

THE BINOMIAL AND MULTINOMIAL COMBINATORIAL CONVOLUTION SUMS

ABSTRACT. In [1] and [3] we can find some formulas of binomial combinatorial convolution sums. And starting from these formulas, we obtain various binomial and multinomial combinatorial convolution sums.

1. INTRODUCTION

Let $\mathbb{N}$ denote the set of positive integers. Further, let $N, d \in \mathbb{N}$, where $s, r, k \in \mathbb{N} \cup \{0\}$. Throughout this paper, we define

$$\begin{aligned}\sigma_s(N) &:= \sum_{d|N} d^s, & \sigma_{s,r}(N; k) &:= \sum_{\substack{d|N \\ d \equiv r \pmod k}} d^s, \\ \sigma_s^*(N) &:= \sum_{\substack{d|N \\ N/d \text{ odd}}} d^s, & (a)_N &:= (a)(a-1) \cdots (a-N+1).\end{aligned}$$

We note that

$$(1) \quad \sigma_s(N) = \sigma_{s,1}(N; 2) + \sigma_{s,0}(N; 2)$$

and

$$(2) \quad \sigma_{s,1}(2N; 2) = \sigma_{s,1}(N; 2), \quad \sigma_{s,0}(2N; 2) = 2^s \sigma_s(N).$$

Moreover we can deduce that the property

$$(3) \quad \sigma_s^*(2N) = 2^s \sigma_s^*(N).$$

For example, we can see that

$$\sum_{m=1}^{N-1} \sigma_1^*(m) \sigma_1^*(N-m) = \frac{1}{4} (\sigma_3^*(N) - N \sigma_1^*(N))$$

and

$$\sum_{m=1}^{N-1} \sigma_1^*(m) \sigma_3^*(N-m) = \frac{1}{16} (\sigma_5^*(N) - N \sigma_3^*(N))$$

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in [2]. We introduce the very important Proposition 1.1, which is the starting point of our paper.

Proposition 1.1. (See [3, Theorem 12.3]) *Let $k, N \in \mathbb{N}$. Then*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \right) \\ &= \frac{2k+3}{4k+2} \sigma_{2k+1}(N) + \left(\frac{k}{6} - N \right) \sigma_{2k-1}(N) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N). \end{aligned}$$

In Section 2 we obtain some results about the binomial combinatoric convolution sums similar to the Proposition 1.1. For instance we have

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m)$$

and

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m)$$

(see (10) and (11)). Similarly, getting an idea from the Proposition 1.1, we obtain some various multinomial combinatorial convolution sums as follows :

Theorem 1.2. *Let N be an even integer with $l \in \mathbb{N}$. Then for odd positive integers a, b, c we have*

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} a \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \frac{(2l+1)N}{32} \{ \sigma_{2l+1}^*(N) - 2N \sigma_{2l-1}^*(N) \}. \end{aligned}$$

Theorem 1.3. *Let $l, N \in \mathbb{N}$ with $4|N$. Then for odd positive integers a, b, c, d, e, f , we have*

$$\begin{aligned} & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} (a+b-1)(a+b)(c+d)(e+f) \binom{2l}{a,b,c,d,e,f} \\ & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\ & \times \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \sigma_d^*(m_4) \sigma_e^*(m_5) \sigma_f^*(m_6) \\ &= 2^{-9} (2l)_4 N \{ \sigma_{2l-3}^*(N) - 4N \sigma_{2l-5}^*(N) \}. \end{aligned}$$

Theorem 1.4. *Let $l, N \in \mathbb{N}$ with $8|N$ and let the condition*

$$C := \left\{ (m_1, m_2, \dots, m_{11}, m_{12}) \in \mathbb{N}^{12} \mid \sum_{i=1}^{12} m_i = N, \quad \begin{aligned} &2|(m_1 + m_2), \quad 2|(m_3 + m_4), \\ &2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), \\ &4|(m_1 + m_2 + m_3 + m_4), \quad 4|(m_5 + m_6 + m_7 + m_8), \\ &8|(m_9 + m_{10} + m_{11} + m_{12}) \end{aligned} \right\}.$$

Then for odd positive integers a_i , we have

$$\begin{aligned} &\sum_{\substack{\sum_{i=1}^{12} a_i = 2l \\ a_i \text{ odd}}} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\ &\quad \times (a_1 + a_2 + a_3 + a_4 - 3)(a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2) \\ &\quad \times (a_9 + a_{10} + a_{11} + a_{12} - 2) \binom{2l}{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}} \\ &\quad \times \sum_C \frac{(-1)^{m_1 + m_3 + m_5 + m_7 + m_9 + m_{11}}}{(m_1 + m_2)(m_3 + m_4)(m_5 + m_6)(m_7 + m_8)(m_9 + m_{10})(m_{11} + m_{12})} \\ &\quad \times \frac{(-1)^{\frac{m_1 + m_2 + m_3 + m_4}{4} + \frac{m_1 + m_2}{2} + \frac{m_5 + m_6}{2} + \frac{m_9 + m_{10}}{2}}}{(m_1 + m_2 + m_3 + m_4)(m_5 + m_6 + m_7 + m_8)(m_9 + m_{10} + m_{11} + m_{12})} \\ &\quad \times \prod_{i=1}^{12} \sigma_{a_i}^*(m_i) \\ &= 2^{-16} (2l)_{10} N \{ \sigma_{2l-9}^*(N) - 8N \sigma_{2l-11}^*(N) \}. \end{aligned}$$

Theorem 1.5. *Let $l, N \in \mathbb{N}$ with $16|N$ and let the condition*

$$D := \left\{ (m_1, m_2, \dots, m_{23}, m_{24}) \in \mathbb{N}^{24} \mid \sum_{i=1}^{24} m_i = N, \quad \begin{aligned} &2|(m_1 + m_2), \quad 2|(m_3 + m_4), \\ &2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), \\ &2|(m_{13} + m_{14}), \quad 2|(m_{15} + m_{16}), \quad 2|(m_{17} + m_{18}), \quad 2|(m_{19} + m_{20}), \\ &2|(m_{21} + m_{22}), \quad 2|(m_{23} + m_{24}), \quad 4|(m_1 + m_2 + m_3 + m_4), \\ &4|(m_5 + m_6 + m_7 + m_8), \quad 4|(m_9 + m_{10} + m_{11} + m_{12}), \\ &4|(m_{13} + m_{14} + m_{15} + m_{16}), \quad 4|(m_{17} + m_{18} + m_{19} + m_{20}), \\ &4|(m_{21} + m_{22} + m_{23} + m_{24}), \quad 8|(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8), \\ &8|(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16}), \\ &16|(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24}) \end{aligned} \right\}.$$

