

THE BINOMIAL AND MULTINOMIAL COMBINATORIAL CONVOLUTION SUMS

ABSTRACT. In [1] and [3] we can find some formulas of binomial combinatorial convolution sums. And starting from these formulas, we obtain various binomial and multinomial combinatorial convolution sums.

1. INTRODUCTION

Let $\mathbb{N}$ denote the set of positive integers. Further, let $N, d \in \mathbb{N}$, where $s, r, k \in \mathbb{N} \cup \{0\}$. Throughout this paper, we define

$$\sigma_s(N) := \sum_{d|N} d^s, \quad \sigma_{s,r}(N; k) := \sum_{\substack{d|N \\ d \equiv r \pmod{k}}} d^s, \\ \sigma_s^*(N) := \sum_{\substack{d|N \\ N/d \text{ odd}}} d^s, \quad (a)_N := (a)(a-1) \cdots (a-N+1).$$

We note that

$$(1) \quad \sigma_s(N) = \sigma_{s,1}(N; 2) + \sigma_{s,0}(N; 2)$$

and

$$(2) \quad \sigma_{s,1}(2N; 2) = \sigma_{s,1}(N; 2), \quad \sigma_{s,0}(2N; 2) = 2^s \sigma_s(N).$$

Moreover we can deduce that the property

$$(3) \quad \sigma_s^*(2N) = 2^s \sigma_s^*(N).$$

For example, we can see that

$$\sum_{m=1}^{N-1} \sigma_1^*(m) \sigma_1^*(N-m) = \frac{1}{4} (\sigma_3^*(N) - N \sigma_1^*(N))$$

and

$$\sum_{m=1}^{N-1} \sigma_1^*(m) \sigma_3^*(N-m) = \frac{1}{16} (\sigma_5^*(N) - N \sigma_3^*(N))$$

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in [2]. We introduce the very important Proposition 1.1, which is the starting point of our paper.

Proposition 1.1. (See [3, Theorem 12.3]) Let $k, N \in \mathbb{N}$. Then

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \right) \\ &= \frac{2k+3}{4k+2} \sigma_{2k+1}(N) + \left(\frac{k}{6} - N \right) \sigma_{2k-1}(N) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N). \end{aligned}$$

In Section 2 we obtain some results about the binomial combinatoric convolution sums similar to the Proposition 1.1. For instance we have

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m)$$

and

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m)$$

(see (10) and (11)). Similarly, getting an idea from the Proposition 1.1, we obtain some various multinomial combinatorial convolution sums as follows :

Theorem 1.2. Let N be an even integer with $l \in \mathbb{N}$. Then for odd positive integers a, b, c we have

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} a \binom{2l+1}{a, b, c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \frac{(2l+1)N}{32} \{ \sigma_{2l+1}^*(N) - 2N \sigma_{2l-1}^*(N) \}. \end{aligned}$$

Theorem 1.3. Let $l, N \in \mathbb{N}$ with $4|N$. Then for odd positive integers a, b, c, d, e, f , we have

$$\begin{aligned} & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} (a+b-1)(a+b)(c+d)(e+f) \binom{2l}{a, b, c, d, e, f} \\ & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\ & \times \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \sigma_d^*(m_4) \sigma_e^*(m_5) \sigma_f^*(m_6) \\ &= 2^{-9} (2l)_4 N \{ \sigma_{2l-3}^*(N) - 4N \sigma_{2l-5}^*(N) \}. \end{aligned}$$

[*, Theorem 10]

Proposition

Theorem 1.4. Let $l, N \in \mathbb{N}$ with $8|N$ and let the condition

$$C := \left\{ (m_1, m_2, \dots, m_{11}, m_{12}) \in \mathbb{N}^{12} \mid \sum_{i=1}^{12} m_i = N, \quad 2|(m_1 + m_2), \quad 2|(m_3 + m_4), \right. \\ 2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), \\ 4|(m_1 + m_2 + m_3 + m_4), \quad 4|(m_5 + m_6 + m_7 + m_8), \\ \left. 8|(m_9 + m_{10} + m_{11} + m_{12}) \right\}.$$

