

A General Inequality Related to Variational Inequalities and its Consequences

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ABSTRACT

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty subset of X . Let $g: K \rightarrow \mathbb{R}$ be a map. In this paper we study the following problem: "Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ ". Some results on this problem have been studied by Takahasi, Park and Kim and Yen. Behera and Panda recognized this problem as an independent problem and proved several results on the existence of solution of this problem in the setting of a Banach space. This problem includes as special cases, many problems on variational inequalities and generalized variational inequalities studied by many authors. In this paper we also introduce and study another problem, "Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ " where $S: K \rightarrow 2^K$ is any point-to-set map. This problem includes as a particular case the classical quasi-variational inequality problem. A generalization of Minty's Lemma is also studied.

Keywords: Partition of unity, upper and lower semicontinuous point-to setmap, monotone operator

1. INTRODUCTION

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty subset of X . Let the value of $u \in X^*$ at $x \in X$ be denoted by (u, x) . Let $g: K \rightarrow \mathbb{R}$ be a map (possibly nonlinear). The classical minimization problem for the pair (g, K) is to find $x_0 \in K$ such that

$$g(x_0) = \min_{y \in K} g(y).$$

If we define a function $f: K \times K \rightarrow \mathbb{R}$ as $f(x, y) = g(y) - g(x)$ for all $x, y \in K$, then the above problem reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

If $T: K \rightarrow X^*$, then the nonlinear variational inequality problem is to find $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$. If we define $f: K \times K \rightarrow \mathbb{R}$ as $f(x, y) = (Tx, y - x)$, then it reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

The above discussions lead to consider the following problem:

P₁ : Given $f: K \times K \rightarrow \mathbb{R}$, find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

35 Special cases

36 1. If $f(x, y) = (Tx, \theta(y, x))$ for some maps $T: K \rightarrow X^*$, and $\theta: K \times K \rightarrow X$, then P_1 reduces to
 37 the problem of finding $x_0 \in K$ such that $(Tx_0, \theta(y, x_0)) \geq 0$ for all $y \in K$, which is the
 38 *variational-like inequality problem* studied by Behera and Panda [2, 4], Dien [9], Noor [15]
 39 and Siddiqi et. al. [21–23]. Yang and Chen [27] also studied this problem calling it a *pre-*
 40 *variational inequality problem* because of the relation of this problem with the *pre-invex*
 41 *functions* (see [6, 27]).

42
 43 2. If $z \in K$ is fixed and $f(x, y) = (T((z+x)/2), y-x)$, then P_1 reduces to the problem of
 44 finding $x_0 \in K$ such that $(T((z+x_0)/2), y-x_0) \geq 0$ for all $y \in K$, which is the *variational-type*
 45 *inequality problem* originally introduced by Behera and Panda [3] in the setting of Banach
 46 spaces and subsequently studied by Panda and Dash [16] in the setting of topological
 47 vector spaces.

48
 49 3. If $f(x, y) = (Tx - Ax, y - x)$ for some maps $T, A: K \rightarrow X^*$, then P_1 reduces to the problem
 50 of finding $x_0 \in K$ such that $(Tx_0, y - x_0) \geq (Ax_0, y - x_0)$ for all $y \in K$ which is the *strongly*
 51 *nonlinear variational inequality problem* studied by Nanda [12] and Noor [13].

52
 53 4. If $f(x, y) = (Tx - Ax, g(y) - g(x))$ for some maps $T, A: K \rightarrow X^*$, $g: K \rightarrow K$, then P_1 reduces to
 54 the problem of finding $x_0 \in K$ such that $(Tx_0, g(y) - g(x_0)) \geq (Ax_0, g(y) - g(x_0))$ for all $y \in K$,
 55 which is the *strongly nonlinear implicit variational inequality problem* introduced and studied
 56 by Noor [13] in connection with the solution of the differential equations of odd order.

57
 58 5. If $f(x, y) = (Tx - Ax, \theta(y, x))$ for some maps $T, A: K \rightarrow X^*$ and $\theta: K \times K \rightarrow X$, then P_1 reduces
 59 to the problem of finding $x_0 \in K$ such that $(Tx_0, \theta(y, x_0)) \geq (Ax_0, \theta(y, x_0))$ for all $y \in K$, which is
 60 the *strongly nonlinear variational-like inequality problem* studied by Panda and Dash [17].

61
 62 Some results on Problem P_1 are available in Takahasi [25], Park and Kim [18] and Yen [28].
 63 Behera and Panda [5] recognized P_1 as an independent problem and proved several results
 64 on the existence of solution of this problem in the setting of Banach spaces. Many authors
 65 used results on Problem P_1 to prove the existence of solution of variational and generalized
 66 variational inequality problems.

