

On commutative \$EKF\$-ring with ascending chain condition on annihilators.

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ABSTRACT (ARIAL, BOLD, 11 FONT, LEFT ALIGNED, CAPS)

We consider the class $\mathcal{E}$ of endo-noetherian modules, i.e. the modules M which satisfying the ascending chain condition for endomorphic kernels: any ascending chain $\text{Ker } f_1 \subset \text{Ker } f_2 \subset \dots \subset \text{Ker } f_n \subset \dots$ is stationary, where $f_i \in \text{End}(M)$. Let $\mathcal{N}$ be the class of Noetherian modules. It is clear that every Noetherian R -module M is endo-noetherian, so $\mathcal{N} \subset \mathcal{E}$, but the converse is not true. The aim of this work, is to characterize commutative rings for which $\mathcal{E}$ and $\mathcal{N}$ are identical.

Keywords : Annihilator; Artinian; EKF -ring; endo-noetherian; hopfian; Noetherian; strongly hopfian. (Arial, inclined, 10 font, justified)

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1. INTRODUCTION (ARIAL, BOLD, 11 FONT, LEFT ALIGNED, CAPS)

The ascending chain condition is a mathematical property on orders, which used by Emmy Noether in 1921 \cite{noe} in the context of commutative algebra. The rings named, Noetherian rings, have been the subject of extensive study to modules. The descending chain condition on ideals was introduced by Artin. Rings in which any descending chain of ideals is finite are called Artinian. We say that a module is Noetherian (resp. Artinian) if any ascending (resp. descending) chain of submodules is stationary. In fact, the finiteness of the dimension of a vector space E is equivalent to the Noetherian or Artinian condition, then we have the following properties:

\begin{itemize}

\item[\rm(i)] any surjective endomorphism is injective;

\item[\rm(ii)] any injective endomorphism is surjective.

\end{itemize}

Condition (i) (resp. (ii)) has inspired, the concept of hopfian (resp. co-hopfian) modules. An R -module is said hopfian if any surjective homomorphism from M to M is injective. This concept began with the introduction of hopfian groups by G. Baumslag in 1963 \cite{bau}. Since then, we focused on other issues relating to hopfian and co-hopfian properties, including:

\begin{enumerate}

\item the characterization of rings over which every hopfian (resp. co-hopfian, resp. fitting) module is Noetherian (resp. Artinian, resp. of finite length);

\item the characterization of rings for which every hopfian (resp. co-hopfian) module is finitely generated.

\end{enumerate}

The question in 1 has been fully resolved, in the commutative case by A. Kaidi and M. Sanghar\{e}, who have established in 1988 the following results in \cite{eks}: Let R be a commutative ring, then the following conditions are equivalent:

\begin{enumerate}

\item[\rm(a)] R is an IS -ring;\footnote{An IS -ring R is a ring such that every hopfian R -module is Noetherian.}

\item[\rm(b)] R is an SS -ring;\footnote{An SS -ring R is a ring such that every co-hopfian R -module is Artinian.}

\item[\rm(c)] R is an Artinian principal ideal ring.

\end{enumerate}

In 1992 M. Sanghar\{e} has established in \cite{sanghare}: Let R be a commutative ring, then the following conditions are equivalent:

\begin{enumerate}

\item[\rm(a)] R is an IS -ring;

\item[\rm(b)] R is an SS -ring;

\item[\rm(b)] R is an FF -ring;\footnote{An FF -ring R is a ring such that every fitting R -module is of finite length.}

\item[\rm(c)] R is an Artinian principal ideal ring.

\end{enumerate}

The characterization of $FGSS$ -rings,\footnote{An $FGSS$ -ring R is a ring for which every hopfian R -module is finitely generated.} was fully resolved in the commutative case, by C.T. Gueye and M. Sanghar\{e} in 2004 \cite{cts}. They

established that: a ring R is an FGS-ring if and only if R is an Artinian principal ideal ring. The characterization of FGI-rings, footnote{An FGI-ring R is a ring for which every co-hopfian R -module is finitely generated.} was fully resolved in the commutative case by M. Barry et al in 2005 \cite{bar}. They established that: a ring R is an FGI-ring if and only if R is an Artinian principal ideal ring.

Nevertheless, the question remains open in the general case. In 2007 A. Hmaimou et al introduced in \cite{hks}, the notions of modules that involve both the conditions of chains and endomorphism, defined as follows: an R -module M is said strongly hopfian (resp. strongly co-hopfian) if for every endomorphism f of M , then $\text{Ker } f \subset \text{Ker } f^2 \subset \dots$ (resp. $\text{Im } f \supset \text{Im } f^2 \supset \dots$) is stationary. They showed that the class of strongly hopfian (resp. strongly co-hopfian) modules is contained the class of Noetherian (resp. Artinian) modules and the hopfian (resp. co-hopfian) modules.

