

On commutative $EKFN$ -ring with ascending chain condition on annihilators.

Mohameth Alassane Ndiaye¹, Cheikh Thiécoumba Gueye^{1 *}

¹Laboratoire d'Algèbre, de Cryptographie,
de Géométrie Algébrique et Applications (LACGAA),
Département de Mathématiques et Informatique,
Faculté des Sciences et Techniques,
Université Cheikh Anta Diop,
BP 5005 Dakar, Sénégal.

Abstract

We consider the class $\mathcal{E}$ of endo-noetherian modules, i.e. the modules M which satisfying the ascending chain condition for endomorphic kernels: any ascending chain $\text{Ker}f_1 \subset \text{Ker}f_2 \subset \dots \subset \text{Ker}f_n \subset \dots$ is stationary, where $f_i \in \text{End}(M)$. Let $\mathcal{N}$ be the class of Noetherian modules. It is clear that every Noetherian R -module M is endo-noetherian, so $\mathcal{N} \subset \mathcal{E}$, but the converse is not true. The aim of this work, is to characterize commutative rings for which $\mathcal{E}$ and $\mathcal{N}$ are identical.

Keywords : Annihilator; Artinian; $EKFN$ -ring; endo-noetherian; hopfian; Noetherian; strongly hopfian.

Introduction

The ascending chain condition is a mathematical property on orders, which used by Emmy Noether in 1921 [10] in the context of commutative algebra. The rings named, Noetherian rings, have been the subject of extensive

*Corresponding authors: E-mail addresses: meta_ndiaye@hotmail.com (M.A. Ndiaye), cheikht.gueye@ucad.edu.sn (C.T. Gueye).

study to modules. The descending chain condition on ideals was introduced by Artin. Rings in which any descending chain of ideals is finite are called **Artinian**. We say that a module is **Noetherian** (resp. **Artinian**) if any ascending (resp. descending) chain of submodules is stationary. In fact, the finiteness of the dimension of a vector space E is equivalent to the **Noetherian or Artinian condition**, then we have the following properties:

- (i) any surjective endomorphism is injective;
- (ii) any injective endomorphism is surjective.

Condition (i) (resp. (ii)) has inspired, the concept of hopfian (resp. co-hopfian) modules. An R -module is said hopfian if any surjective homomorphism from M to M is injective. This concept began with the introduction of hopfian groups by G. Baumslag in 1963 [3]. Since **then**, we focused on other issues relating to hopfian and co-hopfian properties, including:

1. the characterization of rings over which every hopfian (resp. co-hopfian, resp. fitting) module is **Noetherian** (resp. **Artinian**, resp. of finite length);
2. the characterization of rings for which every hopfian (resp. co-hopfian) module is finitely generated.

The question in 1 has been fully resolved, in the commutative case by A. Kaidi and M. Sangharé, who have established in 1988 the following results in [8]: **Let** R be a commutative ring, then the following conditions are equivalent:

- (a) R is an I -ring;¹
- (b) R is an S -ring;²
- (c) R is an **Artinian** principal ideal ring.

In 1992 M. Sangharé has established in [11]: **Let** R be a commutative ring, then the following conditions are equivalent:

- (a) R is an I -ring;
- (b) R is an S -ring;

¹An I -ring R is a ring such that every hopfian R -module is Noetherian.

²An S -ring R is a ring such that every co-hopfian R -module is Artinian.

- (b) R is an F -ring;³
- (c) R is an Artinian principal ideal ring.

The characterization of *FGS*-rings,⁴ was fully resolved in the commutative case, by C.T. Gueye and M. Sangharé in 2004 [5]. They established that: a ring R is an *FGS*-ring if and only if R is an Artinian principal ideal ring. The characterization of *FGI*-rings,⁵ was fully resolved in the commutative case by M. Barry et al in 2005 [2]. They established that: a ring R is an *FGI*-ring if and only if R is an Artinian principal ideal ring. Nevertheless, the question remains open in the general case.