67
 68 In this paper, we prove some results on the existence of solution of Problem P_1 under
 69 different assumptions in the setting of Hausdorff topological vector spaces. We also use a
 70 result on the existence of solution of Problem P_1 for the study the following problem, which is
 71 a generalization of the quasi-variational inequality problem introduced by Benoussan and
 72 Lions [7] in connection with impulse control and subsequently studied by Baicchi and
 73 Capello [1] and Mosco [11].

74 **P_2 :** Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ where $S: K \rightarrow 2^K$ is any point-to-
 75 set map.

76
 77 We also prove a theorem on the uniqueness of solution of Problem P_1 which would serve as
 78 a generalization of the well-known Minty's Lemma (see [8], p.6).

79

2. PRILIMINARIES

In this section we recall some definitions and known results which will be needed in the sequel. Throughout this section, X is a real Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X .

2.1 Definition. A mapping $T: K \rightarrow X^*$ is said to be monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in K$. T is said to be strictly monotone if strict inequality holds whenever $x \neq y$. T is said to be pseudo-monotone if $x, y \in K$ and $(Tx, x - y) \geq 0$ then $(Ty, x - y) \geq 0$.

2.2 Definition. A function $h: K \rightarrow \mathbb{R}$ is said to be upper semi-continuous, if for each real number λ , the set $\{x \in K: h(x) < \lambda\}$ is open; f is said to be lower semi-continuous if $-f$ is upper semi-continuous.

2.3 Definition. A function $h: K \rightarrow \mathbb{R}$ is said to be quasi-convex if for each real number λ , the set $\{x \in K: h(x) < \lambda\}$ is convex; function f is said to be quasi-concave if $-f$ is quasi-convex.

2.4 Definition. Let M and N be two topological spaces, and $h: M \rightarrow 2^N$, a point-to-set map. Then h is said to be upper semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \subset U$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \subset U$; h is said to be lower semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \cap U \neq \emptyset$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \cap U \neq \emptyset$; h is said to be upper semi-continuous (lower semi-continuous) on M if h is upper semi-continuous (lower semi-continuous) at each point of M .

2.5 Theorem. (Isac [10]) Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let f and g be two real valued functions on $K \times K$ having the following properties :

- (a) for all $x, y \in K$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq 0$,
- (b) for each fixed $y \in K$, $f(x, y)$ is a lower semicontinuous function of x on K ,
- (c) for each fixed $x \in K$, $g(x, y)$ is a quasi-concave function of x on K .

Then there exists a point $\hat{x} \in K$ such that $f(\hat{x}, y) \leq 0$ for all $y \in K$.

2.6 Theorem. (Shih and Tan [24]) Let X be a Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X . Let $S: K \rightarrow 2^K$ be an upper semicontinuous point-to-set map such that for each $x \in K$, $S(x)$ is nonempty and bounded. Then for each $p \in X^*$, the map $f_p: K \rightarrow \mathbb{R}$ defined by $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semicontinuous.

2.7 Theorem. (Takahasi [25]) Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X and let $F: K \times K \rightarrow \mathbb{R}$ be a function such that

- (a) for each $y \in K$, the map $x \mapsto F(x, y)$ is upper semicontinuous,
- (b) for each $x \in K$, the map $y \mapsto F(x, y)$ is convex,
- (c) $F(x, x) \geq \alpha$ for all $x \in K$ with some real number α .

Then there exists $x_0 \in K$ such that $F(x_0, y) \geq \alpha$ for all $y \in K$.

2.8 Theorem. (Rudin [19]) Let K be a compact subset of a topological space and let $\{V_1, V_2, \dots, V_n\}$ be a finite open covering of K . Then there exists a family $\{\beta_1, \beta_2, \dots, \beta_n\}$ of continuous real valued function on K such that $\beta_i(x) = 0$ outside V_i , $0 \leq \beta_i(x) \leq 1$ for all $i \in \{1, 2, \dots, n\}$ and for all $x \in K$, and $\sum_{i=1}^n \beta_i(x) = 1$ for all $x \in K$.

2.9 Theorem. (Tarafdar [26]) Let K be a nonempty subset of a Hausdorff topological vector space X and let $F: K \rightarrow 2^K$ be a point-to-set map such that

- 132 (a) for each $x \in K$, $F(x)$ is nonempty,
 133 (b) for each $y \in K$, $F^{-1}(y) = \{x \in K : y \in F(x)\}$ contains a relatively open subset A_y of
 134 K (A_y may be empty for some y),
 135 (c) $\bigcup_{x \in K} A_x = K$,
 136 (d) K contains a nonempty set K_0 contained in a compact convex subset K_1 of K
 137 such that $D = \bigcap_{x \in K_0} A_x^C$ is compact (D may be empty).

138 Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.
 139

140 **2.10 Theorem.** ([24], p. 34). Let X be a Hausdorff topological vector space with dual X^* and
 141 K a nonempty convex subset of X . Let $S: K \rightarrow 2^K$ be a point-to-set upper semi-continuous
 142 map such that for each $x \in K$, $S(x)$ is nonempty and bounded. Then for each $p \in X^*$ the map
 143 $f_p: K \rightarrow \mathbb{R}$ defined by $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semi-continuous.

