

BERNOULLI POLYNOMIALS AND CONVOLUTION SUMS

ABSTRACT. In this paper, we study the summations $\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$ and $\sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c) + f(b+c+d) + f(c+d+a))$. Especially, for the first summation we obtain some relations related to Bernoulli polynomial.

1. INTRODUCTION

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan.

The Bernoulli polynomials $B_m(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!},$$

play an important role in different areas of mathematics, including number theory and the theory of finite differences. The Bernoulli polynomials satisfy the following well-known identities :

$$B_m(x+1) - B_m(x) = mx^{m-1}, \quad m \geq 1$$

and

$$(1) \quad \sum_{j=0}^n j^m = \frac{B_{m+1}(n+1) - B_{m+1}(0)}{m+1}, \quad m \geq 1.$$

We set $B_m = B_m(0)$. It is obvious from the way the polynomials $B_m(x)$ are constructed that all the B_m are rational numbers. It can be shown that $B_{2m+1} = 0$ for $m \geq 1$, and is alternatively positive and negative for even m .

For $n \in \mathbb{N}$ with $k \in \mathbb{N} \cup \{0\}$, we define a summation, which also appear in many areas of number theory:

$$S_k(n) = \sum_{i=1}^{n-1} i^k.$$

In 2000 Huard, Ou, Spearman and Williams proved a far reaching generalization of Liouville's Identity.

2010 *Mathematics Subject Classification.* 11A05.

Key words and phrases. Convolution sum.

Proposition 1.1. [3, Theorem 13.1] *Let $f : \mathbb{Z}^4 \rightarrow \mathbb{C}$ be such that*

$$(2) \quad f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all integers a, b, x and y . Let $n \in \mathbb{N}$. Then

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ & \quad - f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)) \\ &= \sum_{\substack{d \in \mathbb{N} \\ d|n}} \sum_{\substack{x \in \mathbb{N} \\ x < d}} (f(0, \frac{n}{d}, x, d) + f(\frac{n}{d}, 0, d, x) + f(\frac{n}{d}, \frac{n}{d}, d-x, -x) \\ & \quad - f(x, x-d, \frac{n}{d}, \frac{n}{d}) - f(x, d, 0, \frac{n}{d}) - f(d, x, \frac{n}{d}, 0)). \end{aligned}$$

In this paper we will find a special formula of the summation

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$$

and so using this, we will show some relations related to Bernoulli polynomial and the generalized Riemann zeta function :

Lemma 1.2. *Let $n \geq 3$ be a positive integer. Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) = 3 \sum_{k=1}^{n-2} (n-k-1) f(n-k).$$

Lemma 1.3. *Let $n \geq 4$ be a positive integer. Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) = \frac{3}{2} \sum_{k=1}^{n-1} (k-1)(k-2)f(k).$$

Theorem 1.4. *Let $l \in \mathbb{N}$. Then we have*

$$\begin{aligned} A(l, n) &:= \sum_{i=0}^l \binom{l}{i} \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\ & \quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m} \\ &= \frac{(l+1)(B_{l+2}(n) - B_{l+2}) - (l+2)(B_{l+1}(n) - B_{l+1})}{(l+2)(l+1)}. \end{aligned}$$

Furthermore, if $l \in \mathbb{N} \cup \{0\}$, then we obtain

$$A(l, n) = S_{l+1}(n) - S_l(n).$$

Corollary 1.5. *Let $t, l \in \mathbb{N} \cup \{0\}$. Then we have*

(a)

$$\sum_{l=0}^t A(l, n) = S_{t+1}(n) - S_0(n).$$

(b) Let $\overline{A(l, n)} = S_{t+1}(n) + S_l(n)$. So

$$\sum_{l=0}^t A(l, n) \overline{A(l, n)} = S_{t+1}^2(n) - S_0^2(n).$$

2. PROOF OF THE LEMMA 1.2, THEOREM 1.4 AND COROLLARY 1.5

Using the four basic theta functions, one is lead to the formula (see [3, p. 14])(for $k, n \in \mathbb{N}$),

$$(3) \quad \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2n \\ a,b,x,y \text{ odd}}} ((a+b)^{2k} - (a-b)^{2k}) = \sum_{\substack{m \in \mathbb{N} \\ m|n \\ \frac{n}{m} \text{ odd}}} 2^{2k} m^{2k+1}.$$

