribution when $\theta = 1$. In addition, the generalized inverse exponential distribution (GIE (η, θ)) is a special case of BGIW distribution when $\alpha - 1 = \eta$, $\beta = a = b = 1$. Inverse Weibull (IW(α, β)) distribution can be obtained from (7) by making $\theta = a = b = 1$. The BGIW distribution contains the inverse exponential (IE(η)), and inverse Rayleigh (IR(η)) distributions as special models for $\theta = \beta = a = b = 1$, $\alpha - 1 = \eta$, and $\beta = 2$, $\theta = a = b = 1$, $\eta = \alpha 2$, respectively. These special models with their references are summarized in Table 1 and graphically represented in Figure 4.

Furthermore, the BGIW distribution in (7) is a general form of several new beta distributions. The beta inverse exponential distribution (BIE (η, a, b)) is clearly a special case of BGIW when $\alpha - 1 = \eta$, $\theta = \beta = 1$. When $\alpha - 1 = \eta$, $\beta = 1$, the beta generalized inverse exponential distribution (BGIE (η, θ, a, b)) can be obtained. The BGIW distribution can be reduced to the beta inverse Rayleigh distribution (BIR (η, a, b)) when $\theta = 1$, $\beta = 2$, $\eta = \alpha 2$ and to the beta generalized inverse Rayleigh distribution (BGIR (η, θ, a, b)) when $\beta = 2$, $\eta = \alpha 2$.

Table 1. generalized inverse Weibull type distributions

Distribution	***BGIW(α, β, θ, a, b)***					**Main reference**
	α	**θ**	**β**	**a**	**b**	
GIW (*α, β, θ*)	**α**	**θ**	**β**	**1**	**1**	**De Gusmao et al. [3]**
BIW (*α, β, a, b*)	**α**	**1**	**β**	**a**	**b**	**Khan [15]**
GIE (η, *θ*)	**η=α - 1**	**θ**	**1**	**1**	**1**	**Khan [16]**
IW (*α, β*)	**α**	**1**	**β**	**1**	**1**	**Keller and Kamath [17]**
IE (*η*)	**η=α - 1**	**1**	**1**	**1**	**1**	**Keller and Kamath [17]**
IR (η)	**η=α2**	**1**	**2**	**1**	**1**	**Voda [18]**
B ≡ Beta, E ≡ exponential, G ≡ generalized, I ≡ inverse, , R ≡ Rayleigh, W ≡ Weibull						

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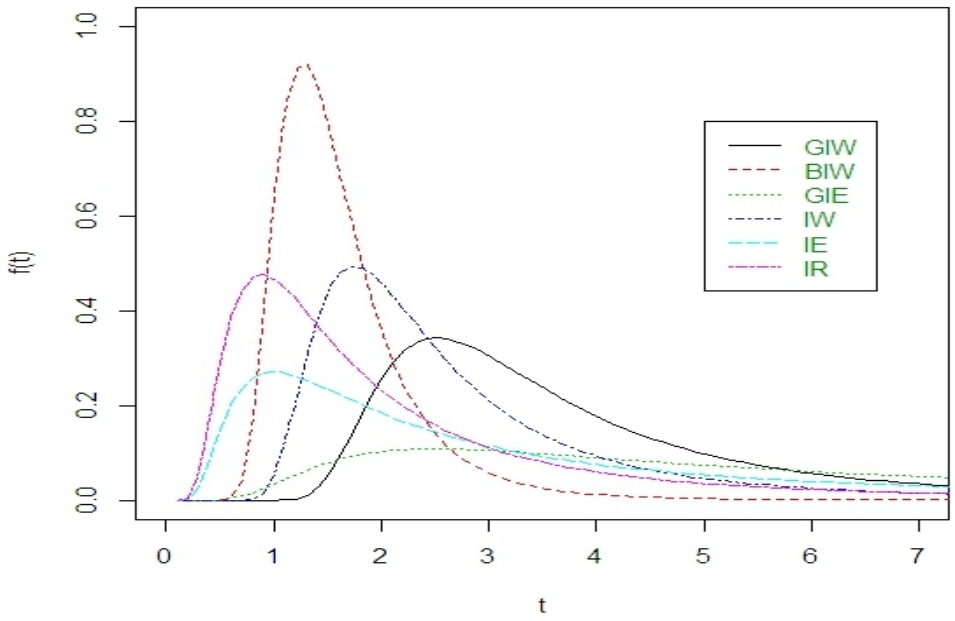