Finally A. Kaidi introduced in 2009 in \cite{amine} the class of endo-noetherian (resp. endo-artinian) modules as follows: an R -module M is said to be endo-noetherian (resp. endo-artinian) if for every endomorphism f_i in M , then $\text{Ker } f_1 \subset \text{Ker } f_2 \subset \dots$ (resp. $\text{Im } f_1 \supset \text{Im } f_2 \supset \dots$) is stationary. He showed that the class of endo-noetherian (resp. endo-artinian) modules is between the class of Noetherian (resp. Artinian) modules and the strongly hopfian (resp. strongly co-hopfian) modules.

The purpose of this paper is to characterize rings on which an endo-noetherian module M is Noetherian. Such rings will be called EKFN-rings. We prove that a commutative ring with ascending chain condition on annihilators is an EKFN-ring if and only if it is an Artinian principal ideal ring.

Our paper is structured as follows: the first section covers the properties and basic aspects of certain classes of modules and rings; special attention is paid to the properties of endo-noetherian modules. In the second section, we characterize commutative EKFN-rings satisfying ascending chain condition on annihilators.

2. PRELIMINARIES

The rings considered in this section are associative with unit. Unless otherwise mentioned, all the modules considered are left unitary modules.

\begin{proposition} (Schur's Lemma)

If S and T are two simple modules. Then every non-zero homomorphism is an isomorphism.

\end{proposition}

\begin{theorem} (\cite{acl}, P.220)

For a ring R the following statements are equivalent:

\begin{enumerate}

\item R is Noetherian and every prime ideal is maximal;

\item R is Artinian.

\end{enumerate}

\end{theorem}

\begin{lemma} (\cite{eks}, P.248)

Let R be an Artinian ring having at least one non-principal ideal. Then R admits a local quotient ring B with maximal ideal J such that $J^2=(0)$, and such that J/J^2 is a B/J -vector space of dimension 2.

\end{lemma}

\begin{lemma} (\cite{eks}, P.248) \label{a}

Let R be an Artinian ring having at least one non-principal ideal. Then R admits a quotient ring $B=C\oplus bC$, where C is a local subring of B , with maximal ideal $rC \neq 0$, and $b \neq 0$ with $r^2=rb=b^2=0$.

96 \end{lemma}

97 \begin{lemma} (\cite{eks}, P.249) \label{1}\

98 Let C be a local ring with maximal ideal $\mathfrak{r} \neq 0$, where $\mathfrak{r}^2 = 0$. Let M be the total ring of fractions of the ring of
99 polynomials $C[X]$, and σ the C -endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$, then:

100 \begin{enumerate}

101 \item $\sigma = \sigma^2 = 0$;

102 \item If f is a C -endomorphism of M commuting with σ , then for all $m \in M$, $f(rm) = rf(1)m$.

103 \end{enumerate}

104 \end{lemma}

105 \begin{proposition} (\cite{eks}, P.250) \label{2}\

106 Let R be an Artinian ring having at least one non-principal ideal. Then there exists an R -module M which is not
107 finitely generated.

108 \end{proposition}

109 \textbf{Proof :} By Lemma \ref{a}, assume that $R = C \oplus bC$, where C is a local subring of R , with maximal ideal
110 $\mathfrak{r} \neq 0$, and $b \neq 0$ with $\mathfrak{r}^2 = \mathfrak{r}b = b^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$
111 and φ the map defined by: $R \rightarrow \text{End}(M)$ where $\alpha + b\beta \mapsto \alpha 1_M + \beta \sigma$;
112 $\alpha, \beta \in C$, 1_M the identity map of M and σ the C -endomorphism defined in Lemma \ref{1}.
113 φ is a ring homomorphism which gives to M an R -module structure and for this structure, R -endomorphisms
114 of M are C -endomorphisms of M which commute with σ . To show that M is not a finitely generated R -
115 module, it suffices to note that M contains infinite direct sum of non-zero submodules $\bigoplus_{n \geq 1} RrX^n$.
116 $\square$

117 \begin{definition}

118 An R -module M is called endo-noetherian if any ascending chain of endomorphic kernels $\text{Ker } f_1 \subset \text{Ker } f_2$
119 $\subset \dots \subset \text{Ker } f_n \subset \dots$ stabilizes, where f_i 's are endomorphisms of M , i.e. there exists a
120 positive integer n such that $\text{Ker } f_n = \text{Ker } f_{n+1}$.