There are several arithmetical formulae due to Liouville. A classical one of Liouville very often used in elementary number theory (completely avoiding the hard analysis) is that (see [1, p. 144]) : For $n \in \mathbb{N}$ and for $f : \mathbb{Z} \rightarrow \mathbb{C}$ an even function, the identity

$$(4) \quad \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2n \\ a,b,x,y \text{ odd}}} (f(a+b) - f(a-b)) = \sum_{\substack{m \in \mathbb{N} \\ m|n \\ \frac{n}{m} \text{ odd}}} (f(2m) - f(0))$$

holds.

An another important classical formula of Liouville is [2, p. 247] : For $n \in \mathbb{N}$ and for $f : \mathbb{Z} \rightarrow \mathbb{C}$ an even function, the identity

$$(5) \quad \begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (f(a-b) - f(a+b)) \\ &= f(0) (\sigma_1(n) - \sigma_0(n)) + \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(1 + \frac{2n}{d} - d \right) f(d) - 2 \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\sum_{v=1}^d f(v) \right). \end{aligned}$$

Proof of the Lemma 1.2. We note that

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\ &= 3 \sum_{a+b+c=n} f(a+b) = 3 \sum_{k \geq 1} f(k) \sum_{\substack{a+b+c=n \\ a+b=k}} 1 \\ &= 3 \{ (n-2)f(n-1) + (n-3)f(n-2) + \cdots + (n-(n-1))f(2) \} \\ &= 3 \sum_{k=1}^{n-2} (n-k-1) f(n-k). \end{aligned}$$

□

We can see the arithmetic function $F_l(n)$ given by

$$F_l(n) := \begin{cases} 1, & l \mid n, \\ 0, & l \nmid n \end{cases}$$

for $l, n \in \mathbb{N}$ in [3, p. 32]. Using this function we introduce the Example 2.1 and the Example 2.2.

Example 2.1. *Let us consider the function $f(x) = F_2(x)$ in the Lemma 1.2. If n is even, then the left hand side of the Lemma 1.2 is*

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n \\ n \text{ even}}} (f(a+b) + f(b+c) + f(c+a)) \\ &= 3 \sum_{a+b+c=2n} F_2(a+b) = 3 \sum_{\substack{a+b+c=2n \\ 2 \mid (a+b)}} 1 \end{aligned}$$

and the right hand side of of the Lemma 1.2 is

$$\begin{aligned} & 3 \sum_{k=1}^{2n-2} (2n-k-1) f(2n-k) = 3 \sum_{k=1}^{2n-2} (2n-k-1) F_2(2n-k) \\ &= 3 \sum_{\substack{k=1 \\ 2 \mid (2n-k)}}^{2n-2} (2n-k-1) = 3 \sum_{\substack{k=1 \\ 2 \mid k}}^{2n-2} (2n-k-1) \\ &= 3 \sum_{k=1}^{n-1} (2n-2k-1) = 3(n-1)^2, \end{aligned}$$

where we use $\sum_{k=1}^n k = \frac{n(n+1)}{2}$. These facts show that

$$(6) \quad \sum_{\substack{a+b+c=2n \\ 2 \mid (a+b)}} 1 = (n-1)^2$$

Also, if n is odd, then we obtain

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n \\ n \text{ odd}}} (f(a+b) + f(b+c) + f(c+a)) \\
 &= 3 \sum_{a+b+c=2n-1} F_2(a+b) = 3 \sum_{\substack{a+b+c=2n-1 \\ 2|(a+b)}} 1 \\
 (7) \quad &= 3 \sum_{k=1}^{2n-3} (2n-2-k) F_2(2n-1-k) = 3 \sum_{\substack{k=1 \\ 2|(2n-1-k)}}^{2n-3} (2n-2-k) \\
 &= 3 \sum_{\substack{k=1 \\ 2 \nmid k}}^{2n-3} (2n-2-k) = 3 \left\{ \sum_{k=1}^{2n-3} (2n-2-k) - \sum_{\substack{k=1 \\ 2|k}}^{2n-3} (2n-2-k) \right\} \\
 &= 3 \left\{ \sum_{k=1}^{2n-3} (2n-2-k) - \sum_{k=1}^{n-2} (2n-2-2k) \right\} = 3(n-1)^2.
 \end{aligned}$$