Then for odd positive integers a_i , we have

$$\sum_{\substack{\sum_{i=1}^{12} a_i = 2l \\ a_i \text{ odd}}} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\ \times (a_1 + a_2 + a_3 + a_4 - 3)(a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2) \\ \times (a_9 + a_{10} + a_{11} + a_{12} - 2) \binom{2l}{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}} \\ \times \sum_C \frac{(-1)^{m_1 + m_3 + m_5 + m_7 + m_9 + m_{11}}}{(m_1 + m_2)(m_3 + m_4)(m_5 + m_6)(m_7 + m_8)(m_9 + m_{10})(m_{11} + m_{12})} \\ \times \frac{(-1)^{\frac{m_1 + m_2 + m_3 + m_4}{4} + \frac{m_1 + m_2}{2} + \frac{m_5 + m_6}{2} + \frac{m_9 + m_{10}}{2}}}{(m_1 + m_2 + m_3 + m_4)(m_5 + m_6 + m_7 + m_8)(m_9 + m_{10} + m_{11} + m_{12})} \\ \times \prod_{i=1}^{12} \sigma_{a_i}^*(m_i) \\ = 2^{-16} (2l)_{10} N \{ \sigma_{2l-9}^*(N) - 8N \sigma_{2l-11}^*(N) \}.$$

Theorem 1.5. Let $l, N \in \mathbb{N}$ with $16|N$ and let the condition

$$D := \left\{ (m_1, m_2, \dots, m_{23}, m_{24}) \in \mathbb{N}^{24} \mid \sum_{i=1}^{24} m_i = N, \quad 2|(m_1 + m_2), \quad 2|(m_3 + m_4), \right. \\ 2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), \\ 2|(m_{13} + m_{14}), \quad 2|(m_{15} + m_{16}), \quad 2|(m_{17} + m_{18}), \quad 2|(m_{19} + m_{20}), \\ 2|(m_{21} + m_{22}), \quad 2|(m_{23} + m_{24}), \quad 4|(m_1 + m_2 + m_3 + m_4), \\ 4|(m_5 + m_6 + m_7 + m_8), \quad 4|(m_9 + m_{10} + m_{11} + m_{12}), \\ 4|(m_{13} + m_{14} + m_{15} + m_{16}), \quad 4|(m_{17} + m_{18} + m_{19} + m_{20}), \\ 4|(m_{21} + m_{22} + m_{23} + m_{24}), \quad 8|(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8), \\ 8|(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16}), \\ \left. 16|(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24}) \right\}.$$

Then for odd positive integers a_i , we have

$$\begin{aligned}
 (4) \quad & \sum_{\substack{i=1 \\ a_i, \text{ odd}}}^{24} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \times (a_{13} + a_{14})(a_{15} + a_{16})(a_{17} + a_{18})(a_{19} + a_{20})(a_{21} + a_{22})(a_{23} + a_{24}) \\
 & \times (a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2)(a_9 + a_{10} + a_{11} + a_{12} - 2) \\
 & \times (a_{13} + a_{14} + a_{15} + a_{16} - 2)(a_{17} + a_{18} + a_{19} + a_{20} - 2)(a_{21} + a_{22} + a_{23} + a_{24} - 2) \\
 & \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 7) \\
 & \times (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 6) \\
 & \times (a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} - 6) \\
 & \times (a_{17} + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} - 6) \\
 & \times \binom{2l}{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23}, a_{24}} \\
 & \times \sum_D (-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}+m_{13}+m_{15}+m_{17}+m_{19}+m_{21}+m_{23}} \\
 & \times (-1)^{\frac{m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8}{8} + \frac{m_1+m_2+m_3+m_4}{4}} \\
 & \times (-1)^{\frac{m_9+m_{10}+m_{11}+m_{12}}{4} + \frac{m_{17}+m_{18}+m_{19}+m_{20}}{4}} \\
 & \times (-1)^{\frac{m_1+m_2+m_5+m_6+m_9+m_{10}+m_{13}+m_{14}+m_{17}+m_{18}+m_{21}+m_{22}}{2}} \\
 & \times \frac{1}{(m_1 + m_2)(m_3 + m_4)(m_5 + m_6)(m_7 + m_8)(m_9 + m_{10})(m_{11} + m_{12})} \\
 & \times \frac{1}{(m_{13} + m_{14})(m_{15} + m_{16})(m_{17} + m_{18})(m_{19} + m_{20})(m_{21} + m_{22})(m_{23} + m_{24})} \\
 & \times \frac{1}{(m_1 + m_2 + m_3 + m_4)(m_5 + m_6 + m_7 + m_8)(m_9 + m_{10} + m_{11} + m_{12})} \\
 & \times \frac{1}{(m_{13} + m_{14} + m_{15} + m_{16})(m_{17} + m_{18} + m_{19} + m_{20})(m_{21} + m_{22} + m_{23} + m_{24})} \\
 & \times \frac{1}{(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8)} \\
 & \times \frac{1}{(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16})} \\
 & \times \frac{1}{(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24})} \prod_{i=1}^{24} \sigma_{a_i}^*(m_i) \\
 & = 2^{-29} (2l)_{22} N \{ \sigma_{2l-21}^*(N) - 16N \sigma_{2l-23}^*(N) \}.
 \end{aligned}$$