Then for odd positive integers a_i , we have

$$\begin{aligned}
 (4) \quad & \sum_{\substack{\sum_{i=1}^{24} a_i = 2l \\ a_i \text{ odd}}} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \times (a_{13} + a_{14})(a_{15} + a_{16})(a_{17} + a_{18})(a_{19} + a_{20})(a_{21} + a_{22})(a_{23} + a_{24}) \\
 & \times (a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2)(a_9 + a_{10} + a_{11} + a_{12} - 2) \\
 & \times (a_{13} + a_{14} + a_{15} + a_{16} - 2)(a_{17} + a_{18} + a_{19} + a_{20} - 2)(a_{21} + a_{22} + a_{23} + a_{24} - 2) \\
 & \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 7) \\
 & \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 6) \\
 & \times (a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} - 6) \\
 & \times (a_{17} + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} - 6) \\
 & \times \binom{2l}{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23}, a_{24}} \\
 & \times \sum_D (-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}+m_{13}+m_{15}+m_{17}+m_{19}+m_{21}+m_{23}} \\
 & \times (-1)^{\frac{m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8}{8} + \frac{m_1+m_2+m_3+m_4}{4}} \\
 & \times (-1)^{\frac{m_9+m_{10}+m_{11}+m_{12}}{4} + \frac{m_{17}+m_{18}+m_{19}+m_{20}}{4}} \\
 & \times (-1)^{\frac{m_1+m_2+m_5+m_6+m_9+m_{10}+m_{13}+m_{14}+m_{17}+m_{18}+m_{21}+m_{22}}{2}} \\
 & \times \frac{1}{(m_1 + m_2)(m_3 + m_4)(m_5 + m_6)(m_7 + m_8)(m_9 + m_{10})(m_{11} + m_{12})} \\
 & \times \frac{1}{(m_{13} + m_{14})(m_{15} + m_{16})(m_{17} + m_{18})(m_{19} + m_{20})(m_{21} + m_{22})(m_{23} + m_{24})} \\
 & \times \frac{1}{(m_1 + m_2 + m_3 + m_4)(m_5 + m_6 + m_7 + m_8)(m_9 + m_{10} + m_{11} + m_{12})} \\
 & \times \frac{1}{(m_{13} + m_{14} + m_{15} + m_{16})(m_{17} + m_{18} + m_{19} + m_{20})(m_{21} + m_{22} + m_{23} + m_{24})} \\
 & \times \frac{1}{(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8)} \\
 & \times \frac{1}{(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16})} \\
 & \times \frac{1}{(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24})} \prod_{i=1}^{24} \sigma_{a_i}^*(m_i) \\
 & = 2^{-29} (2l)_{22} N \{ \sigma_{2l-21}^*(N) - 16N \sigma_{2l-23}^*(N) \}.
 \end{aligned}$$

2. THE BINOMIAL COMBINATORIAL CONVOLUTION SUMS

To prove the Lemma 2.2 we need the Proposition 2.1.

Proposition 2.1. [3, p. 112] *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Let $N \in \mathbb{N}$. Then we obtain*

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} (f(a-b) - f(a+b)) \\ &= f(0) (\sigma_1^*(N) - \sigma_0^*(N)) - \frac{1}{2} \sum_{\substack{d \in \mathbb{N} \\ d|N \\ d \text{ even}}} df(d) - 2 \sum_{\substack{d \in \mathbb{N} \\ d|N \\ d \text{ odd}}} \sum_{\substack{t \in \mathbb{N} \\ 1 \leq t < d \\ t \text{ even}}} f(t). \end{aligned}$$

Lemma 2.2. *Let $k, N \in \mathbb{N}$. Then we have*

$$\begin{aligned} (a) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) = 2^{2k-1} \sigma_{2k+1}\left(\frac{N}{2}\right) \\ & + \frac{1}{4k+2} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} \binom{2k+1}{j} \binom{2k+1-j}{r} 2^j B_j \sigma_{2k+1-j-r,1}(N; 2). \\ (b) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) = 2^{2(k-1)} N \sigma_{2k+1}\left(\frac{N}{2}\right) \\ & + \frac{N}{8k+4} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} \binom{2k+1}{j} \binom{2k+1-j}{r} 2^j B_j \sigma_{2k+1-j-r,1}(N; 2). \end{aligned}$$

Proof. (a) We apply $f(x) = x^{2k}$ in Proposition 2.1. Then the left hand side is

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} (f(a-b) - f(a+b)) = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} ((a-b)^{2k} - (a+b)^{2k}) \\ &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} \left(\sum_{r=0}^{2k} \binom{2k}{r} (-1)^r a^{2k-r} b^r - \sum_{r=0}^{2k} \binom{2k}{r} a^{2k-r} b^r \right) \\ (5) \quad &= -2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} \sum_{\substack{r=0 \\ r \text{ odd}}}^{2k} \binom{2k}{r} a^{2k-r} b^r \\ &= -2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} a^{2k-2s-1} b^{2s+1}. \end{aligned}$$

Here we can observe that

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} a^{2k-2s-1} b^{2s+1} &= \sum_{m=1}^{N-1} \left(\sum_{\substack{a|m \\ a \text{ odd}}} a^{2k-2s-1} \right) \left(\sum_{\substack{b|N-m \\ b \text{ odd}}} b^{2s+1} \right) \\ &= \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(N-m;2). \end{aligned}$$

So (5) becomes

$$(6) \quad -2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(N-m;2).$$

And the right hand side of Proposition 2.1 is

$$(7) \quad -\frac{1}{2} \sum_{\substack{d|N \\ d \text{ even}}} d^{2k+1} - 2 \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{\substack{1 \leq t < d \\ t \text{ even}}} t^{2k}.$$