Therefore, by (6) and (7), we conclude that

$$\sum_{\substack{a+b+c=n \\ 2|(a+b)}} 1 = (n-1)^2.$$

We can prove the Lemma 1.3 by the similar manner to the Lemma 1.2 as follows :

Proof of the Lemma 1.3. We observe that

$$\begin{aligned}
 & \sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) \\
 &= 3 \sum_{a+b+c+d=n} f(a+b+c) = 3 \sum_{k \geq 1} f(k) \sum_{\substack{a+b+c+d=n \\ a+b+c=k}} 1 \\
 &= 3 \sum_{k=3}^{n-1} f(k) \{1 + 2 + \dots + (k-2)\} = 3 \sum_{k=3}^{n-1} f(k) \cdot \frac{(k-1)(k-2)}{2} \\
 &= \frac{3}{2} \sum_{k=1}^{n-1} (k-1)(k-2)f(k).
 \end{aligned}$$

□

Example 2.2. We apply the function $f(x) = F_2(x)$ in the Lemma 1.3. If n is even, then the left hand side of the Lemma 1.3 is

$$\begin{aligned}
 & \sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n \\ n \text{ even}}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) \\
 &= 3 \sum_{a+b+c+d=2n} F_2(a+b+c) = 3 \sum_{\substack{a+b+c+d=2n \\ 2|(a+b+c)}} 1
 \end{aligned}$$

and the right hand side of of the Lemma 1.3 is

$$\begin{aligned}
 & \frac{3}{2} \sum_{k=1}^{2n-1} (k-1)(k-2)f(k) = \frac{3}{2} \sum_{k=1}^{2n-1} (k-1)(k-2)F_2(k) \\
 &= \frac{3}{2} \sum_{\substack{k=1 \\ 2|k}}^{2n-1} (k-1)(k-2) = \frac{3}{2} \sum_{k=1}^{n-1} (2k-1)(2k-2) \\
 &= \frac{3}{2} \sum_{k=1}^{n-1} \{4k^2 - 6k + 2\} = \frac{1}{2} \{4n^3 - 15n^2 + 17n - 6\},
 \end{aligned}$$

where we use $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$. These facts lead us to

$$(8) \quad \sum_{\substack{a+b+c+d=2n \\ 2|(a+b+c)}} 1 = \frac{1}{6} \{4n^3 - 15n^2 + 17n - 6\}.$$

Similarly, if n is odd, then we have

$$\begin{aligned}
 & \sum_{\substack{(a,b,c,d) \in \mathbb{N}^4 \\ a+b+c+d=n \\ n \text{ odd}}} (f(a+b+c) + f(b+c+d) + f(c+d+a)) \\
 &= 3 \sum_{a+b+c+d=2n-1} F_2(a+b+c) = 3 \sum_{\substack{a+b+c+d=2n-1 \\ 2|(a+b+c)}} 1 \\
 (9) \quad &= \frac{3}{2} \sum_{k=1}^{2n-2} (k-1)(k-2)F_2(k) = \frac{3}{2} \sum_{\substack{k=1 \\ 2|k}}^{2n-2} (k-1)(k-2) \\
 &= \frac{3}{2} \sum_{k=1}^{n-1} (2k-1)(2k-2) = \frac{1}{2} \{4n^3 - 15n^2 + 17n - 6\}.
 \end{aligned}$$

Therefore, by (8) and (9), we conclude that

$$\sum_{\substack{a+b+c+d=n \\ 2|(a+b+c)}} 1 = \frac{1}{6} \{4n^3 - 15n^2 + 17n - 6\}.$$

Proof of the Theorem 1.4. Let $l \in \mathbb{N} \cup \{0\}$. We apply $f(x) = x^l$ in Lemma 1.2. Then the left hand side is

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\
 &= 3 \sum_{a+b+c=n} f(a+b) = 3 \sum_{a+b+c=n} (a+b)^l \\
 (10) \quad &= 3 \sum_{a+b+c=n} \sum_{i=0}^l \binom{l}{i} a^{l-i} b^i \\
 &= 3 \sum_{i=0}^l \binom{l}{i} \sum_{a+b+c=n} a^{l-i} b^i c^0.
 \end{aligned}$$