Fig. 4. Plots the BGIW probability density functions of the six special models

4. STATISTICAL PROPERTIES

In this section, some of the basic statistical properties of the BGIW distribution are provided.

Property 4.1 Quantile function, median, and simulation

The quantile function of BGIW (α, β, θ, a, b) distribution corresponding to (7) is

$$t = Q(u) = F^{-1}(u) = \alpha\left(\frac{-1}{\theta}\log\left(I_u^{-1}(a,b)\right)\right)^{\frac{-1}{\beta}}, \qquad (12)$$

where $I_u^{-1}(a,b)$ denotes the inverse of the incomplete beta function with parameters a and b. The quartiles Q1, Q2 (median), and Q3 of the BGIW (α, β, θ, a, b) distribution correspond to the values u= 0.25, 0.50, and 0.75 respectively.

The simulation of the BGIW (α, β, θ, a, b) random variable T is easily obtained if B is a random number following a beta distribution with parameters a and b then

$$t = \alpha\left(\frac{-1}{\theta}\log(B)\right)^{\frac{-1}{\beta}} \text{ follows a BGIW } (\alpha, \beta, \theta, a, b) \text{ distribution.}$$

Property 4.2 Mode of the BGIW distribution

The BGIW (α, β, θ a, b) distribution is unimodal. The mode is located at$f'(t) = 0$. Equivalently, it is the solution of the Equation $\frac{d}{dt}(\ln f(t)) = 0$. That is

$$(\beta+1)t^{\beta} - [(\beta+1)t^{\beta} - a\theta\beta\alpha^{\beta} - b\theta\beta\alpha^{\beta} + \theta\beta\alpha^{\beta}]e^{-\theta\left(\frac{\alpha}{t}\right)^{\beta}} - a\theta\beta\alpha^{\beta} = 0. \qquad (13)$$

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The mode of BGIW distribution is obtained numerically at some selected values of parameters a and b when α=2, β =2.5, θ =2.5. Table 2 shows that the mode values approximately coincide with the graphical mode shown in Figure 1 for all selected values of parameters.

Table 2. Numerical evaluation of the mode of BGIW for each case in Figure1

Parameters	**a = b= 0.5**	**a=0.5, b = 4**	**a = 2, b = 0.5**
Mode	1.97 (I)	1.75 (III)	3.78 (II)
Parameters	$a = b = 5$	$a = 6, b = 3$	$a = 3, b = 5$
Mode	3.15 (IV)	3.83 (V)	2.71 (VI)

Property 4.3 The non-central moments of the BGIW distribution

The r^{th} non-central moment of the BGIW (α, β, θ, a, b) distribution can be written as

$$\acute{\mu}_r = \frac{\alpha^r \theta^{\frac{r}{\beta}}}{b\, B(a,b)} \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j, j+1)} (a+j)^{-\left(1-\frac{r}{\beta}\right)}, \beta > r, r = 1, 2, \ldots \quad (14)$$

Therefore, the mean and the variance of the BGIW (α, β, θ, a, b) distribution can be expressed as

$$\mu = \frac{\alpha\, \theta^{\frac{1}{\beta}}}{b\, B(a,b)} \Gamma\left(1 - \frac{1}{\beta}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j, j+1)} (a+j)^{-\left(1-\frac{1}{\beta}\right)}, \beta > 1, \quad (15)$$

$$\sigma^2 = \frac{\alpha^2 \theta^{\frac{2}{\beta}}}{b\, B(a,b)} \Gamma\left(1 - \frac{2}{\beta}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j, j+1)} (a+j)^{-\left(1-\frac{2}{\beta}\right)} - \mu^2, \beta > 2. \quad (16)$$

