

SUPERSTABILITY OF A GENERALIZATION OF THE COSINE EQUATION

ABSTRACT

The aim of this paper is to investigate the stability problem for the functional equation:

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \quad (E_{g,f})$$

and the superstability of the d'Alembert's equation:

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \quad (A)$$

under the conditions from which the differences of each equation are bounded by $\varphi(x)$, $\psi(x)$ and $\min(\varphi(x), \psi(y))$ where G is an arbitrary group, not necessarily abelian, $f, g : G \rightarrow \mathbb{C}$ and $\varphi, \psi : G \rightarrow \mathbb{R}$ are functions and σ is an involution of G .

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1. INTRODUCTION

There is a strong stability phenomenon which is known as a superstability. An equation of homomorphism is called superstable if each approximate homomorphism is actually a true homomorphism. This property was first observed by J. Baker et al. [4] in the following Theorem: Let V be a vector space. If a function $f : V \rightarrow \mathbb{R}$ satisfies the inequality

$$|f(x+y) - f(x)f(y)| \leq \varepsilon,$$

for some $\varepsilon > 0$ and for all $x, y \in V$. Then either f is a bounded function or

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in V.$$

In light of this result, the stability of a class of functional equations has been investigated by Badora, Baker, Dragomir, Găvruta, Ger, Kabbaj, Kim, Rassias, Roukbi, Tyrala, Székelyhidi, Zeglami etc. The interested reader should refer to [1-11, 13-18, 21-26] for a thorough account on the subject of stability of functional equations.

In [2], R. Badora and R. Ger have improved the superstability problem of the d'Alembert equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G, \quad (C)$$

under the condition

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \quad \text{or} \quad \varphi(y).$$

Namely, the following theorems hold true.

Theorem 1. (R. Badora, R. Ger [2]) *Let $(G, +)$ be an Abelian group, $f : G \rightarrow \mathbb{C}$ and let $\varphi : G \rightarrow \mathbb{R}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x), \quad \text{for all } x, y \in G$$

Then either f is bounded or f satisfies the classical d'Alembert's equation (C).

Theorem 2. (R. Badora, R. Ger [2]) *Let $(G, +)$ be an Abelian group, $f : G \rightarrow \mathbb{C}$ and let $\varphi : G \rightarrow \mathbb{R}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(y), \quad \text{for all } x, y \in G$$

Then either f is bounded or f satisfies the classical d'Alembert's equation (C).

In [18] A. Roukbi, D. Zeglami and S. Kabbaj proved the superstability of the equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \quad (E_{f,g})$$

without imposing any conditions on the group G . Equation $(E_{f,g})$ is called the Wilson equation (see [19]).

In the present paper, we consider, in both abelian and non abelian groups and without any conditions on f , the Hyers-Ulam stability problem of the functional equation

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \quad (E_{g,f})$$

under the conditions $|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x), \psi(y)$ or $\min(\varphi(x), \psi(y))$ where G is any one group and σ is an involution of G , i. e. $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$. The equation $(E_{g,f})$ is called, sometimes, second generalization of the cosine equation.

Furthermore, the obtained results can be extended to the equation

$$f(xy) + f(x\sigma(y)) = \lambda g(x)f(y), \quad x, y \in G.$$

As a consequence, we obtain the superstability of the d'Alembert's functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \quad (A)$$

which proved by Roukbi, Zeglami and Kabbaj [18] on any group, by Redouani, Elqorachi and Rassias [17] on step 2 nilpotent groups and by Baker, Badora and Ger, Gavruta, Kim, etc ([3], [2], [9], [15], ...) in the case where G is an abelian group.

In this paper, let G is any one group, e denote its neutral element, C the field of complex numbers and IR the field of real numbers. We may assume that f and g are complex valued functions on G , $\varphi, \psi : G \rightarrow IR$ are mappings,

λ, δ are nonnegative real constants, and σ is an involution of G . In the case where $f(e) \neq 0$ we put $\tilde{f} = \frac{1}{f(e)} f$.