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2. THE BINOMIAL COMBINATORIAL CONVOLUTION SUMS

To prove the Lemma 2.2 we need the Proposition 2.1.

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(c) We note that

$$\begin{aligned} & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,1}(m; 2) \\ &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\ & \quad - \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2). \end{aligned}$$

So we apply the Corollary 2.6 (a) and (b). □

2. Proof of Theorem 1.5

3. THE MULTINOMIAL COMBINATORIAL CONVOLUTION SUMS

Proof of the Theorem 1.2. Let us consider

$$\begin{aligned} (12) \quad & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} (a+b) \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} \frac{(2l+1)!}{(a+b-1)!c!} \frac{(a+b)!}{a!b!} \\ & \quad \times \sum_{M < \frac{N}{2}} \left(\sum_{m_1+m_2=2M} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \right) \sigma_c^*(N-2M) \\ &= \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} \frac{(2l+1)!}{(a+b-1)!c!} \\ & \quad \times \sum_{M < \frac{N}{2}} \left(\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m_1=1}^{2M-1} (-1)^{m_1+1} \sigma_{2k-2s-1}^*(m_1) \sigma_{2s+1}^*(2M-m_1) \right) \sigma_c^*(N-2M), \end{aligned}$$

where we put $a+b=2k$ and $b=2s+1$. Note that $k \geq 1$. Since

$$(13) \quad \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} (-1)^{m+1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(2N-m) = N \sigma_{2k-1}^*(2N)$$

in [1, (16)], so by (3) we can write (12) as

$$\begin{aligned}
 (14) \quad & \sum_{\substack{2k+c=2l+1 \\ c \text{ odd}}} \frac{(2l+1)!}{(2k-1)!c!} \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(2M) \sigma_c^*(N-2M) \\
 &= \sum_{k=1}^l \binom{2l}{2k-1} (2l+1) \sum_{M < \frac{N}{2}} M \cdot 2^{2k-1} \sigma_{2k-1}^*(M) \cdot 2^{2l+1-2k} \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= 2^{2l} (2l+1) \sum_{k=1}^l \binom{2l}{2k-1} \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right).
 \end{aligned}$$

Then

$$\begin{aligned}
 (15) \quad & \sum_{k=1}^l \binom{2l}{2k-1} \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= \frac{N/2}{2} \sum_{k=1}^l \binom{2l}{2k-1} \sum_{M < \frac{N}{2}} \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right)
 \end{aligned}$$

because

$$\begin{aligned}
 & \sum_{M < \frac{N}{2}} M \sigma_{2k-1}^*(M) \sigma_{2l+1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= \sum_{M < \frac{N}{2}} \left(\frac{N}{2} - M\right) \sigma_{2k-1}^*\left(\frac{N}{2} - M\right) \sigma_{2l+1-2k}^*(M).
 \end{aligned}$$