In 2007 A. Hmaimou et al introduced in [6], the notions of modules that involve both the conditions of chains and endomorphism, defined as follows: an R -module M is said strongly hopfian (resp. strongly co-hopfian) if for every endomorphism f of M , then

$\text{Ker}f \subset \text{Ker}f^2 \subset \dots$ (resp. $\text{Im}f \supset \text{Im}f^2 \supset \dots$) is stationary. They showed that the class of strongly hopfian (resp. strongly co-hopfian) modules is contained the class of Noetherian (resp. Artinian) modules and the hopfian (resp. co-hopfian) modules.

Finally A. Kaidi introduced in 2009 in [7] the class of endo-noetherian (resp. endo-artinian) modules as follows: an R -module M is said to be endo-noetherian (resp. endo-artinian) if for every endomorphism $f_i \in M$, then $\text{Ker}f_1 \subset \text{Ker}f_2 \subset \dots$ (resp. $\text{Im}f_1 \supset \text{Im}f_2 \supset \dots$) is stationary. He showed that the class of endo-noetherian (resp. endo-artinian) modules is between the class of Noetherian (resp. Artinian) modules and the strongly hopfian (resp. strongly co-hopfian) modules.

The purpose of this paper is to characterize rings on which an endo-noetherian module M is Noetherian. Such rings will be called *EKFN*-rings. We prove that a commutative ring with ascending chain condition on annihilators is an *EKFN*-ring if and only if it is an Artinian principal ideal ring.

Our paper is structured as follows: the first section covers the properties and basic aspects of certain classes of modules and rings; special attention is paid to the properties of endo-noetherian modules. In the second section, we characterize commutative *EKFN*-rings satisfying ascending chain condition on annihilators.

³An F -ring R is a ring such that every fitting R -module is of finite length.

⁴An *FGS*-ring R is a ring for which every hopfian R -module is finitely generated.

⁵An *FGI*-ring R is a ring for which every co-hopfian R -module is finitely generated.

1 Preliminaries

The rings considered in this section are associative with unit. Unless otherwise mentioned, all the modules considered are left unitary modules.

Proposition 1.1 (*Schur's Lemma*)

If S and T are two simple modules. Then every non-zero homomorphism is an isomorphism.

Theorem 1.2 ([4], P.220)

For a ring R the following statements are equivalent:

1. R is **Noetherian** and every prime ideal is maximal;
2. R is **Artinian**.

Lemma 1.1 ([8], P.248)

*Let R be an **Artinian** ring having at least one non-principal ideal. Then R admits a local quotient ring B with maximal ideal J such that $J^2 = (0)$, and such that J/J^2 is a B/J -vector space of dimension 2.*

Lemma 1.2 ([8], P.248)

*Let R be an **Artinian** ring having at least one non-principal ideal. Then R admits a quotient ring $B = C \oplus bC$, where C is a local subring of B , with maximal ideal $rC \neq 0$, and $b \neq 0$ with $r^2 = rb = b^2 = 0$.*

Lemma 1.3 ([8], P.249)

*Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C -endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$, **then:***

1. $r\sigma = \sigma^2 = 0$;
2. If F is a C -endomorphism of M commuting with σ , then for all $m \in M$, $F(rm) = rmF(1)$.

Proposition 1.3 ([8], P.250)

*Let R be an **Artinian** ring having at least one non-principal ideal. Then there exists an R -module M which is not finitely generated.*

Proof : By **Lemma 1.2**, assume that $R = C \oplus bC$, where C is a local subring of R , with maximal ideal $rC \neq 0$, and $b \neq 0$ with $r^2 = rb = b^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$ and φ the map defined by: $R \longrightarrow \text{End}(M)$ where $\alpha + b\beta \mapsto \alpha 1_M + \beta\sigma$; $\alpha, \beta \in C$, 1_M

the identity map of M and σ the C -endomorphism defined in Lemma 1.3. φ is a ring homomorphism which gives to M an R -module structure and for this structure, R -endomorphisms of M are C -endomorphisms of M which commute with σ . To show that M is not a finitely generated R -module, it suffices to note that M contains infinite direct sum of non-zero submodules $\bigoplus_{n \geq 1} RrX^n$. $\square$

Definition 1.1 *An R -module M is called endo-noetherian if any ascending chain of endomorphic kernels $\text{Ker}f_1 \subset \text{Ker}f_2 \subset \dots \subset \text{Ker}f_n \subset \dots$ stabilizes, where f_i 's are endomorphisms of M , i.e. there exists a positive integer n such that $\text{Ker}f_n = \text{Ker}f_{n+1}$.*

Proposition 1.4 [7]

$\mathbb{Q}$ is a non-noetherian $\mathbb{Z}$ -module which is endo-noetherian.