A typical example of the involution σ is the group involution $\sigma(x) = x^{-1}, x \in G$. Another is the adjoint $A \rightarrow A^*$ in the matrix group $GL(n, C)$. A third one is

$$\sigma \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

on the Heisenberg group H_3 .

2. SOLUTIONS OF THE EQUATION $(E_{g,f})$

We shall start from solutions of the d'Alembert functional equation: In 2008, Th. Davison [7] proved the following result:

Lemma 1. Let G be a topological group and $f : G \rightarrow C$ a continuous function with $f(e) = 1$ satisfying

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G.$$

Then there is a continuous (group) homomorphism $h : G \rightarrow SL(2, C)$ such that

$$f(x) = \frac{1}{2} \text{tr}(h(x)) \text{ for all } x, y \in G.$$

In [20] H. Stetkær gave solutions of equation (A) introducing the theory of representations. Precisely, he proved that:

Lemma 2. Let S be a semigroup. The non-zero continuous solutions f of (A) on S are the functions of the form

$$f = \frac{1}{2} \chi_\pi \quad (2.1)$$

where π ranges over the 2-dimensional continuous representations of S for which

$$\pi(\sigma(x)) = \text{adj}(\pi(x)) \quad (2.2)$$

for all $x \in S$ and

$$\text{adj} : \text{Mat}_2(C) \rightarrow \text{Mat}_2(C), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Note that the equation (A) is raised by Kannappan in the case where G is abelian [12]. Using Lemma 2 we directly prove the following fact concerning solutions of equation $(E_{g,f})$.

Theorem 3. Let G be any group. Then $f, g : G \rightarrow C$ satisfy the equation $(E_{g,f})$ if and only if

i) $f = 0$ and g is arbitrary, or

ii) $f \neq 0$ and $f(x) = \alpha g(x)$ for all $x \in G$, where $\alpha \in C - \{0\}$ and g is a solution of (A).

Furthermore, the non-zero continuous solutions f, g of $(E_{g,f})$ on G are functions of the form

$$g = \frac{1}{2} \chi_\pi, \quad f = \frac{\alpha}{2} \chi_\pi.$$

where $\alpha \in C - \{0\}$ and π ranges over the 2-dimensional continuous representations of G satisfying (2.2).

Proof. Assume that $f \neq 0$. Setting $y = e$ in $(E_{g,f})$ we have

$$f(x) = g(x)f(0)$$

for all $x \in G$. From above, we conclude that $f(0) \neq 0$. Putting $\alpha := f(0)$ we get

$$f(x) = \alpha g(x) \text{ for all } x \in G.$$

So, from $(E_{g,f})$ we obtain

$$\alpha g(xy) + \alpha g(x\sigma(y)) = 2g(x)\alpha g(y), \quad x, y \in G,$$

for all $x, y \in G$. Then g is a solution of (A) and $f = \alpha g$. The rest of the proof follows from Lemma 2.

3. STABILITY OF THE EQUATION $(E_{g,f})$

Lemma 3. Assume that functions $f, g : G \rightarrow C$ and $\psi : G \rightarrow IR$ satisfy the inequality

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y), \quad \text{for all } x, y \in G \quad (3.1)$$

such that $f \neq 0$. Then f is unbounded if and only if g is unbounded too.

Proof. If $f(e) = 0$. Putting $y = e$ in (3.1) we get $|f(x)| \leq \frac{\psi(e)}{2}$, for all $x \in G$ i.e. f is bounded. Let $M = \sup|f|$ and

choose $a \in G$ such that $f(a) \neq 0$ then we get from the inequality (3.1) that

$$|f(x) - f(e)g(x)| \leq \frac{\psi(e)}{2}$$

Which shows that f is unbounded is equivalent to g is unbounded too.