Thus, by the above fact and

$$\begin{aligned}
 (16) \quad & \sum_{k=0}^{l-1} \binom{2l}{2k+1} \sum_{m=1}^{N-1} \sigma_{2l-2k-1}^*(m) \sigma_{2k+1}^*(N-m) \\
 &= \frac{1}{2} \left\{ \sigma_{2l+1}^*(N) - N \sigma_{2l-1}^*(N) \right\}
 \end{aligned}$$

in [1, (10)], we have (14) as

$$\begin{aligned}
 & 2^{2l} (2l+1) \cdot \frac{N}{4} \sum_{k=0}^{l-1} \binom{2l}{2k+1} \sum_{M < \frac{N}{2}} \sigma_{2k+1}^*(M) \sigma_{2l-1-2k}^*\left(\frac{N}{2} - M\right) \\
 &= \frac{2^{2l} (2l+1) N}{4} \cdot \frac{1}{2} \left\{ \sigma_{2l+1}^*\left(\frac{N}{2}\right) - \frac{N}{2} \sigma_{2l-1}^*\left(\frac{N}{2}\right) \right\}.
 \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} (a+b) \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \frac{(2l+1)N}{16} \{ \sigma_{2l+1}^*(N) - 2N \sigma_{2l-1}^*(N) \}. \end{aligned}$$

And noting that

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} (a+b) \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} 2a \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3), \end{aligned}$$

so, finally, we have

$$\begin{aligned} & \sum_{\substack{a+b+c=2l+1 \\ a,b,c \text{ odd}}} a \binom{2l+1}{a,b,c} \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \\ &= \frac{(2l+1)N}{32} \{ \sigma_{2l+1}^*(N) - 2N \sigma_{2l-1}^*(N) \}. \end{aligned}$$

□

Proof of the Theorem 1.3. First since $4|N$, we have $\frac{N}{2}$ is even. Let $a+b := 2k_1$, $c+d := 2k_2$, and $e+f := 2k_3$. And from the conditions $2|(m_1+m_2)$, $2|(m_3+m_4)$, and $4|(m_5+m_6)$, we can write $m_1+m_2 = 2N_1$, $m_3+m_4 = 2N_2$, and $m_5+m_6 = 2N_3$, respectively. Note that N_3 is even. Then using these facts and (13), the problem can be written as

$$\begin{aligned}
 (17) \quad & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} (a+b-1)(a+b)(c+d)(e+f) \binom{2l}{a,b,c,d,e,f} \\
 & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5+4}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\
 & \times \sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)\sigma_e^*(m_5)\sigma_f^*(m_6) \\
 = & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} \frac{(a+b-1)(2l)!}{(a+b-1)!(c+d-1)!(e+f-1)!} \cdot \frac{(a+b)!}{a!b!} \cdot \frac{(c+d)!}{c!d!} \cdot \frac{(e+f)!}{e!f!} \\
 & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+1}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\
 & \times (-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2) \cdot (-1)^{m_3+1}\sigma_c^*(m_3)\sigma_d^*(m_4) \cdot (-1)^{m_5+1}\sigma_e^*(m_5)\sigma_f^*(m_6) \\
 = & \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3 \geq 1}} \frac{(2k_1-1)(2l)!}{(2k_1-1)!(2k_2-1)!(2k_3-1)!} \\
 & \times \sum_{\substack{N_1+N_2+N_3=\frac{N}{2} \\ N_3 \text{ even} \\ \frac{N}{2} \text{ even}}} \frac{(-1)^{N_1+1}}{2N_1 \cdot 2N_2 \cdot 2N_3} N_1 \sigma_{2k_1-1}^*(2N_1) \cdot N_2 \sigma_{2k_2-1}^*(2N_2) \cdot N_3 \sigma_{2k_3-1}^*(2N_3)
 \end{aligned}$$