Property 4.4 The central moments of the BGIW distribution

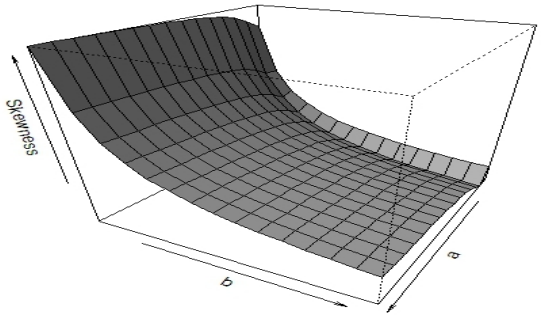
The r^{th} central moment of the BGIW(α, β, θ, a, b) distribution is given by

$$M_r = E[(X - \mu)^r] = \sum_{k=0}^{r} C_k^r (-1)^k \mu^k \mu'_{r-k} \quad (17)$$

The coefficient of skewness $\alpha_3 = \frac{M_3}{M_2^{3/2}}$, the coefficient of kurtosis $\alpha_4 = \frac{M_4}{M_2^2}$, and the excess in kurtosis $\alpha_4 - 3$ can be calculated from (17). Figures 5 and 6 illustrate skewness and kurtosis at some selected values of parameters a and b, for fixed α = 2, β = 2.5, and θ = 2.5. These show that the skewness and kurtosis curves monotonically increase when parameter a increases and monotonically decrease when parameter b increases.

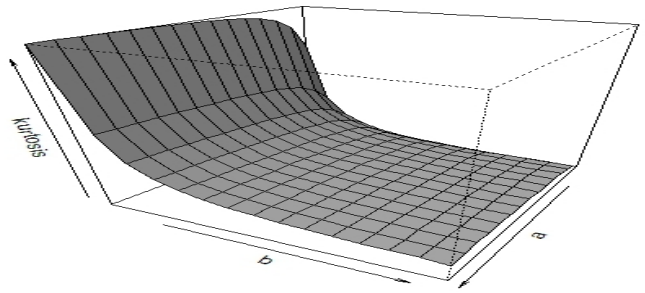
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184 **Fig. 5. Variation of skewness, a = 1,2,…, 15 and b = .1, .45, .8 , …, 5 , for fixed α=2, β =2.5, θ =2.5**



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186 **Fig. 6. Variation of kurtosis, a = 1,2,…, 15 and b = .1, .45, .8 , …, 5 , for fixed α=2, β =2.5, θ =2.5**

187 **Property 4.5 Negative moments of the BGIW distribution**

188 The r^{th} non-central negative integral moment of the BGIW (α, β, θ, a, b) distribution is given
189 by

190 $$\mu'_{-r} = \frac{\beta\theta}{\alpha\, b\, B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j,j+1)} \int_0^{\infty} t^{-r} \left(\frac{\alpha}{t}\right)^{\beta+1} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}\, dt. \quad (18)$$

191 After some manipulations, Equation (18) reduces to

192 $$\mu'_{-r} = \frac{\alpha^{-r}\theta^{-\frac{r}{\beta}}}{b\, B(a,b)}\ \Gamma\left(\frac{r}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j,j+1)}\ (a+j)^{-\left(\frac{r}{\beta}+1\right)}, \beta > r, r = 1, 2, \ldots \quad (19)$$

193 **Property 4.6 Moment generating function of the BGIW distribution**

194 The moment generating function of BGIW (α, β, θ, a, b) distribution has the form

195 $$M_T(u) = \frac{\beta\theta}{\alpha\, b\, B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{B(b-j,j+1)} \int_0^{\infty} e^{t\,u} \left(\frac{\alpha}{t}\right)^{\beta+1} e^{-(a+j)\,\theta\left(\frac{\alpha}{t}\right)^{\beta}}\, dt. \quad (20)$$