Lemma 4. Assume that functions $f, g : G \rightarrow C$ and $\psi : G \rightarrow IR$ satisfy the inequality

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y), \quad \text{for all } x, y \in G$$

Such that $f(e) = 1$. Then

$$i) \quad |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y) + \psi(e), \quad \text{for all } x, y \in G. \quad (3.2)$$

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107 *ii) f is unbounded if and only if g is also unbounded.*

108 **Proof.** i) Assume that $f(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show that

$$|f(x) - g(x)| \leq \frac{\psi(e)}{2} \quad (3.3)$$

110 for all $x \in G$. Let

$$F(x) := f(x) - g(x)$$

112 By virtue of inequality (3.3), we have

$$g(x) = f(x) - F(x) \quad \text{and} \quad |F(x)| \leq \frac{\psi(e)}{2} \quad (3.4)$$

114 for all $x \in G$. By the definition of F and the use of (3.1) we have

$$\begin{aligned} |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| &= |f(xy) - F(xy) + f(x\sigma(y)) - F(x\sigma(y)) - 2g(x)f(y)| \\ &\leq |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| + |F(xy)| + |F(x\sigma(y))| \\ &\leq \psi(y) + \psi(e). \end{aligned}$$

118 *ii) Follows from (3.3) and it is also a particular case of Lemma 1.*

119 **Lemma 5.** Assume that functions $f, g : G \rightarrow C$ and $\varphi : G \rightarrow IR$ satisfy the inequality

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x), \quad (3.5)$$

121 for all $x, y \in G$ such that $f(e) = 1$. Then

$$|g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq 2\varphi(x), \quad \text{for all } x, y \in G. \quad (3.6)$$

123 **Proof.** i) Assume that $f(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show that

$$|f(x) - g(x)| \leq \frac{\varphi(x)}{2} \quad (3.7)$$

126 for all $x \in G$. Let $F(x) := f(x) - g(x)$. By virtue of inequality (3.7), we have

$$g(x) = f(x) - F(x) \quad \text{and} \quad |F(x)| \leq \frac{\varphi(x)}{2} \quad (3.8)$$

128 for all $x \in G$. Using (3.5) and (3.8) we get

$$|g(xy) + g(x\sigma(y)) - 2g(x)f(y)| = |f(xy) - F(xy) + f(x\sigma(y)) - F(x\sigma(y)) - 2g(x)f(y)|$$

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$$\begin{aligned} &\leq |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| + |F(xy)| + |F(x\sigma(y))| \\ &\leq 2\varphi(x) \end{aligned}$$

Theorem 4. Assume that functions $f, g : G \rightarrow C$ and $\psi : G \rightarrow IR$ satisfy the inequality

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y),$$

for all $x, y \in G$ such that $f \neq 0$. Then either g (or f) is bounded or

$$\tilde{f}(xy) + \tilde{f}(x\sigma(y)) + \tilde{f}(yx) + \tilde{f}(\sigma(y)) = 4\tilde{f}(x)\tilde{f}(y), \quad (3.9)$$

for all $x, y \in G$, where $\tilde{f} = \frac{1}{f(e)} f$.

Proof. i) Assume that f, g satisfy the inequality (3.1) such that g is unbounded (which is equivalent - by lemma 1- to f is also unbounded). We firstly start with the following particular case $f(e) = 1$. For all $x, y, z \in G$ we have