Here, we can observe that

$$\begin{aligned}
 & \sum_{a+b+c=n} a^{l-i} b^i c^0 \\
 (11) \quad &= \sum_{a+b=n-1} a^{l-i} b^i 1^0 + \sum_{a+b=n-2} a^{l-i} b^i 2^0 + \cdots + \sum_{a+b=2} a^{l-i} b^i (n-2)^0 \\
 &= \sum_{a+b=n-1} a^{l-i} b^i + \sum_{a+b=n-2} a^{l-i} b^i + \cdots + \sum_{a+b=2} a^{l-i} b^i.
 \end{aligned}$$

Then we can calculate the first term of (11) as :

$$\begin{aligned}
 & \sum_{a+b=n-1} a^{l-i} b^i = \sum_{a=1}^{n-2} a^{l-i} \{(n-1)-a\}^i \\
 &= \sum_{a=1}^{n-2} a^{l-i} \sum_{j=0}^i \binom{i}{j} (n-1)^{i-j} (-1)^j a^j \\
 &= \sum_{j=0}^i \binom{i}{j} (n-1)^{i-j} (-1)^j \sum_{a=1}^{n-2} a^{l-i+j} \\
 &= \sum_{j=0}^i \binom{i}{j} (n-1)^{i-j} (-1)^j \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^m \binom{l+1-i+j}{m} \\
 & \quad \times B_m (n-2)^{l+1-i+j-m} \\
 &= \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
 & \quad \times (n-1)^{i-j} (n-2)^{l+1-i+j-m},
 \end{aligned}$$

where we use the Faulhaber's sum as

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p (-1)^j \binom{p+1}{j} B_j n^{p+1-j}$$

with the Bernoulli number B_j . And we can apply the same method into the other terms of (11). So (11) becomes

$$\begin{aligned}
 & \sum_{a+b+c=n} a^{l-i} b^i c^0 \\
 &= \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
 & \quad \times \left\{ (n-1)^{i-j} (n-2)^{l+1-i+j-m} + (n-2)^{i-j} (n-3)^{l+1-i+j-m} \right. \\
 & \quad \left. + \cdots + (n-(n-2))^{i-j} (n-(n-1))^{l+1-i+j-m} \right\} \\
 &= \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
 & \quad \times \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m}.
 \end{aligned}$$

So Eq. (10) can be written as

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\
 (12) \quad &= 3 \sum_{i=0}^l \binom{l}{i} \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\
 & \quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m}.
 \end{aligned}$$

And the right hand side of Lemma 1.2 is

$$\begin{aligned}
 & 3 \sum_{k=1}^{n-2} (n-k-1)(n-k)^l \\
 &= 3 \sum_{k=1}^{n-2} \{(n-k)^{l+1} - (n-k)^l\} \\
 (13) \quad &= 3 \left[\{(n-1)^{l+1} - (n-1)^l\} + \{(n-2)^{l+1} - (n-2)^l\} \right. \\
 & \quad \left. + \cdots + \{3^{l+1} - 3^l\} + \{2^{l+1} - 2^l\} \right] \\
 &= 3 \left\{ \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \right\}.
 \end{aligned}$$

Equating (12) and (13), we obtain

$$\begin{aligned}
 (14) \quad A(l, n) &:= \sum_{i=0}^l \binom{l}{i} \sum_{j=0}^i \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\
 &\quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m} \\
 &= \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l.
 \end{aligned}$$

For $l \in \mathbb{N} \cup \{0\}$, we can write $A(l, n) = S_{l+1}(n) - S_l(n)$.