144
 145 **2.11 Theorem.** ([20], p.201). Let X be a locally convex Hausdorff topological vector space
 146 with dual X^* and K a nonempty closed and convex subset of X . Let x^* be a point of X not in
 147 K . Then there exists $p \in X^*$ such that $(p, x^*) > \sup_{x \in K} (p, x)$.
 148
 149

150 3. EXISTENCE OF SOLUTION

151
 152 The following theorems on the existence of solution of problem P_1 are the main results of
 153 this paper.
 154

155 **3.1 Theorem.** Let K be a nonempty compact convex subset of a Hausdorff topological vector
 156 space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 157 (a) $f(x, x) \geq 0$ for each $x \in K$,
 158 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
 159 (c) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K .

160 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.
 161

162 *Proof.* Same as the proof of Theorem 2.7. Indeed Takahasi assumed that for each $y \in K$, the
 163 map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is upper semi-continuous and for each $x \in K$, the map $y \mapsto f(x, y)$
 164 of K into $\mathbb{R}$ is convex which are stronger than hypotheses (b) and (c) of this theorem.
 165

166 The following theorem generalizes Theorem 3.1 to an arbitrary nonempty convex subset K of
 167 X .
 168

169 **3.2 Theorem.** Let K be a nonempty convex subset of a Hausdorff topological vector space X
 170 and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 171 (a) $f(x, x) \geq 0$ for each $x \in K$,
 172 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
 173 (c) for each $y \in K$, the set $\{x \in C : f(x, y) < 0\}$ is open in K for every compact
 174 subset C of K ,
 175 (d) there exists a nonempty compact subset L of K such that for each
 176 $x \in K - L$, there exists $u \in L$ for which $f(x, u) < 0$.

177 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.
 178

179 *Proof.* For each $y \in K$, let

180 $A(y) = \{x \in L : f(x, y) \geq 0\}$.
 181 From hypothesis (c) it follows that $A(y)$ is a nonempty compact subset of K . The conclusion
 182 of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. To ascertain this, it is sufficient to show that the
 183 family $\{A(y) : y \in K\}$ has the finite inter-section property.

184 Let $y_1, y_2, \dots, y_n$ be arbitrary elements of K and K_0 , the convex hull of
 185 $L \cup \{y_1, y_2, \dots, y_n\}$. Then K_0 is a nonempty compact convex subset of K . By Theorem 3.1,
 186 there exists $\tilde{x}_0 \in K_0$ such that

$$187 \quad f(\tilde{x}_0, y) \geq 0 \quad \dots(1)$$

188 for all $y \in K_0$. We claim that $\tilde{x}_0 \in L$; for if $\tilde{x}_0 \notin L$, then, $\tilde{x}_0 \in K_0 - L \subset K - L$ and by hypothesis
 189 (d), there exists $u \in L$ such that $f(\tilde{x}_0, u) < 0$ which contradicts (1) when $y = u$. Thus $\tilde{x}_0 \in L$,
 190 and in particular, $\tilde{x}_0 \in A(y_i)$ for $i = 1, 2, \dots, n$; that is $\tilde{x}_0 \in \bigcap_{i=1}^n A(y_i)$. Hence, $\bigcap_{i=1}^n A(y_i) \neq \emptyset$
 191 which proves that the family $\{A(y) : y \in K\}$ has the finite intersection property. This
 192 completes the proof.

193
 194

195 If X is a normed linear space and K is locally compact, we have the following result.

196

197 **3.3 Theorem.** Let K be a nonempty locally compact convex subset of a real normed linear
 198 space X with $0 \in K$, and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 199 (a) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
- 200 (b) for all $y \in K$ and $r > 0$, the set $\{x \in K_r : f(x, y) < 0\}$ is open in K , where
- 201 $K_r = \{x \in K : \|x\| \leq r\}$,
- 202 (c) $f(x, x) = 0$ for each $x \in K$, and for each $r > 0$,
- 203 (d) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

204 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

205 *Proof.* For each $r > 0$, the set K_r is a nonempty compact and convex subset of K . Hence by
 206 Theorem 3.1 there exists $x_r \in K_r$ such that

$$207 \quad f(x_r, y) \geq 0 \quad \dots(2)$$

208 for all $y \in K_r$. We claim that $\|x_r\| < r$; for if $\|x_r\| = r$, then by hypothesis (d), $f(x_r, 0) < 0$
 209 which contradicts (2) when $y = 0$. Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently
 210 small so that

$$211 \quad y_r = \lambda x + (1-\lambda)x_r \in K_r.$$

212 Now by hypotheses (a) and (b)

$$213 \quad \begin{aligned} 0 \leq f(x_r, y_r) &= f(x_r, \lambda x + (1-\lambda)x_r) \\ &\leq \lambda f(x_r, x) + (1-\lambda)f(x_r, x_r) = \lambda f(x_r, x). \end{aligned}$$

214 Since $\lambda > 0$, it follows that $f(x_r, x) \geq 0$. Since x is arbitrary, the proof is complete.

215

216 If X is a reflexive Banach space then local compactness of K can be relaxed from the above
 217 theorem.

218

219 **3.4 Theorem.** Let K be a nonempty closed and convex subset of a reflexive real Banach
 220 space X with $0 \in K$ and $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 221 (a) $f(x, x) = 0$ for each $x \in K$,
- 222 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
- 223 (c) the set $\{x \in C : f(x, y) < 0\}$ is weakly open in K for each $y \in K$, and for each
- 224 weakly compact subset C of K .