$$\begin{aligned} &2|g(z)| |f(xy) + f(x\sigma(y)) + f(yx) + f(\sigma(y)x) - 4f(x)f(y)| \\ &= |2g(z)f(xy) + 2g(z)f(x\sigma(y)) + 2g(z)f(yx) + 2g(z)f(\sigma(y)x) - 8g(z)f(x)f(y)| \\ &\leq |f(zxy) + f(z\sigma(y)\sigma(x)) - 2g(z)f(xy)| \\ &\quad + |f(zx\sigma(y)) + f(zy\sigma(x)) - 2g(z)f(x\sigma(y))| \\ &\quad + |f(zyx) + f(z\sigma(x)\sigma(y)) - 2g(z)f(yx)| \\ &\quad + |f(z\sigma(y)x) + f(z\sigma(x)y) - 2g(z)f(\sigma(y)x)| \\ &\quad + |f(zxy) + f(zx\sigma(y)) - 2g(zx)f(y)| \\ &\quad + |f(zyx) + f(zy\sigma(x)) - 2g(zy)f(x)| \\ &\quad + |f(z\sigma(y)x) + f(z\sigma(y)\sigma(x)) - 2g(z\sigma(y))f(x)| \\ &\quad + |f(z\sigma(x)y) + f(z\sigma(x)\sigma(y)) - 2g(z\sigma(x))f(y)| \\ &\quad + 2|f(y)| |g(zx) + g(z\sigma(x)) - 2g(z)f(x)| \\ &\quad + 2|f(x)| |g(zy) + g(z\sigma(y)) - 2g(z)f(y)|. \end{aligned}$$

By virtue of inequalities (3.1) and (3.2), we have

$$2|g(z)| |f(xy) + f(x\sigma(y)) + f(yx) + f(\sigma(y)x) - 4f(x)f(y)|$$

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$$\leq \psi(xy) + \psi(x\sigma(y)) + \psi(yx) + \psi(\sigma(y))x + 2\psi(x)$$

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$$+ 2\psi(y) + 2|f(y)|(\psi(x) + \psi(e)) + 2|f(x)|(\psi(y) + \psi(e)) \quad (3.10)$$

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If we fix x, y , the right hand side of the above inequality is bounded function of z . Since g is unbounded, from the preceding (3.10), we conclude that f is a solution of the long equation (3.9), which ends the proof in this case. If f is a non-zero function such that $f(e) = 0$ then g is bounded (Lemma 3). For the case $f(e) \neq 0$ is any one complex number, dividing the two sides of the inequality (3.1) by $\alpha = f(e)$ we find that

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$$|\tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y)| \leq \frac{\psi(y)}{|\alpha|}, \quad \text{for all } x, y \in G,$$

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where $\tilde{f} = \frac{1}{\alpha} f$. We see that the pair $(\tilde{f}, g)$ satisfies the inequality (3.1) with $\tilde{f}(e) = 1$ which shows, using the first

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case result that each functions f, g satisfying (3.1) falls into one of the categories (i)-(iii).

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As a consequence of Theorem 4, we have the following result on the superstability of the equation (A) which generalizes the Badora's and Ger's result on the classical d'Alembert's functional equation on an abelian group (Theorem 2).

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Corollary 1. Assume that functions $f : G \rightarrow C$ and $\psi : G \rightarrow IR$ satisfy the inequality

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$$|f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \psi(y), \quad (3.11)$$

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for all $x, y \in G$. Then either f is bounded or f satisfies the d'Alembert's long functional equation (3.9). Further, in the latter case, if G is abelian then f satisfies the equation (A) and so it has the form (2.1).

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Proof. Assume that f is unbounded function satisfying (3.11). On putting $f = g$ in Theorem 4 we get that $\tilde{f}$ is a solution of the equation (3.9). Substituting y by e in (3.11) we obtain $|f(x)(f(e) - 1)| \leq \frac{\psi(e)}{2}$ for all $x \in G$. This inequality shows that $f(e) = 1$ because f is unbounded. So $f = \tilde{f}$ is a solution of (3.9) and if G is abelian then f satisfies the equation (A).

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In the following theorem the Hyers-Ulam stability of the equation $(E_{g,f})$ will be investigated on an arbitrary group. For $f = 0$ the pair (f, g) is a trivial solution of the equation $(E_{g,f})$.

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Theorem 5. Assume that functions $f, g : G \rightarrow C$ and $\phi, \psi : G \rightarrow IR$ satisfy the inequality

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$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \phi(x) \text{ and } \psi(y), \quad (3.12)$$

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for all $x, y \in G$ such that $f \neq 0$. Then

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i) f, g are bounded or

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ii) g is unbounded and $\tilde{f} = \frac{1}{f(e)} f$ satisfies the d'Alembert's long functional equation (3.9), or

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iii) f is unbounded and (f, g) satisfies the equation

$$g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y) \text{ for all } x, y \in G. \quad (3.13)$$

Proof. (ii) is proven in Theorem 4.