121 \end{definition}

122 \begin{proposition} \cite{amine}\

123 $\mathbb{Q}$ is a non-noetherian $\mathbb{Z}$ -module which is endo-noetherian.

124 \end{proposition}

125 \begin{proposition} \cite{amine} \label{3}\

126 A ring R is endo-noetherian if and only if R satisfies the ascending chain condition on principal left annihilators.

127 \end{proposition}

128 \begin{proposition} \cite{amine}\

129 Let M be an endo-noetherian R -module. Then M satisfies the ascending chain condition on direct summands.

130 \end{proposition}

131 \begin{corollary} \label{4} \cite{amine}\

132 Let M be an endo-noetherian R -module that is a direct sum of non-zero submodules, $M = \bigoplus_{i \in I} M_i$. Then the set I is finite.

133

134 $\end{corollary}$

135 $\begin{proposition} \label{5} \text{cite{amine}} \\\$

136 Let M be an R -module and N a submodule of M : if M is endo-noetherian and N is a direct summand then

137 N is endo-noetherian.

138 $\end{proposition}$

139 $\textbf{Proof :}$ Assume that $M = N \oplus L$. Let $\text{Ker } h_1 \subset \text{Ker } h_2 \subset \dots \subset \text{Ker } h_n \subset \dots$ be an

140 ascending chain, where the h_i 's are endomorphisms of N that can be extended to M as $f_i = h_i \oplus 0$, 0

141 being the zero endomorphism of L . Since M is endo-noetherian, there exists $n \in \mathbb{N}$ such that $\text{Ker } f_n =$

142 $\text{Ker } f_{n+1}$. Then $\text{Ker } h_n = \text{Ker } h_{n+1}$. Thus, N is endo-noetherian. $\square$

143 $\begin{proposition} \label{6} \text{cite{amine}} \\\$

144 Let M_i , $i \in I$, be a family of R -modules such that the set $\text{Hom}(M_i, M_j) = \{0\}$ for $i \neq j$. Let $M =$

145 $\bigoplus_{i \in I} M_i$. Then M is endo-noetherian if and only if for every $i \in I$ the module M_i is

146 endo-noetherian.

147 $\end{proposition}$

148 $\textbf{Proof :}$ By Proposition \ref{5}, if M is endo-noetherian, so is M_i .

149 Assume now that every M_i is endo-noetherian. Since $\text{Hom}(M_i, M_j) = 0$ for all $i \neq j$, we have $\text{End}(M) \cong$

150 $\prod_{i \in I} \text{End}(M_i)$. And if $f = (f_i)$ is an endomorphism of M , then we have $f(\sum a_i) = \sum$

151 $f_i(a_i)$: this sum is well defined because of the fact that almost all a_i vanish. Thus $f = \bigoplus_{i \in I} f_i$.

152 For every endomorphism $f \in \text{End}(M)$ we have $\text{Ker } f = \bigoplus_{i \in I} \text{Ker } f_i$. So, every ascending

153 chain of kernels of $\text{End}(M)$ gives a chain of the same type in each $\text{End}(M_i)$, that stabilizes. Hence M is endo-

154 noetherian. $\square$

155 **3. THE MAIN RESULTS**

156 Let R be an associative and commutative ring with identity $1 \neq 0$.

157 $\begin{definition}$

158 R is said to be an EKFN -ring\footnote{\$\text{EKFN}\$ is derived from the two concepts namely Endo Kernel Finite and

159 Noetherian.} if every endo-noetherian R -module is Noetherian.

160 $\end{definition}$

161 $\begin{example}$

162 A semisimple ring and an Artinian ring with principal ideal, are EKFN -rings.

163 $\end{example}$

164 $\begin{proposition} \label{7}$

165 Every homomorphic image of an EKFN -ring is an EKFN -ring.

166 $\end{proposition}$

167 $\begin{proposition}$

168 Any integral domain EKFN -ring is a field.

169 $\end{proposition}$

Proof : Let R be an EKFN-ring. Let K be the field of fractions of R . We know that K has an R -module structure. Let f be an R -endomorphism, then $\text{Ker } f = 0$ or $\text{Ker } f = K$. It follows that any ascending chain for endomorphic kernels of K stabilizes. Then K is Noetherian. We know that the map $\phi: R \rightarrow K$ is a monomorphism, so $R \subset K$. K is Noetherian, then K is a finitely generated R -module, so K is a fractional ideal of R by ([1], P.134). Thus there exists $r \in R$ such that $rK \subset R$. But $rK = K$, hence $K \subset R$. $\square$

$\begin{corollary} \label{8}$

Any prime ideal of an EKFN-ring is maximal.