225 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ under each of the following
 226 conditions:

- 227 (i) K is bounded,
 228 (ii) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

229
 230 *Proof.* Equip K with the weak topology. If K is bounded, then it becomes weakly compact and
 231 in this case, the proof follows from Theorem 3.1.

232
 233 If K is not bounded let

$$K_r = \{x \in K : \|x\| \leq r\}.$$

234 Then K_r is a nonempty closed, convex and bounded subset of K . By the first part of the
 235 theorem, there exists $x_r \in K_r$ such that $f(x_r, y) \geq 0$ for all $y \in K_r$. The remaining part of the
 236 proof follows from the proof of Theorem 3.3 and hence it is omitted.

237 The following theorem generalizes Theorem 3.4 to a Hausdorff topological vector space.

238
 239 **3.5 Theorem.** Let K be a nonempty convex subset of a Hausdorff topological vector space
 240 X and L , a nonempty compact and convex subset of K . Let $f: K \times K \rightarrow \mathbb{R}$ be a map such
 241 that

- 242 (a) $f(x, x) = 0$ for each $x \in L$,
 243 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 244 (c) for each $y \in K$, the set $\{x \in L : f(x, y) < 0\}$ is open in L ,
 245 (d) for every x in the boundary of L , there exists $u \in L$ such that $f(x, u) < 0$.

246 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

247 *Proof.* By theorem 3.1 there exists $x_0 \in L$ such that

248 $f(x_0, y) \geq 0$... (3)
 249 for all $y \in L$. We claim that $x_0 \in \text{int } L$ (where $\text{int } L$ stands for the interior of L), for if $x_0 \notin$
 250 $\text{int } L$, then by hypothesis (d) there exists $u \in L$ such that $f(x_0, u) < 0$ which contradicts (3)
 251 when $y = u$.

252 Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently small so that

$$y_\lambda = \lambda x + (1-\lambda)x_0 \in L.$$

253 By hypotheses (a) and (b)

$$0 \leq f(x_0, y_\lambda) \leq \lambda f(x_0, x) + (1-\lambda)f(x_0, x_0) = \lambda f(x_0, x).$$

254 Since $\lambda > 0$, it follows that $f(x_0, x) \geq 0$. This completes the proof.

255
 256 The following theorem is a variant of Theorem 3.1 in which, the hypotheses (a) and (b) of
 257 Theorem 3.1 have been replaced by a single generalized condition.

258
 259 **3.6 Theorem.** Let K be a nonempty compact convex subset of a Hausdorff topological
 260 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 261 (a) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K ,
 262 (b) for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K and
 263 $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$, $\max_{1 \leq i \leq n} f(u, y_i) \geq 0$.

264 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

265
 266 *Proof.* Assume that the assertion is false. Then for each $x \in K$, there exists $y \in K$ such that
 267 $f(x, y) < 0$. For each $y \in K$, let

268 $A(y) = \{x \in K : f(x, y) < 0\}$.
 269 Then by hypothesis (a), $A(y)$ is open for every $y \in K$. Further $K = \bigcup_{y \in K} A(y)$. Since K is
 270 compact, there exists $\{y_1, y_2, \dots, y_n\} \subset K$ such that $K = \bigcup_{i=1}^n A(y_i)$. Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be a
 271 partition of unity subordinate to the covering $\{A(y_i) : i = 1, 2, \dots, n\}$ and let
 272 $S = \text{conv}(\{y_1, y_2, \dots, y_n\})$. Then S is a compact convex subset of K . Define a continuous map
 273 $p : K \rightarrow K$ as $p(x) = \sum_{i=1}^n \beta_i(x) y_i$. Since $p(x)$ is a convex linear combination of points of the
 274 set S , $p(x) \in S$ for each $x \in K$. In particular, p maps S into S . By Browder's fixed point
 275 theorem p has a fixed point; that is, there exists $\hat{x} \in K$ such that $\hat{x} = p(\hat{x})$.