Then we have

$$-\frac{1}{2} \sum_{\substack{d|N \\ d \text{ even}}} d^{2k+1} = -\frac{1}{2} \sum_{2l|N} (2l)^{2k+1} = -2^{2k} \sum_{l|\frac{N}{2}} l^{2k+1} = -2^{2k} \sigma_{2k+1}\left(\frac{N}{2}\right)$$

and

$$\begin{aligned} -2 \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{\substack{1 \leq t < d \\ t \text{ even}}} t^{2k} &= -2 \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{l=1}^{\frac{d-1}{2}} (2l)^{2k} = -2^{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{l=1}^{\frac{d-1}{2}} l^{2k} \\ &= -2^{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} \frac{1}{2k+1} \sum_{j=0}^{2k} (-1)^j \binom{2k+1}{j} B_j \left(\frac{d-1}{2}\right)^{2k+1-j} \\ &= \sum_{j=0}^{2k} (-1)^{j+1} \binom{2k+1}{j} \frac{2^j B_j}{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} (d-1)^{2k+1-j} \\ &= \sum_{j=0}^{2k} (-1)^{j+1} \binom{2k+1}{j} \frac{2^j B_j}{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{r=0}^{2k+1-j} \binom{2k+1-j}{r} (-1)^r d^{2k+1-j-r} \\ &= \sum_{j=0}^{2k} (-1)^{j+1} \binom{2k+1}{j} \frac{2^j B_j}{2k+1} \sum_{r=0}^{2k+1-j} \binom{2k+1-j}{r} (-1)^r \sigma_{2k+1-j-r,1}(N;2) \\ &= \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r+1} \binom{2k+1}{j} \binom{2k+1-j}{r} \frac{2^j B_j}{2k+1} \sigma_{2k+1-j-r,1}(N;2). \end{aligned}$$

Therefore (7) becomes

$$(8) \quad -2^{2k} \sigma_{2k+1}\left(\frac{N}{2}\right) + \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r+1} \binom{2k+1}{j} \binom{2k+1-j}{r} \frac{2^j B_j}{2k+1} \sigma_{2k+1-j-r,1}(N; 2).$$

Equating (6) and (8), we obtain

$$(9) \quad \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) = 2^{2k-1} \sigma_{2k+1}\left(\frac{N}{2}\right) + \frac{1}{4k+2} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} \binom{2k+1}{j} \binom{2k+1-j}{r} 2^j B_j \sigma_{2k+1-j-r,1}(N; 2).$$

(b) We have

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) \\ &= \frac{N}{2} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(N-m; 2) \sigma_{2s+1,1}(m; 2), \end{aligned}$$

since

$$\begin{aligned} & \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) \\ &= \sum_{m=1}^{N-1} (N-m) \sigma_{2k-2s-1,1}(N-m; 2) \sigma_{2s+1,1}(m; 2). \end{aligned}$$

Therefore, we use the Lemma 2.2 (a). □

Lemma 2.3. *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= 2^{2k} \left\{ \frac{2k+3}{4k+2} \sigma_{2k+1}\left(\frac{N}{2}\right) + \left(\frac{k}{6} - \frac{N}{2}\right) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\ & \quad \left. + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= 2^{2k-1} N \left\{ \frac{2k+3}{4k+2} \sigma_{2k+1}\left(\frac{N}{2}\right) + \left(\frac{k}{6} - \frac{N}{2}\right) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\ & \quad \left. + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \right\}. \end{aligned}$$

Proof. (a) From (2) we obtain

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} 2^{2k-2s-1} \sigma_{2k-2s-1}\left(\frac{m}{2}\right) \cdot 2^{2s+1} \sigma_{2s+1}\left(\frac{N-m}{2}\right) \\ &= 2^{2k} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}\left(\frac{N}{2} - m\right). \end{aligned}$$

So we use the Proposition 1.1.

(b) We obtain

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= \frac{N}{2} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(N-m; 2) \sigma_{2s+1,0}(m; 2), \end{aligned}$$

since

$$\begin{aligned} & \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= \sum_{m=1}^{N-1} (N-m) \sigma_{2k-2s-1,0}(N-m; 2) \sigma_{2s+1,0}(m; 2). \end{aligned}$$

Therefore, we refer to the Lemma 2.3 (a). □

Corollary 2.4. *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m; 2) \\
 &= \frac{1}{24k+12} \left[(2k+1)(k-6N) \sigma_{2k-1}(N) + 4^k (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\
 & \quad + 3 \left\{ 2^{2k+1} \sigma_{2k+1}\left(\frac{N}{2}\right) + (2k+3) \sigma_{2k+1}(N) \right. \\
 & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) + 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) \\
 & \quad \left. \left. - \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N; 2) \right\} \right].
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,1}(N-m; 2) \\
 &= \frac{1}{24k+12} \left[(2k+1)(k-6N) \sigma_{2k-1}(N) - 4^k (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\
 & \quad - 3 \left\{ 2^{2(k+1)} (k+1) \sigma_{2k+1}\left(\frac{N}{2}\right) - (2k+3) \sigma_{2k+1}(N) \right. \\
 & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) - 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) \\
 & \quad \left. \left. + \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N; 2) \right\} \right].
 \end{aligned}$$

Proof. (a) From (1), we can write

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(N-m;2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \{ \sigma_{2k-2s-1}(m) - \sigma_{2k-2s-1,0}(m;2) \} \\
&\quad \times \{ \sigma_{2s+1}(N-m) - \sigma_{2s+1,0}(N-m;2) \} \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \\
&\quad - 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m;2) \\
&\quad + \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2) \sigma_{2s+1,0}(N-m;2),
\end{aligned}$$

since

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m;2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2) \sigma_{2s+1}(N-m).
\end{aligned}$$

Thus we use the Proposition 1.1, the Lemma 2.2 (a), and the Lemma 2.3 (a).

(b) Since

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,0}(N-m;2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m;2) \\
&\quad - \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2) \sigma_{2s+1,0}(N-m;2),
\end{aligned}$$

therefore we refer to the Lemma 2.3 (a) and the Corollary 2.4 (a).