Then, by using the fact (3), we have (17) as

$$\begin{aligned}
 (18) \quad & 2^{2l-6}(2l)(2l-1)(2l-2) \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3 \geq 1}} (2k_1-1) \binom{2l-3}{2k_1-1, 2k_2-1, 2k_3-1} \\
 & \times \sum_{\substack{N_1+N_2+N_3=\frac{N}{2} \\ N_3 \text{ even} \\ \frac{N}{2} \text{ even}}} (-1)^{N_1+1} \sigma_{2k_1-1}^*(N_1) \sigma_{2k_2-1}^*(N_2) \sigma_{2k_3-1}^*(N_3).
 \end{aligned}$$

So we can apply the Theorem 1.2 directly in (18) by replacing $l \rightarrow l-2$ and $N \rightarrow \frac{N}{2}$. Thus we have

$$\begin{aligned}
 & \sum_{\substack{a+b+c+d+e+f=2l \\ a,b,c,d,e,f \text{ odd}}} (a+b-1)(a+b)(c+d)(e+f) \binom{2l}{a,b,c,d,e,f} \\
 & \times \sum_{\substack{m_1+m_2+m_3+m_4+m_5+m_6=N \\ 2|(m_1+m_2) \\ 2|(m_3+m_4) \\ 4|(m_5+m_6)}} \frac{(-1)^{\frac{m_1+m_2}{2}+m_1+m_3+m_5+4}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)} \\
 & \times \sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)\sigma_e^*(m_5)\sigma_f^*(m_6) \\
 & = 2^{-9}(2l)(2l-1)(2l-2)(2l-3)N \{ \sigma_{2l-3}^*(N) - 4N\sigma_{2l-5}^*(N) \} \\
 & = 2^{-9}(2l)_4 N \{ \sigma_{2l-3}^*(N) - 4N\sigma_{2l-5}^*(N) \},
 \end{aligned}$$

where we use the definition of $(a)_N$. □

Proof of the Theorem 1.4. First since $8|N$, we can know that $4|\frac{N}{2}$. Let $a_1 + a_2 := 2k_1$, $a_3 + a_4 := 2k_2$, $a_5 + a_6 := 2k_3$, $a_7 + a_8 := 2k_4$, $a_9 + a_{10} := 2k_5$, and $a_{11} + a_{12} := 2k_6$. And the condition

$$\begin{aligned}
 C := \Big\{ (m_1, m_2, \dots, m_{11}, m_{12}) \in \mathbb{N}^{12} \Big| & \sum_{i=1}^{12} m_i = N, \quad 2|(m_1+m_2), \quad 2|(m_3+m_4), \\
 & 2|(m_5+m_6), \quad 2|(m_7+m_8), \quad 2|(m_9+m_{10}), \quad 2|(m_{11}+m_{12}), \\
 & 4|(m_1+m_2+m_3+m_4), \quad 4|(m_5+m_6+m_7+m_8), \\
 & 8|(m_9+m_{10}+m_{11}+m_{12}) \Big\}
 \end{aligned}$$

tells us that

$$\begin{aligned}
 m_1 + m_2 = 2N_1, \quad m_3 + m_4 = 2N_2, \quad m_5 + m_6 = 2N_3, \\
 m_7 + m_8 = 2N_4, \quad m_9 + m_{10} = 2N_5, \quad m_{11} + m_{12} = 2N_6
 \end{aligned}$$

and these facts and the leaving conditions of C show that

$$(19) \quad 2|(N_1 + N_2), \quad 2|(N_3 + N_4), \quad 4|(N_5 + N_6).$$

Considering (19), we investigate the given problem as follows :