276
 277 Since $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a partition of unity, $\beta_i(\hat{x}) > 0$ for at least one i . But $\beta_i(\hat{x}) > 0$ implies
 278 that $\hat{x} \in A(y_i)$ so that
 279
$$f(\hat{x}, y_i) < 0. \quad \dots(4)$$

280 If
 281 then
$$J = \{i : 1 \leq i \leq n \text{ and } \beta_i(\hat{x}) > 0\}$$

so that $\hat{x} \in \text{conv}\{y_i : i \in J\}$. Now by hypothesis (b), there exists $i_0 \in J$ such that $f(\hat{x}, y_{i_0}) \geq 0$
 283 which contradicts (4). This contradiction proves the theorem.
 284

285 The following theorem generalizes Theorem 3.6 to an arbitrary nonempty convex subset K of
 286 X .

287
 288 **3.7 Theorem.** Let X be a locally convex Hausdorff topological vector space, K a nonempty
 289 convex subset of X , K_0 a nonempty compact convex subset of K , and L a nonempty
 290 compact subset of K . Let $f : K \times K \rightarrow \mathbb{R}$ be a map such that

- 291 (a) for each $y \in K$, the set $\{x \in L : f(x, y) \geq 0\}$ is closed in K ,
 292 (b) for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K , and
 293 $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$ $\max_{1 \leq i \leq n} f(u, y_i) \geq 0$.
 294 (c) for each $x \in K - L$, there exists $u \in \text{conv}(K_0 \cup \{x\})$ such that $f(x, u) < 0$.

295 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

296
 297 *Proof.* For each $y \in K$, let

$$298 \quad A(y) = \{x \in L : f(x, y) \geq 0\}.$$

299 From (b) it follows that $A(y)$ is closed and consequently compact for each $y \in K$.
 300 The assertion of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. For this, it is sufficient to prove that the
 301 family $\{A(y) : y \in K\}$ has the finite intersection property.

302
 303 Let $\{y_1, y_2, \dots, y_n\} \subset K$ and $S = \text{conv}(K_0 \cup \{y_1, y_2, \dots, y_n\})$. Clearly S is a non-empty compact
 304 convex subset of K . By Theorem 3.6 there exists $\hat{x} \in S$ such that

305
$$f(\hat{x}, y) \geq 0 \quad \dots(5) \text{ for all } y \in S. \text{ We claim that } \hat{x} \in L; \text{ for}$$

 306 if $\hat{x} \notin L$, that is, $\hat{x} \in S - L \subset K - L$, then there exists $u \in \text{conv}(K_0 \cup \{\hat{x}\})$ such that $f(\hat{x}, u) < 0$
 307 which contradicts (5) when $y = u$.

308 Thus $\hat{x} \in L$, and in particular, $\hat{x} \in A(y_i)$ for each $i \in \{1, 2, \dots, n\}$, that is, $\hat{x}_0 \in \bigcap_{i=1}^n A(y_i)$.
 309 Hence, $\bigcap_{i=1}^n A(y_i) \neq \emptyset$ showing that the family $\{A(y) : y \in K\}$ has the finite intersection
 310 property. This completes the proof.

311
 312 The following theorem also deals with the existence of solution of problem P_1 under pseudo-
 313 monotonicity type assumptions.
 314

315 **3.8 Theorem.** Let K be a nonempty closed and convex subset of a Hausdorff topological
 316 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 317 (a) $f(x, x) \geq 0$ for each $x \in K$,
 318 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex and lower semicontinuous for each
 319 $x \in K$,
 320 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line
 321 segments of K to the usual topology of $\mathbb{R}$,
 322 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$,
 323 (e) there exists a nonempty set K_0 contained in a compact convex subset K_1 of
 324 K such that $D = \bigcap_{y \in K_0} \{y \in K: f(x, y) \leq 0\}$ is either compact or empty.

325 Then there exists $x_0 \in K$ such that

$$326 \quad f(x_0, y) \geq 0 \quad \dots(6)$$

327 for all $y \in K$.

328 **Note:** If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then condition (d) of the
 329 above theorem is equivalent to the fact that T is pseudo-monotone.
 330

331 To prove the above theorem, we need the following lemma.

332 **3.9 Lemma.** Let K be a nonempty closed and convex subset of a Hausdorff topological
 333 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that
 334

- 335 (a) $f(x, x) \geq 0$ for each $x \in K$,
 336 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 337 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line segments
 338 of K to the usual topology of $\mathbb{R}$,
 339 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$.