Proposition 1.5 [7]

A ring R is endo-noetherian if and only if R satisfies the ascending chain condition on principal left annihilators.

Proposition 1.6 [7]

Let M be an endo-noetherian R -module. Then M satisfies the ascending chain condition on direct summands.

Corollary 1.1 [7]

Let M be an endo-noetherian R -module that is a direct sum of non-zero submodules, $M = \bigoplus_{i \in I} M_i$. Then the set I is finite.

Proposition 1.7 [7]

Let M be an R -module and N a submodule of M : if M is endo-noetherian and N is a direct summand then N is endo-noetherian.

Proof : Assume that $M = N \oplus L$. Let $\text{Ker}h_1 \subset \text{Ker}h_2 \subset \dots \subset \text{Ker}h_n \dots$ be an ascending chain, where the h_i 's are endomorphisms of N that can be extended to M as $f_i = h_i \oplus 0$, 0 being the zero endomorphism of L . Since M is endo-noetherian, there exists $n \in \mathbb{N}$ such that $\text{Ker}f_n = \text{Ker}f_{n+1}$. Then $\text{Ker}h_n = \text{Ker}h_{n+1}$. Thus, N is endo-noetherian. $\square$

Proposition 1.8 [7]

Let $M_i, i \in I$, be a family of R -modules such that the set $\text{Hom}(M_i, M_j) = \{0\}$ for $i \neq j$. Let $M = \bigoplus_{i \in I} M_i$. Then M is endo-noetherian if and only if for every $i \in I$ the module M_i is endo-noetherian.

Proof : By Proposition 1.7, if M is endo-noetherian, so is M_i . Assume now that every M_i is endo-noetherian. Since $\text{Hom}(M_i, M_j) = 0$ for all $i \neq j$, we have $\text{End}(M) \cong \prod_{i \in I} \text{End}(M_i)$. And if $f = (f_i)$ is an endomorphism of M , then we have $f(\sum a_i) = \sum f_i(a_i)$: this sum is well defined because of the fact that almost all a_i vanish. Thus $f = \bigoplus_{i \in I} f_i$. For every endomorphism $f \in \text{End}(M)$ we have $\text{Ker } f = \bigoplus_{i \in I} \text{Ker } f_i$. So, every ascending chain of kernels of $\text{End}(M)$ gives a chain of the same type in each $\text{End}(M_i)$, that stabilizes. Hence M is endo-noetherian. $\square$

2 The main results

Let R be an associative and commutative ring with identity $1 \neq 0$.

Definition 2.1 R is said to be an *EKFN-ring*⁶ if every endo-noetherian R -module is Noetherian.

Example 2.1 A semisimple ring and an Artinian ring with principal ideal, are EKFN-rings.

Proposition 2.1 Every homomorphic image of an EKFN-ring is an EKFN-ring.

Proposition 2.2 Any integral domain EKFN-ring is a field.

Proof : Let R be an EKFN-ring. Let K be the field of fractions of R . We know that K has an R -module structure. Let f be an R -endomorphism, then $\text{Ker } f = 0$ or $\text{Ker } f = K$. It follows that any ascending chain for endomorphic kernels of K stabilizes. Then K is Noetherian. We know that the map $i : R \rightarrow K$ is a monomorphism, so $R \subset K$. K is Noetherian, then K is a finitely generated R -module, so K is a fractional ideal of R by ([9], P.134). Thus there exists $r \in R$ such that $rK \subset R$. But $rK = K$, hence $K \subset R$. $\square$

Corollary 2.1 Any prime ideal of an EKFN-ring is maximal.