$$\begin{aligned}
 & (20) \\
 & \sum_{\substack{i=1 \\ a_i \text{ odd}}}^{12} (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \quad \times (a_1 + a_2 + a_3 + a_4 - 3)(a_1 + a_2 + a_3 + a_4 - 2)(a_5 + a_6 + a_7 + a_8 - 2) \\
 & \quad \times (a_9 + a_{10} + a_{11} + a_{12} - 2) \binom{2l}{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}} \\
 & \quad \times \sum_C \frac{(-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)(m_7+m_8)(m_9+m_{10})(m_{11}+m_{12})} \\
 & \quad \times \frac{(-1)^{\frac{m_1+m_2+m_3+m_4}{4} + \frac{m_1+m_2}{2} + \frac{m_5+m_6}{2} + \frac{m_9+m_{10}}{2}}}{(m_1+m_2+m_3+m_4)(m_5+m_6+m_7+m_8)(m_9+m_{10}+m_{11}+m_{12})} \\
 & \quad \times \prod_{i=1}^{12} \sigma_{a_i}^*(m_i) \\
 = & \sum_{\substack{i=1 \\ a_i \text{ odd}}}^{12} (a_1 + a_2 + a_3 + a_4 - 3)(a_1 + a_2 + a_3 + a_4 - 2) \\
 & \quad \times (a_5 + a_6 + a_7 + a_8 - 2)(a_9 + a_{10} + a_{11} + a_{12} - 2) \\
 & \quad \times (a_1 + a_2)(a_3 + a_4)(a_5 + a_6)(a_7 + a_8)(a_9 + a_{10})(a_{11} + a_{12}) \\
 & \quad \times \frac{2l!}{(a_1 + a_2)!(a_3 + a_4)!(a_5 + a_6)!(a_7 + a_8)!(a_9 + a_{10})!(a_{11} + a_{12})!} \\
 & \quad \times \frac{(a_1 + a_2)!}{a_1!a_2!} \cdot \frac{(a_3 + a_4)!}{a_3!a_4!} \cdot \frac{(a_5 + a_6)!}{a_5!a_6!} \cdot \frac{(a_7 + a_8)!}{a_7!a_8!} \cdot \frac{(a_9 + a_{10})!}{a_9!a_{10}!} \cdot \frac{(a_{11} + a_{12})!}{a_{11}!a_{12}!} \\
 & \quad \times \sum_{\substack{N_1+N_2+N_3+N_4+N_5+N_6=\frac{N}{2} \\ 2|(N_1+N_2) \\ 2|(N_3+N_4) \\ 4|(N_5+N_6)}} \frac{(-1)^{m_1+1}(-1)^{m_3+1}(-1)^{m_5+1}(-1)^{m_7+1}(-1)^{m_9+1}(-1)^{m_{11}+1}}{2N_1 \cdot 2N_2 \cdot 2N_3 \cdot 2N_4 \cdot 2N_5 \cdot 2N_6} \\
 & \quad \times \frac{(-1)^{\frac{N_1+N_2}{2}+1}(-1)^{N_1+1}(-1)^{N_3+1}(-1)^{N_5+1}}{2(N_1+N_2) \cdot 2(N_3+N_4) \cdot 2(N_5+N_6)} \prod_{i=1}^{12} \sigma_{a_i}^*(m_i).
 \end{aligned}$$

Then applying the Eq. (13) as the following form

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \frac{(-1)^{m+1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(2N-m)}{N} = \sigma_{2k-1}^*(2N)$$

into (20), we have

$$\begin{aligned}
 (21) \quad & \sum_{\substack{\sum_{i=1}^6 k_i = l \\ k_i \geq 1}} (2k_1 + 2k_2 - 3)(2k_1 + 2k_2 - 2)(2k_3 + 2k_4 - 2)(2k_5 + 2k_6 - 2) \\
 & \times 2k_1 \cdot 2k_2 \cdot 2k_3 \cdot 2k_4 \cdot 2k_5 \cdot 2k_6 \times \frac{2l!}{2k_1! \cdot 2k_2! \cdot 2k_3! \cdot 2k_4! \cdot 2k_5! \cdot 2k_6!} \\
 & \times \sum_{\substack{N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = \frac{N}{2} \\ 2|(N_1 + N_2) \\ 2|(N_3 + N_4) \\ 4|(N_5 + N_6)}} \frac{(-1)^{\frac{N_1 + N_2}{2} + 1} (-1)^{N_1 + 1} (-1)^{N_3 + 1} (-1)^{N_5 + 1}}{2^9 (N_1 + N_2)(N_3 + N_4)(N_5 + N_6)} \\
 & \times \prod_{i=1}^6 \sigma_{2k_i - 1}^*(2N_i) \\
 = & 2l(2l - 1)(2l - 2)(2l - 3)(2l - 4)(2l - 5) \cdot 2^{2l - 15} \\
 & \times \sum_{\substack{\sum_{i=1}^6 k_i = l \\ k_i \geq 1}} (2(k_1 + k_2 - 1) - 1)(2(k_1 2k_2 - 1))(2(k_3 + k_4 - 1))(2(k_5 + k_6 - 1)) \\
 & \times \binom{2l - 6}{2k_1 - 1, 2k_2 - 1, 2k_3 - 1, 2k_4 - 1, 2k_5 - 1, 2k_6 - 1} \\
 & \times \sum_{\substack{N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = \frac{N}{2} \\ 2|(N_1 + N_2) \\ 2|(N_3 + N_4) \\ 4|(N_5 + N_6)}} \frac{(-1)^{\frac{N_1 + N_2}{2} + 1} (-1)^{N_1 + 1} (-1)^{N_3 + 1} (-1)^{N_5 + 1}}{2^9 (N_1 + N_2)(N_3 + N_4)(N_5 + N_6)} \\
 & \times \prod_{i=1}^6 \sigma_{2k_i - 1}^*(N_i).
 \end{aligned}$$