196 Equation (20) can be expressed as

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$$M_T(u) = \sum_{k=0}^{\infty}\left(\frac{\theta}{b\,B(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\alpha^k}{B(b-j,j+1)}\Gamma\left(1-\frac{k}{\beta}\right)[(a+j)\theta]^{-\left(1-\frac{k}{\beta}\right)}\right)\frac{u^k}{k!}, \beta > k. \quad (21)$$

From (21), the kth non-central moment can easily be computed as the coefficient of $\frac{u^k}{k!}$ in the power series expansion of MT (u) into powers of u.

5. PARAMETER ESTIMATION

In this section, the five parameters of BGIW distribution will be estimated using the method of moments and the method of maximum likelihood.

5.1. Moment method of estimation

In order to estimate of the parameters α, β, θ, a, b of BGIW distribution based on the method of moment, where the first 5 non-central population moments ($\mu'_r, r = 1, 2, \ldots, 5$) are equated to the first 5 non-central sample moments (mr, r=1, 2, …, 5). That is $\mu'_r = m_r, r = 1, 2, \ldots, 5$. The resulting equations are:

$$\frac{\hat{\alpha}^r\,\hat{\theta}^{\frac{r}{\beta}}}{\hat{b}\,B(\hat{a},\hat{b})}\,\Gamma\left(1-\frac{r}{\hat{\beta}}\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{B(\hat{b}-j,j+1)}(\hat{a}+j)^{-\left(1-\frac{r}{\hat{\beta}}\right)} = \frac{1}{n}\sum_{i=1}^{n}t_i^r, \beta > r,\ r = 1, \ldots, 5. \quad (22)$$

Because of the complexity of the existing equations, it is difficult to obtain a closed form analytic solution of the system in Equation (22). Therefore a suitable numerical technique can be used to obtain the required moment estimates of the parameters α, β, θ, a, and b.

5.2. Maximum likelihood method

Suppose that $t_1, t_2, \ldots, t_n$ are the n lifetimes observed arising from f(t) defined in Equation (8). The likelihood function $L(v) = L(\alpha, \beta, \theta, a, b) = \prod_{i=1}^{n} f(t_i; v), v = (\alpha, \beta, \theta, a, b)$ is given by

$$L(v) = \frac{\alpha^{n\beta}\beta^n\theta^n}{\prod_{i=1}^{n}t_i^{\beta+1}B^n(a,b)}e^{-a\,\theta\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta}}\prod_{i=1}^{n}\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^{b-1}. \quad (23)$$

It is usually easier to maximize the natural logarithm of the likelihood function. So, the natural logarithm of the likelihood function can be written as

$$\ln L = n\beta\ln\alpha + n\ln\beta + n\ln\theta - n\ln B(a,b) - (\beta+1)\sum_{i=1}^{n}\ln t_i - a\,\theta\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta} +$$

$$(b-1)\sum_{i=1}^{n}\ln\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right). \quad (24)$$

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The maximum likelihood estimates, $\hat{v} = (\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{a}, \hat{b})$, are obtained by taking the first derivative of the natural logarithm of the likelihood function with respect to $\alpha, \beta, \theta, a, b$ and equating to zero as follows

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n\beta}{\alpha} - a\theta\beta\alpha^{\beta-1}\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta} + \theta\beta\alpha^{\beta-1}(b-1)\sum_{i=1}^{n}\frac{e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\left(\frac{1}{t_i}\right)^{\beta}}{1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}},$$

$$\frac{\partial \ln L}{\partial \beta} = n\ln\alpha + \frac{n}{\beta} - \sum_{i=1}^{n}\ln t_i - a\theta\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta}\ln\left(\frac{\alpha}{t_i}\right) + \theta\,(b-1)\sum_{i=1}^{n}\frac{e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\left(\frac{\alpha}{t_i}\right)^{\beta}\ln\left(\frac{\alpha}{t_i}\right)}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)},$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - a\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta} + (b-1)\sum_{i=1}^{n}\frac{e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\left(\frac{\alpha}{t_i}\right)^{\beta}}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)},$$

$$\frac{\partial \ln L}{\partial a} = -n\,\psi(a) + n\,\psi(a+b) - \theta\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta},$$

$$\frac{\partial \ln L}{\partial b} = -n\,\psi(b) + n\,\psi(a+b) + \sum_{i=1}^{n}\ln\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right).$$