Now, let $l \in \mathbb{N}$. Then, by (1) and $B_l = B_l(0)$, the right hand side of (14) is

$$\begin{aligned}
 (15) \quad &\sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \\
 &= \frac{B_{l+2}(n) - B_{l+2}(0)}{l+2} - \frac{B_{l+1}(n) - B_{l+1}(0)}{l+1} \\
 &= \frac{(l+1)(B_{l+2}(n) - B_{l+2}) - (l+2)(B_{l+1}(n) - B_{l+1})}{(l+2)(l+1)},
 \end{aligned}$$

so that we have

$$A(l, n) = \frac{(l+1)(B_{l+2}(n) - B_{l+2}) - (l+2)(B_{l+1}(n) - B_{l+1})}{(l+2)(l+1)}.$$

□

Remark 2.3. It is well-known that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and the generalized Riemann zeta function $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}$. We can change Eq. (13) as

$$\begin{aligned}
 (16) \quad &3 \sum_{k=1}^{n-2} (n-k-1)(n-k)^l \\
 &= 3 \sum_{k=1}^{n-2} \{(n-k)^{l+1} - (n-k)^l\} \\
 &= 3 \{\zeta(-l, n) - \zeta(-l-1, n) - \zeta(-l) + \zeta(-l-1)\},
 \end{aligned}$$

since

$$\begin{aligned}
 &\sum_{k=1}^{n-2} \{(n-k)^{l+1} - (n-k)^l\} \\
 &= \{1^{l+1} + 2^{l+1} + 3^{l+1} + \cdots + (n-2)^{l+1} + (n-1)^{l+1}\} \\
 &\quad - \{1^l + 2^l + 3^l + \cdots + (n-2)^l + (n-1)^l\}
 \end{aligned}$$

with

$$\begin{aligned}\zeta(-l-1) - \zeta(-l-1, n) &= \sum_{n=1}^{\infty} n^{l+1} - \sum_{k=0}^{\infty} (k+n)^{l+1} \\ &= \{1^{l+1} + 2^{l+1} + \dots + (n-1)^{l+1} + n^{l+1} + (n+1)^{l+1} + \dots\} \\ &\quad - \{n^{l+1} + (n+1)^{l+1} + (n+2)^{l+1} + (n+3)^{l+1} + \dots\} \\ &= 1^{l+1} + 2^{l+1} + 3^{l+1} + \dots + (n-2)^{l+1} + (n-1)^{l+1}\end{aligned}$$

and

$$\begin{aligned}\zeta(-l) - \zeta(-l, n) &= \sum_{n=1}^{\infty} n^l - \sum_{k=0}^{\infty} (k+n)^l \\ &= 1^l + 2^l + 3^l + \dots + (n-2)^l + (n-1)^l.\end{aligned}$$

Thus we can rewrite (14) as

$$A(l, n) = \zeta(-l, n) - \zeta(-l-1, n) - \zeta(-l) + \zeta(-l-1).$$

Example 2.4. If $l = 3$ in the Theorem 1.4, then we have

$$A(3, n) = \frac{1}{60}n(12n^4 - 45n^3 + 50n^2 - 15n - 2).$$

Proof of the Corollary 1.5. (a) From the Theorem 1.4, we obtain

$$\begin{aligned}\sum_{l=0}^t A(l, n) &= \sum_{l=0}^t \left\{ \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \right\} = \sum_{k=1}^{n-1} \sum_{l=0}^t \{k^{l+1} - k^l\} \\ &= \sum_{k=1}^{n-1} \{(k-1) + (k^2 - k) + (k^3 - k^2) + \dots + (k^t - k^{t-1}) + (k^{t+1} - k^t)\} \\ &= \sum_{k=1}^{n-1} \{k^{t+1} - 1\} = S_{t+1}(n) - S_0(n).\end{aligned}$$

(b) By the definition of $\overline{A(l, n)} = S_{l+1}(n) + S_l(n)$, we have

$$\begin{aligned}\sum_{l=0}^t A(l, n) \overline{A(l, n)} &= \sum_{l=0}^t \{S_{l+1}^2(n) - S_l^2(n)\} \\ &= \sum_{l=0}^t \{(S_1^2(n) - S_0^2(n)) + (S_2^2(n) - S_1^2(n)) + (S_3^2(n) - S_2^2(n)) + \dots \\ &\quad + (S_t^2(n) - S_{t-1}^2(n)) + (S_{t+1}^2(n) - S_t^2(n))\} \\ &= \sum_{l=0}^t \{S_{t+1}^2(n) - S_0^2(n)\}.\end{aligned}$$

□

REFERENCES

- [1] J. Liouville, *Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres*, J. Math. Pures Appl., **3** (1858), 143–152.
- [2] J. Liouville, *Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres*, J. Math. Pures Appl., **3** (1858), 241–250.
- [3] K. S. Williams, *Number Theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, (2011).