$\end{corollary}$

Proof : Let P be a prime ideal of R . Let π be the canonical surjection defined by $R \rightarrow R/P$ where $r \mapsto \tilde{r}$. By Proposition [7] the quotient ring of the integral domain R/P is an EKFN-ring, then R/P is a field. So P is maximal. $\square$

$\begin{proposition}$

Let R be an EKFN-ring. Then the set of all maximal ideals of R is finite.

$\end{proposition}$

Proof : Let $\mathcal{L}$ be the set of all primes ideals of R . Let's show that $|\text{Hom}_R(R/P_i, R/P_j)| = \{0\}$ for all $i \neq j$. Let $f \in \text{Hom}_R(R/P_i, R/P_j)$. For every prime ideal P_i , R/P_i is a simple R -module, and by Schur's Lemma f is zero or an isomorphism. Assume that f is an isomorphism, we know that $\tilde{1}_{P_j} \in R/P_j$, so there exists a unique $r \notin P_i$ such that $f(\tilde{r}_{P_i}) = \tilde{1}_{P_j}$. For all $\alpha \in P_i$, $f(\alpha \tilde{r}_{P_i}) = \alpha f(\tilde{r}_{P_i}) = \alpha \tilde{1}_{P_j} = \tilde{\alpha}_{P_j}$, but $f(\alpha \tilde{r}_{P_i}) = f(r \tilde{\alpha}_{P_i}) = f(\tilde{0}_{P_i}) = \tilde{0}_{P_j}$, then $\tilde{\alpha}_{P_j} = \tilde{0}_{P_j}$, so $\alpha \in P_j$. Thus $P_i \subset P_j$. Conversely for all $\alpha \in P_j$, $f(\alpha \tilde{r}_{P_i}) = \tilde{\alpha}_{P_j} = \tilde{0}_{P_j} = f(\tilde{0}_{P_i})$, but $f(\alpha \tilde{r}_{P_i}) = f(\tilde{\alpha}_{P_i}) = \tilde{\alpha}_{P_j}$, then $\tilde{\alpha}_{P_j} = \tilde{0}_{P_j}$, so $\alpha \in P_i$, hence $\alpha \in P_i$. Thus $P_j \subset P_i$. Finally $P_j = P_i$, contradiction. Then $f = 0$, so $|\text{Hom}_R(R/P_i, R/P_j)| = \{0\}$ for all $i \neq j$. Let $M = \bigoplus_{i \in I} R/P_i$, let's show that M is endo-noetherian. We know that R/P_i is endo-noetherian. By Proposition [6], M is endo-noetherian. Now it follows from

Corollary [4] that I is finite. $\square$

$\begin{proposition}$

Let C be a local ring with maximal ideal $\mathfrak{m} \neq 0$, where $\mathfrak{m}^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C -endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$. Then every non-zero C -endomorphism F of M commuting with σ is such that either $\text{Ker } F = \mathfrak{m}M$, or is a monomorphism.

$\end{proposition}$

Proof : Note that a ring element $m \in M$ is invertible if and only if $m \notin \mathfrak{m}M$. Indeed, let $m \in \mathfrak{m}M$, and assume that m is invertible in M i.e. there exists $m' \in M$ such that $mm' = m'm = 1$. Since $m = r_1m_1$ where $m_1 \in M$, we have $r_1m_1m' = 1$, then $r_1^2 m_1 m' = r = 0$. Contradiction as $r \neq 0$. Conversely, if $m \notin \mathfrak{m}M$, then $m \notin \mathfrak{m}C$. As $\mathfrak{m}C$ is a maximal ideal of the local ring C , then m is invertible in C , hence in M .