Proof : Let P be a prime ideal of R . Let s be the canonical surjection defined by $R \rightarrow R/P$ where $r \mapsto \tilde{r}$. By Proposition 2.1 the quotient ring of the integral domain R/P is an EKFN-ring, then R/P is a field. So P is maximal. $\square$

⁶EKFN is derived from the two concepts namely Endo Kernel Finite and Noetherian.

Proposition 2.3 *Let R be an EKFN-ring. Then the set of all maximal ideals of R is finite.*

Proof : Let $\mathcal{L}$ be the set of all primes ideals of R . Let's show that $\text{Hom}_R(R/P_i, R/P_j) = \{0\}$ for all $i \neq j$. Let $f \in \text{Hom}_R(R/P_i, R/P_j)$. For every prime ideal P_i , R/P_i is a simple R -module, and by Schur's Lemma f is zero or an isomorphism. Assume that f is an isomorphism, we know that $\tilde{1}_{P_j} \in R/P_j$, so there exists a unique $r \notin P_i$ such that $f(\tilde{r}_{P_i}) = \tilde{1}_{P_j}$. For all $\alpha \in P_i$, $f(\alpha\tilde{r}_{P_i}) = \alpha f(\tilde{r}_{P_i}) = \alpha\tilde{1}_{P_j} = \tilde{\alpha}_{P_j}$, but $f(\alpha\tilde{r}_{P_i}) = f(r\tilde{\alpha}_{P_i}) = f(\tilde{0}_{P_i}) = \tilde{0}_{P_j}$, then $\tilde{\alpha}_{P_j} = \tilde{0}_{P_j}$, so $\alpha \in P_j$. Thus $P_i \subset P_j$. Conversely for all $\alpha \in P_j$, $f(\alpha\tilde{r}_{P_i}) = \tilde{\alpha}_{P_j} = \tilde{0}_{P_j}$, but $f(\alpha\tilde{r}_{P_i}) = f(\tilde{\alpha}r_{P_i}) = \tilde{0}_{P_j}$, then $\tilde{\alpha}r_{P_i} \in P_i$, so $\alpha r \in P_i$, hence $\alpha \in P_i$. Thus $P_j \subset P_i$. Finally $P_j = P_i$, contradiction. Then $f = 0$, so $\text{Hom}_R(R/P_i, R/P_j) = \{0\}$ for all $i \neq j$. Let $M = \bigoplus_{i \in I} R/P_i$, let's

show that M is endo-noetherian. We know that R/P_i is endo-noetherian. By Proposition 1.8, M is endo-noetherian. Now it follows from Corollary 1.1 that I is finite. $\square$

Proposition 2.4 *Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C -endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$. Then every non-zero C -endomorphism F of M commuting with σ is such that either $\text{Ker}F = rM$, or is a monomorphism.*

Proof : Note that a ring element $m \in M$ is invertible if and only if $m \notin rM$. Indeed, let $m \in rM$, and assume that m is invertible in M i.e. there exists $m' \in M$ such that $mm' = m'm = 1$. Since $m = rm_1$ where $m_1 \in M$, we have $rm_1m' = 1$, then $r^2m_1m' = r = 0$. Contradiction as $r \neq 0$. Conversely, if $m \notin rM$, then $m \notin rC$. As rC is a maximal ideal of the local ring C , then m is invertible in C , hence in M .

If $F(1)$ is not invertible, then $F(1) \in rM$. For any $m \in rM$ there exists $q \in M$ such that $m = rq$. Hence by Lemma 1.3, $F(m) = F(rq) = rqF(1) = mF(1) = 0$. So $rM \subset \text{Ker}F$ (*). Let's show that in this case for all $m \notin rM$, then $F(m) \neq 0$. We have for all $m \in M \setminus rM$, $rm \in rM \setminus \{0\}$, then $F(rm) = rmF(1) = 0$. Or $F(rm) = rF(m) = 0$, so $F(m) \in rM$ i.e. there exists $\alpha \in M$ such that $F(m) = r\alpha$. Assume that $\alpha \in rM$, so $F(m) = 0$. Then $m \in \text{Ker}F$, i.e. $M \setminus rM \subset \text{Ker}F$ (**). Hence (*) and (**) imply that $M \subset \text{Ker}F$, then F is identically zero (impossible by assumption). Hence, it follows that $\alpha \notin rM$. So $F(m) = r\alpha \neq 0$. Then, for all $m \notin rM$ $F(m) \neq 0$. Hence $m \in \text{Ker}F \Rightarrow m \in rM$ so $\text{Ker}F \subset rM$ (***). Thus (*) and (***) imply that $\text{Ker}F = rM$.