340 Then the following are equivalent:

- 341 (A) $x_0 \in K$ and $f(x_0, y) \geq 0$ for all $y \in K$.
 342 (B) $x_0 \in K$ and $f(y, x_0) \leq 0$ for all $y \in K$.
 343

344 **Note:** If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$, then Lemma 3.9 reduces
 345 to Minty's Lemma ([8], p. 6).
 346

347 **Proof.** Assume that (A) holds. Then for some $x_0 \in K$ and for all $y \in K, f(x_0, y) \geq 0$. By
 348 hypothesis (d), $f(y, x_0) \leq 0$ for all $y \in K$ from which (B) follows.
 349

350 Now assume that (B) holds. Then for some $x_0 \in K, f(y, x_0) \leq 0$ for all $y \in K$. Fix $y \in K$,
 351 $t \in (0, 1)$ and set $y_t = (1 - t)x_0 + ty$. Then $y_t \in K$ and by (B), $f(y_t, x_0) \leq 0$ for each $t \in (0, 1)$.
 352 Now it follows from hypotheses (a) and (b) that

$$0 \leq f(y_t, y_t) \leq (1 - t)f(y_t, x_0) + tf(y_t, y).$$

353 Thus

$$f(y_t, y) \geq -\frac{1-t}{t}f(y_t, x_0) \geq 0$$

354 for each $t \in (0, 1)$. Since the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous
 355 from the line segments of K to the usual topology of $\mathbb{R}$, taking limit $t \rightarrow 0^+$ in the above
 356 inequality, we get $f(x_0, y) \geq 0$. Since $y \in K$ is arbitrary (B) follows. This completes the proof
 357 of the lemma.
 358

359 **3.10 Proof of Theorem 3.8.** Assume that inequality (6) has no solution. Then, for each
 360 $x \in K$, there exists $u \in K$ such that

$$f(x, u) < 0 \quad \dots(7)$$

and by Lemma 3.9, there exists $v \in K$ such that

$$f(v, x) > 0. \quad \dots(8)$$

Thus for each $x \in K$, it follows from (7) that the set

$$F(x) = \{y \in K: f(x, y) < 0\}$$

is nonempty, which is also convex by hypothesis (b) of the theorem. Again for each $y \in K$, since

$$F^{-1}(y) = \{x \in K: f(x, y) < 0\},$$

it follows that

$$\begin{aligned} [F^{-1}(y)]^c &= \{x \in K: f(x, y) \geq 0\} \\ &\subset \{x \in K: f(y, x) \leq 0\} \quad (\text{by (d)}) \\ &= G(y) \quad (\text{say}). \end{aligned}$$

Thus

$$F^{-1}(y) \supset [G(y)]^c = A(y) \quad (\text{say}).$$

Since for each $x \in K$, the map $y \rightarrow f(x, y)$ is lower semicontinuous, it follows that $G(y)$ is closed in K . Thus for each $y \in K$, $A(y)$ is open in K .

Now, let $x \in K$ be arbitrary. By (8) there exists $v \in K$ such that $f(v, x) > 0$, so that $x \in A(v)$. Thus $\bigcup_{y \in K} A(y) = K$. Finally, $D = \bigcap_{y \in K_0} G(y) = \bigcap_{y \in K_0} [A(y)]^c$ is either empty or compact by hypothesis (e), so that all the conditions of Theorem 2.9 are fulfilled by F . Thus there exists $x_0 \in K$ such that $x_0 \in F(x_0)$, which is tantamount to $f(x_0, x_0) < 0$, a contradiction to hypothesis (a) of the theorem. Hence the conclusion of the theorem follows.

4. UNIQUENESS OF SOLUTIONS

The following theorem characterizes the uniqueness of solution of the Problem P_1 under certain conditions.

4.1 Theorem. Let K be a nonempty subset of a topological vector space X and let $f: K \times K \rightarrow \mathbb{R}$ a map such that

- (a) for all $x, y \in K$, $f(x, y) + f(y, x) \leq 0$,
- (b) equality does not hold in (a) unless $x = y$.

Then, if problem P_1 is solvable, it has a unique solution.

Proof. If possible let x_1, x_2 be two solutions of P_1 . Then

$$f(x_1, y) \geq 0$$

and

$$f(x_2, y) \geq 0$$

for all $y \in K$. Putting $y = x_1$ in the former inequality and $y = x_2$ in the latter, we get

$$f(x_1, x_2) \geq 0$$

and

$$f(x_2, x_1) \geq 0.$$

Adding the last two inequalities we get,

$$f(x_1, x_2) + f(x_2, x_1) \geq 0.$$

This inequality combined with hypothesis (a) gives

$$f(x_1, x_2) + f(x_2, x_1) = 0.$$

Now an application of hypothesis (b) yields $x_1 = x_2$. This completes the proof.

Note: If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then condition (a) of the above theorem is equivalent to the fact that T is monotone.

The following examples illustrate Theorem 4.1. Example 1 shows that the fulfillment of conditions (a) and (b) does not guarantee the existence of a solution to problem P_1 . Examples (2) and (3) show that if conditions (a) and (b) are fulfilled and problem P_1 is solvable, then it has a unique solution.

Example 1. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define let $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = -|x - y|.$$

Then

$$f(x, y) + f(y, x) = -2|x - y|$$

and

$$f(x, y) + f(y, x) = 0$$

if $|x - y| = 0$, that is $x = y$. Thus hypotheses (a) and (b) of Theorem 4.1 are fulfilled by f . But $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $|x_0 - y| \leq 0$ for all $y \geq 0$. Thus in this case no solution exists.