Assume that f, g satisfy inequality (3.12). If f is a nonzero function such that $f(e) = 0$, we have seen in Proof of Lemma 3 that f and g are bounded so we are in case (i). For the case $f(e) = 1$: If f is unbounded then for all $x, y, z \in G$ we have

$$\begin{aligned} & 2|f(z)| |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \\ &= |2f(z)g(xy) + 2f(z)g(x\sigma(y)) - 4f(z)g(x)f(y)| \\ &\leq |f(xyz) + f(xy\sigma(z)) - 2g(xy)f(z)| \\ &\quad + |f(x\sigma(y)z) + f(x\sigma(y)\sigma(z)) - 2g(x\sigma(y))f(z)| \\ &\quad + |f(xyz) + f(x\sigma(z)\sigma(y)) - 2g(x)f(yz)| \\ &\quad + |f(xy\sigma(z)) + f(xz\sigma(y)) - 2g(x)f(y\sigma(z))| \\ &\quad + |f(x\sigma(y)z) + f(x\sigma(z)y) - 2g(x)f(\sigma(z)y)| \\ &\quad + |f(x\sigma(y)\sigma(z)) + f(xzy) - 2g(x)f(zy)| \\ &\quad + |f(x\sigma(z)y) + f(x\sigma(z)\sigma(y)) - 2g(x\sigma(z))f(y)| \\ &\quad + |f(xzy) + f(xz\sigma(y)) - 2g(xz)f(y)| \\ &\quad + |2g(x)\{f(yz) + f(y\sigma(z)) + f(zy) + f(\sigma(z)y) - 4f(y)f(z)\}| \\ &\quad + 2|f(y)| |g(xz) + g(x\sigma(z)) - 2g(x)f(z)| \end{aligned}$$

In virtue of inequalities (3.12) and (3.6), we obtain

$$\begin{aligned} & 2|f(z)| |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \\ &\leq \varphi(xy) + \varphi(x\sigma(y)) + 4\varphi(x) + 2\psi(y) + 4\varphi(x)|f(y)| \\ &\quad + 2|g(x)| |f(yz) + f(y\sigma(z)) + f(zy) + f(\sigma(z)y) - 4f(y)f(z)|. \end{aligned}$$

Since g is unbounded (which is equivalent to f is unbounded) hence according to Theorem 4, g is a solution of the equation (3.9). So we conclude that

$$2|f(z)| |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(xy) + \varphi(x\sigma(y)) + 4\varphi(x) + 2\psi(y) + 4\varphi(x)|f(y)|. \quad (3.14)$$

Again the right hand side of (3.14) as a function of z is bounded for all fixed x, y . Since f is unbounded, from the preceding (3.14), we see that f satisfies the equation

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$$g(xy) + g(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, .$$

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Now we suppose that $f(e) \notin \{0,1\}$. Dividing the two sides of the inequality (3.12) by $\alpha = f(e)$ we find that

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$$\left| \tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y) \right| \leq \frac{\varphi(x)}{|\alpha|} \text{ and } \frac{|\psi(y)|}{|\alpha|} \text{ for all } x, y \in G ,$$

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where $\tilde{f} = \frac{1}{\alpha} f$. We see that the pair $(\tilde{f}, g)$ satisfies the inequality (3.12) with $f(e) = 1$ which shows, using the first

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case result that each pair of functions f, g satisfying (3.12) falls into one of the categories (i)-(iii). which finished this

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proof.

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In Lemma 3 we have seen that if $f \neq 0$ then f is unbounded is equivalent to g is unbounded too. So we present modified and simpler formula of Theorem 5.