(25)

Here, $\psi(x)$ is the digamma function defined as follows

$$\psi(x) = \frac{d}{dx}[\ln\Gamma(x)] = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Setting these 5 expressions in Equation (25) to zero and solving the resulting system of non-linear equations to obtain the maximum likelihood estimators of the unknown parameters, α, β, θ, a and b of BGIW distribution.

5.3. Asymptotic confidence bounds

Since the MLE's of the vector of unknown parameters, v, are not obtained in closed forms, then it is not possible to derive the exact distributions of the MLE's $\hat{v}$. The observed information matrix is obtained for approximate confidence intervals and hypothesis tests of the vector v. This is because the expected information matrix of α, β, θ, a, and b involve numerical integration. The observed 5X5 information matrix I is given by

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234
$$I = \begin{pmatrix} l_{\alpha\alpha} & l_{\alpha\beta} & l_{\alpha\theta} & l_{\alpha a} & l_{\alpha b} \\ l_{\beta\alpha} & l_{\beta\beta} & l_{\beta\theta} & l_{\beta a} & l_{\beta b} \\ l_{\theta\alpha} & l_{\theta\beta} & l_{\theta\theta} & l_{\theta a} & l_{\theta b} \\ l_{a\alpha} & l_{a\beta} & l_{a\theta} & l_{aa} & l_{ab} \\ l_{b\alpha} & l_{b\beta} & l_{b\theta} & l_{ba} & l_{bb} \end{pmatrix},$$

235 where $l_{v_i v_j} = -\frac{\partial^2 \ln L}{\partial v_i \, \partial v_j}, i, j = 1, \ldots, 5,$ which are given in the Appendix.

236 The asymptotic distribution of $\sqrt{n}(\hat{v} - v)$ is $N_5(0, I^{-1})$, that can be used to create
237 approximate intervals and confidence regions for the parameters and for the hazard and the
238 survival functions. An asymptotic confidence bounds with significance level γ for each
239 parameter v_i is given by $(\hat{v}_i \pm z_{\frac{\gamma}{2}}\sqrt{v^{i,j}})$, where $v^{i,i}$ is the ith diagonal element of I^{-1} for i= 1, …,
240 5 and $z_{\frac{\gamma}{2}}$ is the (1-γ/2) percentile of the standard normal distribution.

241 **6. SIMULATION STUDY**

242 Monte Carlo simulation study is performed to investigate the maximum likelihood estimators
243 (MLE's) is studied with respect to their relative absolute bias (RAB) and relative mean
244 squared error (RMSE).
245 The relative absolute bias of the estimate $\hat{v}_j$ is defined as

$$\mathrm{RAB}(\hat{v}_j) = \left|\frac{\mathrm{Bias}(\hat{v}_j)}{v_j}\right|, \quad j = 1, 2, 3, 4, 5$$

246 The relative mean squared error of the estimate $\hat{v}_j$ is defined as

$$\mathrm{RMSE}(\hat{v}_j) = \left|\frac{\mathrm{MSE}(\hat{v}_j)}{v_j}\right|, \quad j = 1, 2, 3, 4, 5$$

247 where $\hat{v}_j$denotes the estimate of v_j.
248 All computations are performed using R packages.
249

250 **6.1. Numerical results for the maximum likelihood estimates**
251 MLE of α, β, θ, a, and b is obtained by solving nonlinear equations which are given in Equation
252 (25) simultaneously, using nlminb routine in R which uses variants of
253 the Newton-Raphson algorithm. Then, the performances of the MLE's are studied mainly for
254 different sample sizes (n=10, 20, 30, 40, 50) and given values of the parameters α= 3, β = θ = a
255 = 2.5 and b= 2.4. Repeat each computation over 10000 replications for different cases. It is
256 observed that the RAB and RMSE decrease as the sample size increases see Table 3.