Example 2. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = x(y - x).$$

Then

$$f(x, y) + f(y, x) = -(x - y)^2 \leq 0$$

and

$$f(x, y) + f(y, x) = 0$$

if and only if $(x - y)^2 = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $x_0(y - x_0) \geq 0$ for all $y \geq 0$, so that $x_0 = 0$ is the only solution in this case.

Example 3. Let $X = K = \mathbb{R}$, and define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = -x^2|x - y|.$$

Then

$$f(x, y) + f(y, x) = -(x^2 + y^2)|x - y| \leq 0$$

and

$$f(x, y) + f(y, x) = 0$$

if and only if $|x - y| = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $x_0^2|x_0 - y| = 0$ for all $y \in \mathbb{R}$ so that $x_0 = 0$ is the only solution in this case also.

5. SOME CONSEQUENCES

In this section, we discuss some consequences of the theorems proved in Section 3.

The following theorem deals with the existence of solution of problem P_2 .

5.1 Theorem. Let X be a locally convex Hausdorff topological vector space with dual X^* , K a nonempty compact convex subset of X , $f: K \times K \rightarrow \mathbb{R}$ a point-to-point map, and $S: K \rightarrow 2^K$, a point-to-set upper semicontinuous map such that

- (a) $f(x, x) \geq 0$ for each $x \in K$,
- (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi convex for each $x \in K$,
- (c) the set $\{x \in K: \sup_{y \in S(x)} f(x, y) < 0\}$ is empty or open in K .

Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $f(x_0, y) \geq 0$ for all $y \in S(x_0)$.

Proof. Assume that the assertion is false. Then for each $x \in K$ either $x \notin S(x)$, or there exists $y \in K$ such that $f(x, y) < 0$. If $x \notin S(x)$, then by Theorem 2.11, there exists $p \in X^*$ such that

$$(p, x) - \sup_{y \in S(x)} (p, y) > 0.$$

446 Let

$$V_0 = \left\{ x \in K: \sup_{y \in S(x)} f(x, y) < 0 \right\},$$

447 and for each $p \in X^*$ let

$$V(p) = \left\{ x \in K: (p, x) - \sup_{y \in S(x)} (p, y) > 0 \right\}.$$

448 Then by hypothesis (c), V_0 is open in K . By Theorem 2.6, since the map $x \mapsto \sup_{y \in S(x)} (p, y)$
 449 is upper semicontinuous for each $p \in X^*$, it follows that the map $x \mapsto (p, x) - \sup_{y \in S(x)} (p, y)$ is
 450 lower semicontinuous. Hence $V(p)$ is open in K for each $p \in X^*$. Further, $K = V_0 \cup$
 451 $\bigcup_{p \in X^*} V(p)$. Since K is compact, there exists $\{p_1, p_2, \dots, p_n\} \subset X^*$ such that $K = V_0 \cup$
 452 $[\bigcup_{i=1}^n V(p_i)]$. Let $\{\beta_0, \beta_1, \dots, \beta_n\}$ be a partition of unity subordinate to the covering
 453 $\{V_0, V(p_1), \dots, V(p_n)\}$, and define $g: K \times K \rightarrow \mathbb{R}$ as

$$g(x, y) = \beta_0(x)f(x, y) + \sum_{i=1}^n \beta_i(x)(p_i, y - x).$$

454 Then it is clear that for each $x \in K$, $g(x, x) \geq 0$, the map $y \mapsto g(x, y)$ is quasi-convex, for
 455 each $y \in K$, the map $x \mapsto g(x, y)$ is upper semicontinuous so that the set $\{x \in K: g(x, y) <$
 456 $0\}$ is open in K . Thus all the conditions of Theorem 3.1 are fulfilled. Hence there exists $\hat{x} \in K$
 457 such that

$$g(\hat{x}, y) \geq 0 \quad \dots(9)$$

458 for all $y \in K$, which is equivalent to

$$\beta_0(\hat{x})f(\hat{x}, y) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, y - \hat{x}) \geq 0.$$

460 Since $\{\beta_0, \beta_1, \dots, \beta_n\}$ is a partition of unity $\beta_i(\hat{x}) > 0$ for at least one i . If $\beta_0(\hat{x}) > 0$ then $\hat{x} \in V_0$
 461 so that $\sup_{y \in S(\hat{x})} f(\hat{x}, y) < 0$. Let $\hat{y} \in S(\hat{x})$ be such that $f(\hat{x}, \hat{y}) < 0$. If $\beta_i(\hat{x}) > 0$ then $\hat{x} \in$
 462 $V(p_i)$ so that

$$(p_i, \hat{x}) - \sup_{y \in S(\hat{x})} (p_i, y) > 0$$

463 from which it follows that

$$(p_i, \hat{x}) > \sup_{y \in S(\hat{x})} (p_i, y) > (p_i, \hat{y})$$

464 which implies that $(p_i, \hat{y} - \hat{x}) < 0$. Thus,

$$g(\hat{x}, \hat{y}) = \beta_0(\hat{x})f(\hat{x}, \hat{y}) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, \hat{y} - \hat{x}) < 0$$

465 which contradicts (9) when $y = \hat{y}$. This contradiction proves the theorem.