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Corollary 2. Assume that functions $f, g : G \rightarrow C$ and $\varphi, \psi : G \rightarrow IR$ satisfy the inequality (3.1) such that $f \neq 0$. Then either f (or g) is bounded or the pair (f, g) satisfies the equation

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$$g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y), \quad x, y \in G.$$

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Furthermore in the latter case the function $\tilde{f}$ satisfies the equation (3.9).

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As another consequence of Theorem 5, we have the following result on the superstability of the equation (A) on both

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abelian and non-Abelian groups which generalizes the Baker's result on the classical d'Alembert functional equation on an abelian group [3, Theorem 5].

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Corollary 3. Let δ be given. Assume that the function $f : G \rightarrow C$ satisfies the inequality

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$$\left| f(xy) + f(x\sigma(y)) - 2f(x)f(y) \right| \leq \delta ,$$

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for all $x, y \in G$. Then either f is bounded or f is a solution of the equation (A). Further, in the latter case f has the form (2.1).

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Proof. Using similar techniques as in Proof of Corollary 1 we see that if f is unbounded then we have $f(e) = 1$ implying that $\tilde{f} = f$. The rest of the proof follows on putting $f = g$ in Theorem 5 (iii).

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From above Theorems we get also the superstability of the equation $(E_{g,f})$ on two particular cases:

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Corollary 3. Let G be an Abelian group (or at least f is central), and let $f, g : G \rightarrow C$ and $\varphi, \psi : G \rightarrow IR$ satisfy the inequality

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$$\left| f(xy) + f(x\sigma(y)) - 2g(x)f(y) \right| \leq \varphi(x) \text{ and } \psi(y) , \quad (3.15)$$

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for all $x, y \in G$. Then there are the following possibilities:

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i) If $f = 0$, then g is arbitrary.

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ii) If $g = 0$, then f is bounded.

232 *iii) If $f \neq 0 \neq g$ and f is bounded, then g is bounded, too.*

233 *iv) If $g \neq 0$ and f is unbounded, then g is unbounded, too. Moreover g is a solutions of the equation (A) and the*
 234 *pair (f, g) satisfies equations $(E_{f,g})$ and $(E_{g,f})$.*

235 **Proof.** (ii) If $g = 0$ then the inequality (3.15) has a form $|f(xy) + f(x\sigma(y))| \leq \varphi(x) \text{ and } \psi(y)$ for all $x, y \in G$. Put
 236 $y = e$, we get $|f(x)| \leq \frac{\psi(e)}{2}$ for all $x \in G$ i.e. f is bounded.

237 (iii) If $f \neq 0 \neq g$ and f is bounded, let $M = \sup|f|$ and choose $a \in G$ such that $f(a) \neq 0$ then we get from the
 238 inequality (3.15) that $|g(x)| \leq \frac{1}{2|f(a)|} (2M + \psi(a))$ for all $x \in G$, i.e. g is bounded too.

239 To get (iv) we use Theorem 5 in which we have seen that if f is unbounded then $f(e) \neq 0$,

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$$g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y), \quad x, y \in G$$

241 and

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$$\tilde{f}(xy) + \tilde{f}(x\sigma(y)) + \tilde{f}(yx) + \tilde{f}(\sigma(y)) = 4\tilde{f}(x)\tilde{f}(y), \quad \text{for all } x, y \in G.$$

243 If G is abelian or at least f is central (i.e. $f(xy) = f(yx)$ for all $x, y \in G$) then we get

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$$\tilde{f}(xy) + \tilde{f}(x\sigma(y)) = 2\tilde{f}(x)\tilde{f}(y), \tag{3.16}$$

245 for all $x, y \in G$. Dividing the two sides of the inequality (3.15) by $\alpha = f(e)$ we find that

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$$|\tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y)| \leq \frac{\varphi(x)}{|\alpha|} \quad \text{and} \quad \frac{\psi(y)}{|\alpha|}, \tag{3.17}$$

247 for all $x, y \in G$. When we substitute (3.16) into (3.17) we get that

248
$$|2\tilde{f}(y)(\tilde{f}(x) - g(x))| \leq \frac{\varphi(x)}{|\alpha|} \quad \text{and} \quad \frac{\psi(y)}{|\alpha|}, \tag{3.18}$$

249 for all $x, y \in G$. Since f is unbounded then so is $\tilde{f}$. Consequently (3.18) implies $\tilde{f} = g$ that g is a solution of (A).
 250 Substituting f by g on the second (resp. the last) Factor of the right hand side of (3.16) the expression reduces to
 251 $(E_{f,g})$ and $(E_{g,f})$.