□

Remark 2.5. We can rewrite the Corollary 2.4 (a) and (b) as follows :

$$\begin{aligned}
 (10) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \cdot 2^{2s+1} \sigma_{2s+1}\left(\frac{N-m}{2}\right) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (11) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m).
 \end{aligned}$$

Corollary 2.6. *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\
 &= \frac{2^{2k-1}}{3(2k+1)} \left\{ (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) + (6k+9) \sigma_{2k+1}\left(\frac{N}{2}\right) \right. \\
 & \quad \left. + 6 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \right\}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2) \\
 &= \frac{4^k}{24k+12} \left[(2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) + 4^k (2k+1) \left(k - \frac{3}{2}N\right) \sigma_{2k-1}\left(\frac{N}{4}\right) \right. \\
 & \quad + 3 \left\{ 2^{2k+1} \sigma_{2k+1}\left(\frac{N}{4}\right) + (2k+3) \sigma_{2k+1}\left(\frac{N}{2}\right) \right. \\
 & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{4}\right) + 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \\
 & \quad \left. \left. - \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}\left(\frac{N}{2}; 2\right) \right\} \right].
 \end{aligned}$$

(c)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,1}(m; 2) \\
 &= \frac{1}{48k+24} \left[16^k \{2k(3N-1) + 3N - 4k^2\} \sigma_{2k-1}\left(\frac{N}{4}\right) \right. \\
 & \quad + 2^{2k+1} \left\{ (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) - 3 \cdot 2^{2k+1} \sigma_{2k+1}\left(\frac{N}{4}\right) + 3(2k+3) \sigma_{2k+1}\left(\frac{N}{2}\right) \right. \\
 & \quad - 3 \cdot 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{4}\right) + 6 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \\
 & \quad \left. \left. + 3 \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}\left(\frac{N}{2}; 2\right) \right\} \right].
 \end{aligned}$$

Proof. (a) We can see that

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m) \\
 & \quad - \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m).
 \end{aligned}$$

Thus we refer to (10) and (11).

(b) Since

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2) \\
 &= 2^{2k} \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}\left(\frac{N}{2} - m\right) \sigma_{2s+1}\left(\frac{m}{2}\right) \\
 &= 2^{2k} \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{4}} \sigma_{2k-2s-1}\left(\frac{N}{2} - 2m\right) \sigma_{2s+1}(m),
 \end{aligned}$$

therefore we use (10).

(c) We note that

$$\begin{aligned} & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,1}(m; 2) \\ &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\ & \quad - \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2). \end{aligned}$$

So we apply the Corollary 2.6 (a) and (b). □

3. THE MULTINOMIAL COMBINATORIAL CONVOLUTION SUMS

Proof of the Theorem 1.2. Let us consider

$$\begin{aligned} (12) \quad & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} (a+b) \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} \frac{(2l+1)!}{(a+b-1)!c!} \frac{(a+b)!}{a!b!} \\ & \quad \times \sum_{M < \frac{N}{2}} \left(\sum_{m_1+m_2=2M} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \right) \sigma_c^*(N-2M) \\ &= \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} \frac{(2l+1)!}{(a+b-1)!c!} \\ & \quad \times \sum_{M < \frac{N}{2}} \left(\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m_1=1}^{2M-1} (-1)^{m_1+1} \sigma_{2k-2s-1}^*(m_1) \sigma_{2s+1}^*(2M-m_1) \right) \sigma_c^*(N-2M), \end{aligned}$$

where we put $a+b=2k$ and $b=2s+1$. Note that $k \geq 1$. Since

$$(13) \quad \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} (-1)^{m+1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(2N-m) = N \sigma_{2k-1}^*(2N)$$

in [1, (16)], so by (3) we can write (12) as

$$\begin{aligned}
 (14) \quad & \sum_{\substack{2k+c=2l+1 \\ c \text{ odd}}} \frac{(2l+1)!}{(2k-1)!c!} \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(2M) \sigma_c^*(N-2M) \\
 &= \sum_{k=1}^l \binom{2l}{2k-1} (2l+1) \sum_{M < \frac{N}{2}} M \cdot 2^{2k-1} \sigma_{2k-1}^*(M) \cdot 2^{2l+1-2k} \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= 2^{2l} (2l+1) \sum_{k=1}^l \binom{2l}{2k-1} \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right).
 \end{aligned}$$

Then

$$\begin{aligned}
 (15) \quad & \sum_{k=1}^l \binom{2l}{2k-1} \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= \frac{N/2}{2} \sum_{k=1}^l \binom{2l}{2k-1} \sum_{M < \frac{N}{2}} \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right)
 \end{aligned}$$

because

$$\begin{aligned}
 & \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= \sum_{M < \frac{N}{2}} \left(\frac{N}{2} - M\right) \sigma_{2k-1}^*\left(\frac{N}{2} - M\right) \sigma_{2l+1-2k}^*(M).
 \end{aligned}$$

Thus, by the above fact and

$$\begin{aligned}
 (16) \quad & \sum_{k=0}^{l-1} \binom{2l}{2k+1} \sum_{m=1}^{N-1} \sigma_{2l-2k-1}^*(m) \sigma_{2k+1}^*(N-m) \\
 &= \frac{1}{2} \left\{ \sigma_{2l+1}^*(N) - N \sigma_{2l-1}^*(N) \right\}
 \end{aligned}$$

in [1, (10)], we have (14) as

$$\begin{aligned}
 & 2^{2l} (2l+1) \cdot \frac{N}{4} \sum_{k=0}^{l-1} \binom{2l}{2k+1} \sum_{M < \frac{N}{2}} \sigma_{2k+1}^*(M) \sigma_{2l-1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= \frac{2^{2l} (2l+1) N}{4} \cdot \frac{1}{2} \left\{ \sigma_{2l+1}^*\left(\frac{N}{2}\right) - \frac{N}{2} \sigma_{2l-1}^*\left(\frac{N}{2}\right) \right\}.
 \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} (a+b) \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \frac{(2l+1)N}{16} \{ \sigma_{2l+1}^*(N) - 2N \sigma_{2l-1}^*(N) \}. \end{aligned}$$

And noting that

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} (a+b) \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} 2a \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3), \end{aligned}$$

so, finally, we have

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} a \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \frac{(2l+1)N}{32} \{ \sigma_{2l+1}^*(N) - 2N \sigma_{2l-1}^*(N) \}. \end{aligned}$$