If $F(1)$ is not invertible, then $F(1) \in \mathfrak{m}M$. For any $m \in \mathfrak{m}M$ there exists $q \in M$ such that $m = rq$. Hence by Lemma [1], $F(m) = F(rq) = rF(q) = mF(1) = 0$. So $\mathfrak{m}M \subset \text{Ker } F$ (last). Let's show that in this case for all $m \notin \mathfrak{m}M$, then $F(m) \neq 0$. We have for all $m \in M \setminus \mathfrak{m}M$, $rm \in \mathfrak{m}M \setminus \{0\}$, then $F(rm) = rF(1) = 0$. Or $F(rm) = rF(m) = 0$, so $F(m) \in \mathfrak{m}M$ i.e. there exists $\alpha \in M$ such that $F(m) = r\alpha$. Assume that $\alpha \in \mathfrak{m}M$, so $F(m) = 0$. Then $m \in \text{Ker } F$, i.e. $M \setminus \mathfrak{m}M \subset \text{Ker } F$ (last last). Hence (last) and (last last) imply that $M \subset \text{Ker } F$, then F is identically zero (impossible by assumption). Hence, it follows that $\alpha \notin \mathfrak{m}M$. So $F(m) = r\alpha \neq 0$. Then, for all $m \notin \mathfrak{m}M$ $F(m) \neq 0$. Hence $m \in \text{Ker } F \rightarrow m \in \mathfrak{m}M$ so $\text{Ker } F \subset \mathfrak{m}M$ (last last last). Thus (last) and (last last last) imply that $\text{Ker } F = \mathfrak{m}M$.

216 If $F(1)$ is invertible, i.e. $F(1) \notin rM$. Let m be a non-zero element of M . Let's show that $F(m) \neq 0$. In case
 217 $m \in rM$, then $F(m) = mF(1)$. Hence $F(m) \neq 0$. If now $m \notin rM$, then $rm \neq 0$, so $F(rm) = rF(m)$,
 218 hence $F(rm) = rF(m) = rmF(1) \neq 0$. Then $F(m) \neq 0$. Thus F is a monomorphism. $\square$

219 $\begin{theorem} \textbf{Characterization theorem} \end{theorem}$

220 Let R be a ring satisfying acc on annihilators, the following conditions are equivalent:

221 $\begin{enumerate}$

222 $\text{item } R$ is an EKFN-ring;

223 $\text{item } R$ is an Artinian principal ideal ring.

224 $\end{enumerate}$

225 $\end{theorem}$

226 $\textbf{Proof :}$

227 $\begin{itemize}$

228 item Assume that R is an EKFN-ring. Since R satisfies the acc on annihilators, then by Proposition \ref{3}, R is
 229 endo-noetherian, so R is Noetherian. We know by Corollary \ref{8} that any prime ideal of an EKFN-ring is maximal.
 230 Thus R is Artinian. Assume that R has at least one non-principal ideal, so there exists an R -module M which is
 231 not finitely generated by Proposition \ref{2}. So M is non-noetherian. Let $Ker f_1 \subset Ker f_2 \subset \dots$
 232 $\subset Ker f_n \subset \dots$ be an ascending chain for R -endomorphisms of M , where $f_i \in End_R(M)$. Note that
 233 R -endomorphisms of M are C -endomorphisms of M which commute with σ defined in Lemma \ref{1}
 234 (See the Proof of Proposition \ref{2}). Hence any ascending chain for R -endomorphisms of M stabilizes. So M is
 235 endo-noetherian, contradiction. Thus R is an Artinian principal ideal ring.

236 item Assume that R is an Artinian principal ideal ring. Then, R is an EKFN-ring. $\square$

237 $\end{itemize}$

238 **COMPETING INTERESTS**

239

240 Declaration of competing interest should be placed here. All authors must disclose any financial and personal
 241 relationships with other people or organizations that could inappropriately influence (bias) their work. Examples of
 242 potential conflicts of interest include employment, consultancies, honoraria, paid expert testimony, patent
 243 applications/registrations, and grants or other funding. If no such declaration has been made by the authors, SDI reserves
 244 to assume and write this sentence: "Authors have declared that no competing interests exist."

245

246 **AUTHORS' CONTRIBUTIONS**

247

248 Mohameth Alassane Ndiaye designed the study, performed the statistical analysis, wrote the protocol, and wrote the first
 249 draft of the manuscript. Cheikh Thi\{e\}coumba Gueye managed the analyses of the study.

250

251 **CONSENT (WHERE EVER APPLICABLE)**

252

253 All authors declare that 'written informed consent was obtained from the patient (or other approved parties) for publication
 254 of this case report and accompanying images. A copy of the written consent is available for review by the Editorial
 255 office/Chief Editor/Editorial Board members of this journal."

256

257 **ETHICAL APPROVAL (WHERE EVER APPLICABLE)**

258

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272 declare that "Principles of laboratory animal care" (NIH publication No. 85-23, revised 1985) were followed, as well as
273 specific national laws where applicable. All experiments have been examined and approved by the appropriate ethics
274 committee”

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279 copy of the IRB or Ethical Committee Approval at any stage of publication (Pre of post publication stage).

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282 hereby declare that all experiments have been examined and approved by the appropriate ethics committee and have
283 therefore been performed in accordance with the ethical standards laid down in the 1964 Declaration of Helsinki.”

284
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