252 **Corollary 4.** Let G be any group, and let $f, g : G \rightarrow C$ satisfy the inequality (3.15) such that $g(\sigma(x)) = g(x)$ for all
 253 $x \in G$. Then if f is unbounded, then g is unbounded, too. Moreover g is a solutions of (A) and (f, g) satisfies
 254 equations $(E_{f,g})$ and $(E_{g,f})$.

Proof. Suppose that f is unbounded and $g(\sigma(x)) = g(x)$ for all $x \in G$. By putting $x = e$ in the equality $g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y)$. It is easy to see that $g(y) = g(e)\tilde{f}(y)$. The case $g(e) = 0$ does not occur here due to our assumption that f is unbounded. Thus $\tilde{f}(y) = \frac{1}{g(e)}g(y)$. Using this equality and Theorem 5 (iii) we get

$$\begin{aligned}\tilde{f}(xy) + \tilde{f}(x\sigma(y)) &= \frac{1}{g(e)}(g(xy) + g(x\sigma(y))) \\ &= \frac{1}{2g(e)}(2g(x)\tilde{f}(y)) \\ &= 2\tilde{f}(x)\tilde{f}(y),\end{aligned}$$

and the rest of the proof runs along the same lines as in proof of Corollary 3 (iv).

Remarks.

i) In the case where G is an abelian and f, g satisfy the inequality (3.15) we know –according to Corollary 3– that if f is unbounded then g is a solutions of (A) but does not always f as shown by the example: Let $f, g : IR \rightarrow IR$ be functions with $f(x) = 3ch(x)$ and $g(x) = ch(x)$ for all $x \in IR$, and let $\sigma(x) = -x$. Then

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| = 0,$$

but f is unbounded and f does not satisfy the equation (A).

ii) Let $f, g : IR \rightarrow IR$ be functions with $f(x) = x^2 + 1$ and $g(x) = 1$ for all $x \in IR$, and let $\sigma(x) = -x$. Then

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| = 2x^2 = \varphi(x),$$

and $f(0) = 1$ but f is unbounded and f, g do not satisfy the equation

$$g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y). \quad (3.19)$$

This shows that the condition

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y), \quad x, y \in G \quad (3.20)$$

is essential in the case (iii) of Theorem 5. This example shows also that the condition (3.20) is essential in Theorem 4.

iii) Let $f, g : IR \rightarrow IR$ be functions with $f(x) = ch(x)$ and $g(x) = 1 + ch(x)$ for all $x \in IR$, and let $\sigma(x) = -x$. Then

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| = 2ch(y) = \psi(y),$$

And $f(0) = 1$ but f is unbounded and f, g do not satisfy the equation (3.19). This shows that the condition

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x), \quad x, y \in G,$$

is essential in the case (iii) of Theorem 5.

iv) The obtained results in this paper can be extended to the equation

$$f(xy) + f(x\sigma(y)) = \lambda g(x)f(y), \quad x, y \in G.$$

v) If we apply the combinaison of cases

(a) $g = f$ or $g \neq f$.

(b) $\sigma(x)$, $\sigma(x) = x$, or $\sigma(x) = -x$.

(c) $\varphi(x) = \psi(x) = c$ or $\varphi(x) \neq \psi(x) \neq c$.

(d) The group G is abelian or non abelian.

to Theorem 4 and Theorem 5, we obtain some results of the papers [1-5, 8-11, 13-18, 21-25].

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