□

Proof of the Theorem 1.3. First since $4|N$, we have $\frac{N}{2}$ is even. Let $a+b := 2k_1$, $c+d := 2k_2$, and $e+f := 2k_3$. And from the conditions $2|(m_1+m_2)$, $2|(m_3+m_4)$, and $4|(m_5+m_6)$, we can write $m_1+m_2 = 2N_1$, $m_3+m_4 = 2N_2$, and $m_5+m_6 = 2N_3$, respectively. Note that N_3 is even. Then using these facts and (13), the problem can be written as

$$\begin{aligned}
 (17) \quad & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} (a+b-1)(a+b)(c+d)(e+f) \binom{2l}{a,b,c,d,e,f} \\
 & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5+4}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\
 & \times \sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)\sigma_e^*(m_5)\sigma_f^*(m_6) \\
 = & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} \frac{(a+b-1)(2l)!}{(a+b-1)!(c+d-1)!(e+f-1)!} \cdot \frac{(a+b)!}{a!b!} \cdot \frac{(c+d)!}{c!d!} \cdot \frac{(e+f)!}{e!f!} \\
 & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+1}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\
 & \times (-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2) \cdot (-1)^{m_3+1}\sigma_c^*(m_3)\sigma_d^*(m_4) \cdot (-1)^{m_5+1}\sigma_e^*(m_5)\sigma_f^*(m_6) \\
 = & \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3 \geq 1}} \frac{(2k_1-1)(2l)!}{(2k_1-1)!(2k_2-1)!(2k_3-1)!} \\
 & \times \sum_{\substack{N_1+N_2+N_3=\frac{N}{2} \\ N_3 \text{ even} \\ \frac{N}{2} \text{ even}}} \frac{(-1)^{N_1+1}}{2N_1 \cdot 2N_2 \cdot 2N_3} N_1\sigma_{2k_1-1}^*(2N_1) \cdot N_2\sigma_{2k_2-1}^*(2N_2) \cdot N_3\sigma_{2k_3-1}^*(2N_3)
 \end{aligned}$$

Then, by using the fact (3), we have (17) as

$$\begin{aligned}
 (18) \quad & 2^{2l-6}(2l)(2l-1)(2l-2) \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3 \geq 1}} (2k_1-1) \binom{2l-3}{2k_1-1, 2k_2-1, 2k_3-1} \\
 & \times \sum_{\substack{N_1+N_2+N_3=\frac{N}{2} \\ N_3 \text{ even} \\ \frac{N}{2} \text{ even}}} (-1)^{N_1+1} \sigma_{2k_1-1}^*(N_1) \sigma_{2k_2-1}^*(N_2) \sigma_{2k_3-1}^*(N_3).
 \end{aligned}$$

So we can apply the Theorem 1.2 directly in (18) by replacing $l \rightarrow l-2$ and $N \rightarrow \frac{N}{2}$. Thus we have

$$\begin{aligned}
 & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} (a+b-1)(a+b)(c+d)(e+f) \binom{2l}{a,b,c,d,e,f} \\
 & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5+4}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\
 & \times \sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)\sigma_e^*(m_5)\sigma_f^*(m_6) \\
 & = 2^{-9}(2l)(2l-1)(2l-2)(2l-3)N \{ \sigma_{2l-3}^*(N) - 4N\sigma_{2l-5}^*(N) \} \\
 & = 2^{-9}(2l)_4 N \{ \sigma_{2l-3}^*(N) - 4N\sigma_{2l-5}^*(N) \},
 \end{aligned}$$

where we use the definition of $(a)_N$. □

Proof of the Theorem 1.4. First since $8|N$, we can know that $4|\frac{N}{2}$. Let $a_1 + a_2 := 2k_1$, $a_3 + a_4 := 2k_2$, $a_5 + a_6 := 2k_3$, $a_7 + a_8 := 2k_4$, $a_9 + a_{10} := 2k_5$, and $a_{11} + a_{12} := 2k_6$. And the condition

$$\begin{aligned}
 C := \left\{ (m_1, m_2, \dots, m_{11}, m_{12}) \in \mathbb{N}^{12} \middle| \sum_{i=1}^{12} m_i = N, \right. & 2|(m_1 + m_2), \quad 2|(m_3 + m_4), \\
 2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), & \\
 4|(m_1 + m_2 + m_3 + m_4), \quad 4|(m_5 + m_6 + m_7 + m_8), & \\
 \left. 8|(m_9 + m_{10} + m_{11} + m_{12}) \right\} &
 \end{aligned}$$

tells us that

$$\begin{aligned}
 m_1 + m_2 = 2N_1, \quad m_3 + m_4 = 2N_2, \quad m_5 + m_6 = 2N_3, \\
 m_7 + m_8 = 2N_4, \quad m_9 + m_{10} = 2N_5, \quad m_{11} + m_{12} = 2N_6
 \end{aligned}$$

and these facts and the leaving conditions of C show that

$$(19) \quad 2|(N_1 + N_2), \quad 2|(N_3 + N_4), \quad 4|(N_5 + N_6).$$

Considering (19), we investigate the given problem as follows :

$$\begin{aligned}
 & (20) \\
 & \sum_{\substack{\sum_{i=1}^{12} a_i = 2l \\ a_i \text{ odd}}} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \quad \times (a_1 + a_2 + a_3 + a_4 - 3)(a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2) \\
 & \quad \times (a_9 + a_{10} + a_{11} + a_{12} - 2) \binom{2l}{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}} \\
 & \quad \times \sum_C \frac{(-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)(m_7+m_8)(m_9+m_{10})(m_{11}+m_{12})} \\
 & \quad \times \frac{(-1)^{\frac{m_1+m_2+m_3+m_4}{4} + \frac{m_1+m_2}{2} + \frac{m_5+m_6}{2} + \frac{m_9+m_{10}}{2}}}{(m_1+m_2+m_3+m_4)(m_5+m_6+m_7+m_8)(m_9+m_{10}+m_{11}+m_{12})} \\
 & \quad \times \prod_{i=1}^{12} \sigma_{a_i}^*(m_i) \\
 = & \sum_{\substack{\sum_{i=1}^{12} a_i = 2l \\ a_i \text{ odd}}} (a_1 + a_2 + a_3 + a_4 - 3)(a_1 + a_2 + a_3 + a_4 - 2) \\
 & \quad \times (a_5 + a_6 + a_7 + a_8 - 2)(a_9 + a_{10} + a_{11} + a_{12} - 2) \\
 & \quad \times (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \quad \times \frac{2l!}{(a_1 + a_2)!(a_3 + a_4)!(a_5 + a_6)!(a_7 + a_8)!(a_9 + a_{10})!(a_{11} + a_{12})!} \\
 & \quad \times \frac{(a_1 + a_2)!}{a_1!a_2!} \cdot \frac{(a_3 + a_4)!}{a_3!a_4!} \cdot \frac{(a_5 + a_6)!}{a_5!a_6!} \cdot \frac{(a_7 + a_8)!}{a_7!a_8!} \cdot \frac{(a_9 + a_{10})!}{a_9!a_{10}!} \cdot \frac{(a_{11} + a_{12})!}{a_{11}!a_{12}!} \\
 & \quad \times \sum_{\substack{N_1+N_2+N_3+N_4+N_5+N_6=\frac{N}{2} \\ 2|(N_1+N_2) \\ 2|(N_3+N_4) \\ 4|(N_5+N_6)}} \frac{(-1)^{m_1+1}(-1)^{m_3+1}(-1)^{m_5+1}(-1)^{m_7+1}(-1)^{m_9+1}(-1)^{m_{11}+1}}{2N_1 \cdot 2N_2 \cdot 2N_3 \cdot 2N_4 \cdot 2N_5 \cdot 2N_6} \\
 & \quad \times \frac{(-1)^{\frac{N_1+N_2}{2}+1}(-1)^{N_1+1}(-1)^{N_3+1}(-1)^{N_5+1}}{2(N_1+N_2) \cdot 2(N_3+N_4) \cdot 2(N_5+N_6)} \prod_{i=1}^{12} \sigma_{a_i}^*(m_i).
 \end{aligned}$$