So we insert $l \rightarrow l - 3$ and $N \rightarrow \frac{N}{2}$ in the Theorem 1.3 and we apply this induced formula into (21), thus (21) becomes

$$\begin{aligned}
 & 2^{2l - 15} \cdot 2l(2l - 1)(2l - 2)(2l - 3)(2l - 4)(2l - 5) \\
 & \times 2^{-9} \cdot 2(l - 3)(2(l - 3) - 1)(2(l - 3) - 2)(2(l - 3) - 3) \\
 & \times \frac{N}{2} \left\{ \sigma_{2(l - 3) - 3}^*\left(\frac{N}{2}\right) - 2N\sigma_{2(l - 3) - 5}^*\left(\frac{N}{2}\right) \right\} \\
 = & 2^{-16}(2l)(2l - 1)(2l - 2)(2l - 3)(2l - 4)(2l - 5)(2l - 6)(2l - 7) \\
 & \times (2l - 8)(2l - 9)N \left\{ \sigma_{2l - 9}^*(N) - 8N\sigma_{2l - 11}^*(N) \right\} \\
 = & 2^{-16}(2l)_{10}N \left\{ \sigma_{2l - 9}^*(N) - 8N\sigma_{2l - 11}^*(N) \right\},
 \end{aligned}$$

where we use (3) and the definition of $(a)_N$.

□

Proof of the Theorem 1.5. It is similar to the Theorem 1.4. From $16|N$, we have $8|\frac{N}{2}$. Let $a_1 + a_2 := 2k_1$, $a_3 + a_4 := 2k_2$, $a_5 + a_6 := 2k_3$, $a_7 + a_8 := 2k_4$,

So we insert $l \rightarrow l-6$ and $N \rightarrow \frac{N}{2}$ in the Theorem 1.4 and we apply this induced formula into (24), thus (24) becomes

$$\begin{aligned} & 2^{-21}(2l)(2l-1)(2l-2)(2l-3)(2l-4)(2l-5)(2l-6) \\ & \quad \times (2l-7)(2l-8)(2l-9)(2l-10)(2l-11) \\ & \quad \times 2^{-16} \cdot 2(l-6)(2(l-6)-1)(2(l-6)-2)(2(l-6)-3) \\ & \quad \times (2(l-6)-4)(2(l-6)-5)(2(l-6)-6) \\ & \quad \times (2(l-6)-7)(2(l-6)-8)(2(l-6)-9) \\ & \quad \times 2^{2l-12} \cdot \frac{N}{2} \left\{ \sigma_{2(l-6)-9}^*\left(\frac{N}{2}\right) - 8 \cdot \frac{N}{2} \sigma_{2(l-6)-11}^*\left(\frac{N}{2}\right) \right\} \\ & = 2^{-29}(2l)_{22}N \left\{ \sigma_{2l-21}^*(N) - 16N \sigma_{2l-23}^*(N) \right\}, \end{aligned}$$

where we use (3) and the definition of $(a)_N$.

□

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