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Table 3. Numerical evaluation of maximum likelihood estimates of the 5-parameters BGIW

n	Parameters	α=3	β=2.5	θ=2.5	a=2.5	b=2.4
10	**MLE**	4.713	2.659	2.662	2.658	2.648
	RAB	0.571	0.064	0.063	0.063	0.103
	RMSE	1.299	0.129	0.133	0.131	0.150
20	**MLE**	4.569	2.580	2.571	2.573	2.575
	RAB	0.523	0.032	0.029	0.029	0.073
	RMSE	0.957	0.055	0.053	0.053	0.064
30	**MLE**	4.522	2.548	2.546	2.544	2.544
	RAB	0.507	0.019	0.018	0.017	0.060
	RMSE	0.858	0.033	0.033	0.032	0.041
40	**MLE**	4.499	2.533	2.533	2.539	2.535
	RAB	0.500	0.013	0.013	0.015	0.056
	RMSE	0.811	0.024	0.023	0.024	0.031
50	**MLE**	4.485	2.526	2.530	2.526	2.526
	RAB	0.495	0.010	0.012	0.010	0.053
	RMSE	0.783	0.018	0.018	0.019	0.026

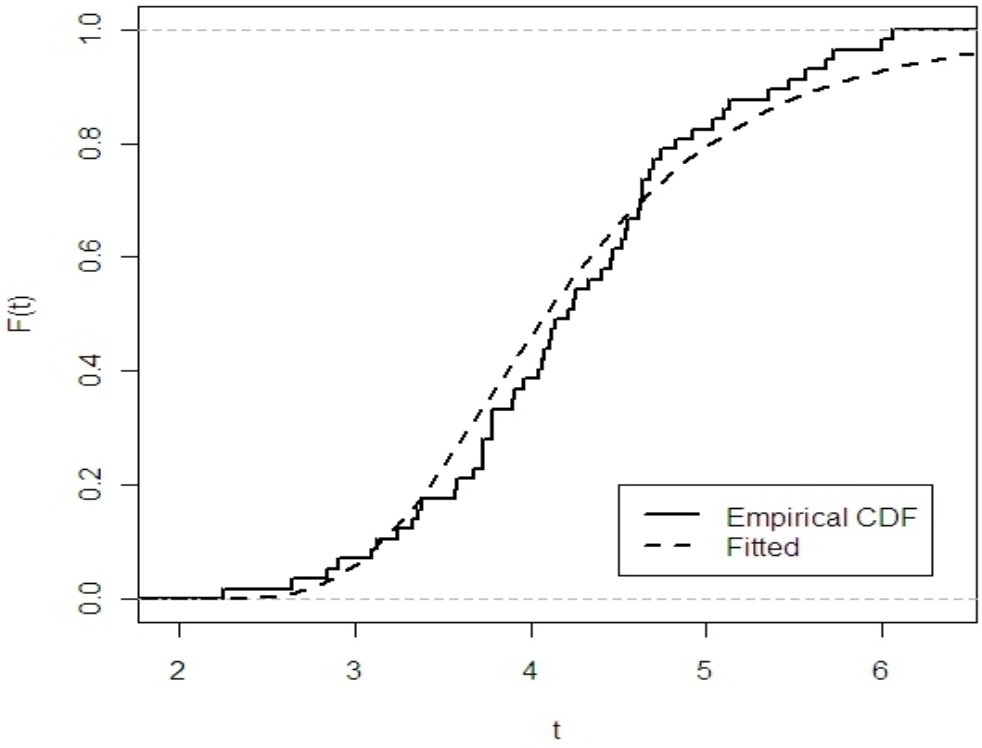
7. APPLICATION

In this section, the utility use of the BIGW distribution may be demonstrated by using a data set. The data set of breaking strengths of carbon fibbers from sample size 57 and length 1 is used to fit the BGIW by the method of maximum likelihood, see Lawless [19].