Then applying the Eq. (13) as the following form

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \frac{(-1)^{m+1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(2N-m)}{N} = \sigma_{2k-1}^*(2N)$$

into (20), we have

$$\begin{aligned}
 (21) \quad & \sum_{\substack{\sum_{i=1}^6 k_i = l \\ k_i \geq 1}} (2k_1 + 2k_2 - 3)(2k_1 + 2k_2 - 2)(2k_3 + 2k_4 - 2)(2k_5 + 2k_6 - 2) \\
 & \times 2k_1 \cdot 2k_2 \cdot 2k_3 \cdot 2k_4 \cdot 2k_5 \cdot 2k_6 \times \frac{2l!}{2k_1! \cdot 2k_2! \cdot 2k_3! \cdot 2k_4! \cdot 2k_5! \cdot 2k_6!} \\
 & \times \sum_{\substack{N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = \frac{N}{2} \\ 2|(N_1 + N_2) \\ 2|(N_3 + N_4) \\ 4|(N_5 + N_6)}} \frac{(-1)^{\frac{N_1 + N_2}{2} + 1} (-1)^{N_1 + 1} (-1)^{N_3 + 1} (-1)^{N_5 + 1}}{2^9 (N_1 + N_2)(N_3 + N_4)(N_5 + N_6)} \\
 & \times \prod_{i=1}^6 \sigma_{2k_i - 1}^*(2N_i) \\
 = & 2l(2l - 1)(2l - 2)(2l - 3)(2l - 4)(2l - 5) \cdot 2^{2l - 15} \\
 & \times \sum_{\substack{\sum_{i=1}^6 k_i = l \\ k_i \geq 1}} (2(k_1 + k_2 - 1) - 1)(2(k_1 2k_2 - 1))(2(k_3 + k_4 - 1))(2(k_5 + k_6 - 1)) \\
 & \times \binom{2l - 6}{2k_1 - 1, 2k_2 - 1, 2k_3 - 1, 2k_4 - 1, 2k_5 - 1, 2k_6 - 1} \\
 & \times \sum_{\substack{N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = \frac{N}{2} \\ 2|(N_1 + N_2) \\ 2|(N_3 + N_4) \\ 4|(N_5 + N_6)}} \frac{(-1)^{\frac{N_1 + N_2}{2} + 1} (-1)^{N_1 + 1} (-1)^{N_3 + 1} (-1)^{N_5 + 1}}{2^9 (N_1 + N_2)(N_3 + N_4)(N_5 + N_6)} \\
 & \times \prod_{i=1}^6 \sigma_{2k_i - 1}^*(N_i).
 \end{aligned}$$

So we insert $l \rightarrow l - 3$ and $N \rightarrow \frac{N}{2}$ in the Theorem 1.3 and we apply this induced formula into (21), thus (21) becomes

$$\begin{aligned}
 & 2^{2l - 15} \cdot 2l(2l - 1)(2l - 2)(2l - 3)(2l - 4)(2l - 5) \\
 & \times 2^{-9} \cdot 2(l - 3)(2(l - 3) - 1)(2(l - 3) - 2)(2(l - 3) - 3) \\
 & \times \frac{N}{2} \left\{ \sigma_{2(l - 3) - 3}^*\left(\frac{N}{2}\right) - 2N \sigma_{2(l - 3) - 5}^*\left(\frac{N}{2}\right) \right\} \\
 = & 2^{-16} (2l)(2l - 1)(2l - 2)(2l - 3)(2l - 4)(2l - 5)(2l - 6)(2l - 7) \\
 & \times (2l - 8)(2l - 9)N \left\{ \sigma_{2l - 9}^*(N) - 8N \sigma_{2l - 11}^*(N) \right\} \\
 = & 2^{-16} (2l)_{10} N \left\{ \sigma_{2l - 9}^*(N) - 8N \sigma_{2l - 11}^*(N) \right\},
 \end{aligned}$$

where we use (3) and the definition of $(a)_N$.