If $F(1)$ is invertible, i.e. $F(1) \notin rM$. Let m be a non-zero element of

M . Let's show that $F(m) \neq 0$. In case $m \in rM$, then $F(m) = mF(1)$. Hence $F(m) \neq 0$. If now $m \notin rM$, then $rm \neq 0$, so $F(rm) = rmF(1)$, hence $F(rm) = rF(m) = rmF(1) \neq 0$. Then $F(m) \neq 0$. Thus F is a monomorphism. $\square$

Theorem 2.5 *Characterization theorem*

Let R be a ring satisfying acc on annihilators, the following conditions are equivalent:

1. R is an *EKFN*-ring;
2. R is an *Artinian* principal ideal ring.

Proof :

- Assume that R is an *EKFN*-ring. Since R satisfies the acc on annihilators, then by Proposition 1.5, R is endo-noetherian, so R is Noetherian. We know by Corollary 2.1 that any prime ideal of an *EKFN*-ring is maximal. Thus R is Artinian. Assume that R has at least one non-principal ideal, so there exists an R -module M which is not finitely generated by Proposition 1.3. So M is non-noetherian. Let $\text{Ker} f_1 \subset \text{Ker} f_2 \subset \dots \subset \text{Ker} f_n \subset \dots$ be an ascending chain for R -endomorphisms of M , where $f_i \in \text{End}_R(M)$. Note that R -endomorphisms of M are C -endomorphisms of M which commute with σ defined in Lemma 1.3 (See the Proof of Proposition 1.3). Hence any ascending chain for R -endomorphisms of M stabilizes. So M is endo-noetherian, contradiction. Thus R is an *Artinian* principal ideal ring.
- Assume that R is an *Artinian* principal ideal ring. Then, R is an *EKFN*-ring. $\square$

References

- [1] Anderson FW, Fuller KR. *Rings and Categories of Modules*, Grad. Texts in Math vol.13 Springer-Verlag, New York 1992.
- [2] Barry M, Diankha O, Sangharé M. On commutative *FGI*-rings. *Math.Sci.Res.J.* 2005; **9** No.4: 87-91.
- [3] Baumslag G. Hopficity and abelian groups, Topics in abelian groups. *Scott Foresman and Company.* 1963: 331-335.
- [4] Chambert-Loir A. *Algèbre commutative*. Version du 29 novembre 2001, 16h 51. French.

- [5] Gueye CT, Sangharé M. On commutative FGS -rings. *Comm. Algebra*. 2004; **32** No.5: 1715-1727.
- [6] Hmaimou A, Kaidi A, Sánchez Campos E. Generalized Fitting modules and rings. *Journal of Algebra*. 2007; **308**: 199-214.
- [7] Kaidi A. Modules with chain conditions on Endoimages Endokernels. preprint.
- [8] Kaidi A, Sangharé M. Une caractérisation des anneaux artiniens à idéaux principaux. *Lecture Notes in Math (Springer-Verlag)*. 1988; **1328**: 245-254. French.
- [9] Malliavin MP. *Algèbre commutative: Applications en géométrie et théorie des nombres*, Masson, 1985. French.
- [10] Noether E. Idealtheorie in Ringbereichen. *Mathematische Annalen*. 1921; **83**: 24-66. German.
- [11] Sangharé M. Sur quelques classes d'anneaux liées au lemme de Fitting. *Rend.sem.Mat.Univ.Padova*. 1992; **87**: 29-37. French.