The kolmogorov-Smirnov (K.S.) goodness-of-fit test is (K.S.= 0.1053)) and the corresponding (P = 0.91), which indicate that the BGIW model can be used to analyze this data set.

Also, the K.S.goodness of fit test is used to verify the fitted of the cumulative distribution function of MLE and the empirical distribution function are displayed in Figure 7.

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269

270 **Fig. 7. Estimate cumulative function and the empirical for breaking strength of single carbon**
271 **fibbers of length**

272 The maximum likelihood estimates of the parameters with relative mean square error
273 and the Akaike information criterion (AIC) for the fitted models are illustrated in Table 4. The
274 (AIC) of the fitted BIGW,GIW,BIW,IW indicates that the suggested distribution is the more
275 appropriate distribution than the other three sub models.

276

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Table 4. MLE s of the model parameters for the breaking strength of single carbon fibbers data, the relative mean square error (in brackets) and the criteria's AIC measure.

Distribution	α	θ	β	a	b	AIC
BGIW	2.479 (0.115)	2.693 (0.036)	2.708 (0.028)	2.987 (0.110)	3.00101 (0.105)	160.699
GIW	2.99404 (0.494)	2.15517 (0.025)	2.977 (0.000176)	1 (-)	1 (-)	181.25
BIW	3.0010 (0.501)	1 (-)	2.968668 (0.00045)	3.988 (0.0714)	2.232 (0.027)	161.3827
IW	2.989 (0.489)	1 (-)	3.0011 (4.03 e-7)	1 (-)	1 (-)	205.742

8. CONCLUDING REMARKS

The beta generalized inverse Weibull distribution is proposed. This generalizes the beta inverse distribution discussed by Khan [15] and represents a generalization of other several models previously studied in the literature. The general mathematical properties of this distribution are studied. The moment generating function is derived and infinite sums for the moments of BGIW distribution are provided. The ML estimators and moment estimators of the BGIW parameters are obtained. The information matrix and the asymptotic confidence bounds of the parameters are derived. Monte Carlo simulation studies were carried out under different sample sizes to study the theoretical performances of the MLE of the parameters. Real data set was analyzed and the BIGW has provide a good fit for the data and was more appropriate model compared to the GIW,BIW, and IW sub-models.

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APPENDIX

The elements of the observed 5X5 information matrix I are given by

$$l_{\alpha\alpha} = -\frac{\partial^2 \ln L}{\partial\, \alpha^2} = \frac{n\beta}{\alpha^2} + a\,\theta\beta(\beta-1)\alpha^{\beta-2}\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta} - (b-1)\theta\beta\alpha^{\beta-2}\sum_{i=1}^{n}\frac{\left(\left(\frac{1}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)\left((\beta-1)+\theta\beta\alpha^{\beta-1}\left(\frac{1}{t_i}\right)^{\beta}\right)}{\left(1-\, e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)} - (b-1)\theta^2\beta^2\alpha^{2\beta-2}\sum_{i=1}^{n}\frac{\left(\left(\frac{1}{t_i}\right)^{2\beta} e^{-2\,\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}{\left(1-\, e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^{2}}$$

$$l_{\beta\alpha} = -\frac{\partial^2 \ln L}{\partial\, \beta\partial\alpha} = -\frac{n}{\alpha} + a\theta\alpha^{\beta-1}\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta}\left[1+\beta ln\alpha+\beta ln\left(\frac{1}{t_i}\right)\right] - (b - 1)\theta\alpha^{\beta-1}\sum_{i=1}^{n}\frac{\left(\left(\frac{1}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}{\left(1-\, e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}\left[1+\beta ln\alpha+\beta ln\left(\frac{1}{t_i}\right)\right] - (b-1)\theta\beta\alpha^{\beta-1}\sum_{i=1}^{n}\frac{\left(\left(\frac{1}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}{\left(1-\, e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^{2}}$$