□

Proof of the Theorem 1.5. It is similar to the Theorem 1.4. From $16|N$, we have $8|\frac{N}{2}$. Let $a_1 + a_2 := 2k_1$, $a_3 + a_4 := 2k_2$, $a_5 + a_6 := 2k_3$, $a_7 + a_8 := 2k_4$,

$a_9 + a_{10} := 2k_5$, $a_{11} + a_{12} := 2k_6$, $a_{13} + a_{14} := 2k_7$, $a_{15} + a_{16} := 2k_8$, $a_{17} + a_{18} := 2k_9$,
 $a_{19} + a_{20} := 2k_{10}$, $a_{21} + a_{22} := 2k_{11}$, $a_{23} + a_{24} := 2k_{12}$. And the condition

$$D := \left\{ (m_1, m_2, \dots, m_{24}) \in \mathbb{N}^{24} \mid \sum_{i=1}^{24} m_i = N, \quad \begin{aligned} &2|(m_1 + m_2), \quad 2|(m_3 + m_4), \\ &2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), \\ &2|(m_{13} + m_{14}), \quad 2|(m_{15} + m_{16}), \quad 2|(m_{17} + m_{18}), \quad 2|(m_{19} + m_{20}), \\ &2|(m_{21} + m_{22}), \quad 2|(m_{23} + m_{24}), \quad 4|(m_1 + m_2 + m_3 + m_4), \\ &4|(m_5 + m_6 + m_7 + m_8), \quad 4|(m_9 + m_{10} + m_{11} + m_{12}), \\ &4|(m_{13} + m_{14} + m_{15} + m_{16}), \quad 4|(m_{17} + m_{18} + m_{19} + m_{20}), \\ &4|(m_{21} + m_{22} + m_{23} + m_{24}), \quad 8|(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8), \\ &8|(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16}), \\ &16|(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24}) \end{aligned} \right\}$$

shows us that

$$\begin{aligned} m_1 + m_2 &= 2N_1, & m_3 + m_4 &= 2N_2, & m_5 + m_6 &= 2N_3, \\ m_7 + m_8 &= 2N_4, & m_9 + m_{10} &= 2N_5, & m_{11} + m_{12} &= 2N_6, \\ m_{13} + m_{14} &= 2N_7, & m_{15} + m_{16} &= 2N_8, & m_{17} + m_{18} &= 2N_9, \\ m_{19} + m_{20} &= 2N_{10}, & m_{21} + m_{22} &= 2N_{11}, & m_{23} + m_{24} &= 2N_{12} \end{aligned}$$

and these facts and the leaving conditions of D lead that

$$(22) \quad E := \left\{ (N_1, N_2, \dots, N_{11}, N_{12}) \in \mathbb{N}^{12} \mid \sum_{i=1}^{12} N_i = \frac{N}{2}, \quad \begin{aligned} &2|(N_1 + N_2), \quad 2|(N_3 + N_4), \\ &2|(N_5 + N_6), \quad 2|(N_7 + N_8), \quad 2|(N_9 + N_{10}), \quad 2|(N_{11} + N_{12}), \\ &4|(N_1 + N_2 + N_3 + N_4), \quad 4|(N_5 + N_6 + N_7 + N_8), \\ &8|(N_9 + N_{10} + N_{11} + N_{12}) \end{aligned} \right\}.$$

Considering (22), we can write (4) as follows :

$$\begin{aligned}
 (23) \quad & \sum_{\substack{\sum_{i=1}^{24} a_i = 2l \\ a_i \text{ odd}}} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \times (a_{13} + a_{14})(a_{15} + a_{16})(a_{17} + a_{18})(a_{19} + a_{20})(a_{21} + a_{22})(a_{23} + a_{24}) \\
 & \times (a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2)(a_9 + a_{10} + a_{11} + a_{12} - 2) \\
 & \times (a_{13} + a_{14} + a_{15} + a_{16} - 2)(a_{17} + a_{18} + a_{19} + a_{20} - 2)(a_{21} + a_{22} + a_{23} + a_{24} - 2) \\
 & \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 7) \\
 & \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 6) \\
 & \times (a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} - 6) \\
 & \times (a_{17} + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} - 6) \\
 & \times \frac{2l!}{(a_1 + a_2)!(a_3 + a_4)!(a_5 + a_6)!(a_7 + a_8)!(a_9 + a_{10})!(a_{11} + a_{12})!} \\
 & \times \frac{1}{(a_{13} + a_{14})!(a_{15} + a_{16})!(a_{17} + a_{18})!(a_{19} + a_{20})!(a_{21} + a_{22})!(a_{23} + a_{24})!} \\
 & \times \frac{(a_1 + a_2)!}{a_1!a_2!} \cdot \frac{(a_3 + a_4)!}{a_3!a_4!} \cdot \frac{(a_5 + a_6)!}{a_5!a_6!} \cdot \frac{(a_7 + a_8)!}{a_7!a_8!} \cdot \frac{(a_9 + a_{10})!}{a_9!a_{10}!} \cdot \frac{(a_{11} + a_{12})!}{a_{11}!a_{12}!} \\
 & \times \frac{(a_{13} + a_{14})!}{a_{13}!a_{14}!} \cdot \frac{(a_{15} + a_{16})!}{a_{15}!a_{16}!} \cdot \frac{(a_{17} + a_{18})!}{a_{17}!a_{18}!} \cdot \frac{(a_{19} + a_{20})!}{a_{19}!a_{20}!} \cdot \frac{(a_{21} + a_{22})!}{a_{21}!a_{22}!} \cdot \frac{(a_{23} + a_{24})!}{a_{23}!a_{24}!} \\
 & \times \sum_D \frac{(-1)^{m_1+1}}{(m_1 + m_2)} \cdot \frac{(-1)^{m_3+1}}{(m_3 + m_4)} \cdot \frac{(-1)^{m_5+1}}{(m_5 + m_6)} \cdot \frac{(-1)^{m_7+1}}{(m_7 + m_8)} \cdot \frac{(-1)^{m_9+1}}{(m_9 + m_{10})} \cdot \frac{(-1)^{m_{11}+1}}{(m_{11} + m_{12})} \\
 & \times \frac{(-1)^{m_{13}+1}}{(m_{13} + m_{14})} \cdot \frac{(-1)^{m_{15}+1}}{(m_{15} + m_{16})} \cdot \frac{(-1)^{m_{17}+1}}{(m_{17} + m_{18})} \cdot \frac{(-1)^{m_{19}+1}}{(m_{19} + m_{20})} \cdot \frac{(-1)^{m_{21}+1}}{(m_{21} + m_{22})} \cdot \frac{(-1)^{m_{23}+1}}{m_{23} + m_{24}} \\
 & \times (-1)^{\frac{m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8}{8} + \frac{m_1+m_2+m_3+m_4}{4}} \\
 & \times (-1)^{\frac{m_9+m_{10}+m_{11}+m_{12}}{4} + \frac{m_{17}+m_{18}+m_{19}+m_{20}}{4}} \\
 & \times (-1)^{\frac{m_1+m_2+m_5+m_6+m_9+m_{10}+m_{13}+m_{14}+m_{17}+m_{18}+m_{21}+m_{22}}{2}} \\
 & \times \frac{1}{(m_1 + m_2 + m_3 + m_4)(m_5 + m_6 + m_7 + m_8)(m_9 + m_{10} + m_{11} + m_{12})} \\
 & \times \frac{1}{(m_{13} + m_{14} + m_{15} + m_{16})(m_{17} + m_{18} + m_{19} + m_{20})(m_{21} + m_{22} + m_{23} + m_{24})} \\
 & \times \frac{1}{(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8)} \\
 & \times \frac{1}{(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16})} \\
 & \times \frac{1}{(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24})} \prod_{i=1}^{24} \sigma_{a_i}^*(m_i).
 \end{aligned}$$

Then applying the Eq. (13) as the following form

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \frac{(-1)^{m+1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(2N-m)}{N} = \sigma_{2k-1}^*(2N)$$

into (23), we obtain

$$\begin{aligned} (24) \quad & \sum_{\substack{\sum_{i=1}^{12} k_i = l \\ k_i \geq 1}} (2(k_1 + k_2 + k_3 + k_4 - 2) - 3) (2(k_1 + k_2 + k_3 + k_4 - 2) - 2) \\ & \times (2(k_5 + k_6 + k_7 + k_8 - 2) - 2) (2(k_9 + k_{10} + k_{11} + k_{12} - 2) - 2) \\ & \times 2(k_1 + k_2 - 1) \cdot 2(k_3 + k_4 - 1) \cdot 2(k_5 + k_6 - 1) \cdot 2(k_7 + k_8 - 1) \\ & \times 2(k_9 + k_{10} - 1) \cdot 2(k_{11} + k_{12} - 1) \\ & \times 2k_1 \cdot 2k_2 \cdot 2k_3 \cdot 2k_4 \cdot 2k_5 \cdot 2k_6 \cdot 2k_7 \cdot 2k_8 \cdot 2k_9 \cdot 2k_{10} \cdot 2k_{11} \cdot 2k_{12} \\ & \times \frac{2l!}{2k_1! \cdot 2k_2! \cdot 2k_3! \cdot 2k_4! \cdot 2k_5! \cdot 2k_6! \cdot 2k_7! \cdot 2k_8! \cdot 2k_9! \cdot 2k_{10}! \cdot 2k_{11}! \cdot 2k_{12}!} \\ & \times \frac{1}{2^{21}} \sum_E \frac{(-1)^{N_1+N_3+N_5+N_7+N_9+N_{11}}}{(N_1+N_2)(N_3+N_4)(N_5+N_6)(N_7+N_8)(N_9+N_{10})(N_{11}+N_{12})} \\ & \times \frac{(-1)^{\frac{N_1+N_2+N_3+N_4}{4} + \frac{N_1+N_2}{2} + \frac{N_5+N_6}{2} + \frac{N_9+N_{10}}{2}}}{(N_1+N_2+N_3+N_4)(N_5+N_6+N_7+N_8)(N_9+N_{10}+N_{11}+N_{12})} \\ & \times \prod_{i=1}^{12} \sigma_{a_{2k_i-1}}^*(2N_i) \\ & = \frac{1}{2^{21}} (2l)(2l-1)(2l-2)(2l-3)(2l-4)(2l-5)(2l-6) \\ & \times (2l-7)(2l-8)(2l-9)(2l-10)(2l-11) \\ & \times \sum_{\substack{\sum_{i=1}^{12} k_i = l \\ k_i \geq 1}} 2(k_1 + k_2 - 1) \cdot 2(k_3 + k_4 - 1) \cdot 2(k_5 + k_6 - 1) \\ & \times 2(k_7 + k_8 - 1) \cdot 2(k_9 + k_{10} - 1) \cdot 2(k_{11} + k_{12} - 1) \\ & \times (2(k_1 + k_2 + k_3 + k_4 - 2) - 3) (2(k_1 + k_2 + k_3 + k_4 - 2) - 2) \\ & \times (2(k_5 + k_6 + k_7 + k_8 - 2) - 2) (2(k_9 + k_{10} + k_{11} + k_{12} - 2) - 2) \\ & \times \binom{2(l-6)}{2k_1-1, 2k_2-1, 2k_3-1, 2k_4-1, \dots, 2k_{11}-1, 2k_{12}-1} \\ & \times \sum_E \frac{(-1)^{N_1+N_3+N_5+N_7+N_9+N_{11}}}{(N_1+N_2)(N_3+N_4)(N_5+N_6)(N_7+N_8)(N_9+N_{10})(N_{11}+N_{12})} \\ & \times \frac{(-1)^{\frac{N_1+N_2+N_3+N_4}{4} + \frac{N_1+N_2}{2} + \frac{N_5+N_6}{2} + \frac{N_9+N_{10}}{2}}}{(N_1+N_2+N_3+N_4)(N_5+N_6+N_7+N_8)(N_9+N_{10}+N_{11}+N_{12})} \\ & \times 2^{2l-12} \prod_{i=1}^{12} \sigma_{a_{2k_i-1}}^*(N_i). \end{aligned}$$

So we insert $l \rightarrow l-6$ and $N \rightarrow \frac{N}{2}$ in the Theorem 1.4 and we apply this induced formula into (24), thus (24) becomes

$$\begin{aligned}
 & 2^{-21}(2l)(2l-1)(2l-2)(2l-3)(2l-4)(2l-5)(2l-6) \\
 & \quad \times (2l-7)(2l-8)(2l-9)(2l-10)(2l-11) \\
 & \quad \times 2^{-16} \cdot 2(l-6)(2(l-6)-1)(2(l-6)-2)(2(l-6)-3) \\
 & \quad \times (2(l-6)-4)(2(l-6)-5)(2(l-6)-6) \\
 & \quad \times (2(l-6)-7)(2(l-6)-8)(2(l-6)-9) \\
 & \quad \times 2^{2l-12} \cdot \frac{N}{2} \left\{ \sigma_{2(l-6)-9}^*\left(\frac{N}{2}\right) - 8 \cdot \frac{N}{2} \sigma_{2(l-6)-11}^*\left(\frac{N}{2}\right) \right\} \\
 & = 2^{-29}(2l)_{22}N \left\{ \sigma_{2l-21}^*(N) - 16N \sigma_{2l-23}^*(N) \right\},
 \end{aligned}$$

where we use (3) and the definition of $(a)_N$.

□

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