$$l_{\theta\alpha} = -\frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha} = a\,\beta\,\alpha^{\beta-1}\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta} - (b-1)\beta\alpha^{\beta-1}\sum_{i=1}^{n}\frac{\left(\left(\frac{1}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)} + (b - 1)\theta\beta\alpha^{2\beta-1}\sum_{i=1}^{n}\frac{\left(\frac{1}{t_i}\right)^{2\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^{2}},$$

$$l_{a\alpha} = -\frac{\partial^2 \ln L}{\partial a\,\partial\alpha} = \theta\,\beta\alpha^{\beta-1}\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta}$$

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$$l_{b\alpha} = -\frac{\partial^2 \ln L}{\partial\, b\, \partial\alpha} = -\theta\beta\alpha^{\beta-1}\sum_{i=1}^{n}\frac{\left(\frac{1}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}}{\left(1-\ e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}$$

$$\mathrm{I}_{\beta\beta} = -\frac{\partial^2 \ln L}{\partial\beta^2} = \frac{n}{\beta^2} + a\theta\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta}\left[\ln\left(\frac{\alpha}{t_i}\right)\right]^2 - \theta(b - 1)\sum_{i=1}^{n}\frac{\left[\ln\left(\frac{\alpha}{t_i}\right)\right]^2\left(\frac{\alpha}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}\right),}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^2}$$

$$l_{\theta\beta} = -\frac{\partial^2 \ln L}{\partial\theta\partial\beta} = a\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta}\ln\left(\frac{\alpha}{t_i}\right) - (b-1)\sum_{i=1}^{n}\frac{e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\ln\left(\frac{\alpha}{t_i}\right)}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)} + (b-1)\theta\sum_{i=1}^{n}\frac{e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\ln\left(\frac{\alpha}{t_i}\right)\left(\frac{\alpha}{t_i}\right)^{\beta}}{\left(1-e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^2}$$

$$l_{a\beta} = -\frac{\partial^2 \ln L}{\partial a\ \partial\beta} = \theta\sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta} ln\left(\frac{\alpha}{t_i}\right),$$

$$l_{b\beta} = -\frac{\partial^2 \ln L}{\partial\, b\, \partial\beta} = -\theta\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{t_i}\right)^{\beta}\ln\left(\frac{\alpha}{t_i}\right)e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}}{\left(1-\ e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}$$

$$l_{\theta\theta} = -\frac{\partial^2 \ln L}{\partial\theta^2} = \frac{n}{\theta^2} + (b-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{t_i}\right)^{2\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}}{\left(1-\ e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)^2}$$

$$l_{a\theta} = -\frac{\partial^2 \ln L}{\partial a\ \partial\theta} = \sum_{i=1}^{n}\left(\frac{\alpha}{t_i}\right)^{\beta}$$

$$l_{b\theta} = -\frac{\partial^2 \ln L}{\partial\, b\, \partial\theta} = -\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{t_i}\right)^{\beta} e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}}{\left(1-\ e^{-\theta\left(\frac{\alpha}{t_i}\right)^{\beta}}\right)}$$

366 $$l_{aa} = -\frac{\partial^2 \ln L}{\partial\, a^2} = n\,\psi'(a) - n\,\frac{\partial\psi(a+b)}{\partial a},$$

367 $$l_{ba} = -\frac{\partial^2 \ln L}{\partial a\ \partial b} = -n\,\frac{\partial\psi(a+b)}{\partial b},$$

368 $$l_{bb} = -\frac{\partial^2 \ln L}{\partial\, b^2} = n\,\psi'(b) - n\,\frac{\partial\psi(a+b)}{\partial b}.$$