466

467 The following result on the existence of solution of the quasi-variational inequality problem is
 468 a direct consequence of the above theorem.

469

470 **5.2 Corollary.** Let X be a locally convex Hausdorff topological vector space with dual X^* , K a
 471 nonempty compact convex subset of X , $T: K \rightarrow X^*$ an operator and $S: K \rightarrow 2^K$ a point-to-set
 472 upper semi-continuous map such that the set $\{x \in K: \sup_{y \in S(x)} (Tx, y - x) < 0\}$ is empty or
 473 open in K . Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $(Tx_0, y - x_0) \geq 0$ for
 474 all $y \in S(x_0)$.

475

476 *Proof.* Follows directly from Theorem 5.1 with $f(x, y) = (Tx, y - x)$.

477

478

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483

484 **REFERENCES**

485

- 486 1. Baiocchi C, Capelo A. Variational and quasi variational inequalities, John Wiley and
487 Sons: New York; 1984.
- 488 2. Behera A, Panda GK. A generalization of Browder's theorem. Bull Inst
489 Math Acad Sinica. 1993; 21(2): 183–186.
- 490 3. Behera A, Panda GK. A variational-type Inequality and Minty's like lemma. Opsearch.
491 1996; 33(4): 213–220.
- 492 4. Behera A, Panda GK. Variational inequality problem in Hausdorff topological vector
493 space. Indian J Pure Appl Math, 1997; 28(2): 181–187.
- 494 5. Behera A, Panda GK. An inequality associated with variational inequalities in Banach
495 spaces and its consequences. Bull Inst Math. Acad Sinica. 1999; 27(3): 227–245.
- 496 6. Ben-Israel A, Mond B. What is invexity?. J Austral Math Soc: Ser B. 1986; 28:
497 1–9.
- 498 7. Bensoussan A, Lions JL. Nouvelle formulation des problemes de controle impulsionnel et
499 applications. C R Acad Sci: Ser A. 1973; 29: 1189–1192. French
- 500 8. Chipot M. Variational Inequalities and Flow in Porous Media. Appl Math Sci. 52,
501 Springer-Verlag: New York/Berlin; 1984.
- 502 9. Dien N.H. Some remarks on variational-like and quasi-variational-like inequalities. Bull
503 Austral Math Soc. 1992; 46: 335–342.
- 504 10. Isac G. Complementarity problems: Lecture notes in Math. **1528**, Springer-Verlag: New
505 York; 1992.
- 506 11. Mosco U. Implicit variational problems and quasi-variational inequalities. Nonlinear
507 Operators and the Calculus of Variations. Lecture notes in Mathematics, No. **453**.
508 Springer-Verlag: New York/Berlin. 1976; 83–156.
- 509 12. Nanda S. Strongly nonlinear variational inequality problem and its alternative form and
510 complementarity problem. Indian J Pure Appl Math. 1991; 22(5): 355 – 359.
- 511 13. Noor M A. Strongly nonlinear variational inequalities, C R Math Rep Acad Sci Canada
512 **4**: 1982; 213–218.
- 513 14. Noor M A. An iterative algorithm for variational inequalities. J Math Anal Appl. 1991;
514 158(2): 448–455.
- 515 15. Noor M A. Variational-like inequalities. Optimization. 1994; 30(4): 323–331.
- 516 16. Panda GK, Dash N. A variational-type inequality and its complementarity problem in
517 Hausdorff topological vector spaces. Indian J Pure Appl Math. 2000; 31(2): 153–160.
- 518 17. Panda GK, Dash N. Strongly non-linear variational-like inequalities. Indian J Pure Appl
519 Math. 2000; 31(7): 797–808.
- 520 18. Park S, Kim I. Nonlinear variational inequalities and fixed-point theorems. Bull Koer Math
521 Soc. 1989; 26(2): 139–149.
- 522 19. Rudin W. Real and Complex Analysis, 2nd ed., Tata McGraw-Hill: New Delhi; 1981.
- 523 20. Royden HL. Real Analysis, Macmillan Publishing Company: New York; 1988.
- 524 21. Siddiqi AH, Ansari QH, Ahmad R. On generalized variational-like inequalities. Indian J
525 Pure Appl Math. 1995; 26(10): 1135–1141.

- 526 22. Siddiqi AH, Ansari QH, Khan MF, Variational-like inequalities for multivalued maps,
527 Indian J Pure Appl Math. 1999; 30(2): 161–166.
- 528 23. Siddiqi AH, KhaliqA, Ansari QH. On variational-like inequalities, Ann Sci Math Quebec.
529 1994; 18(1): 39–48.
- 530 24. Shih MH, Tan KK, Generalized quasi-variational inequalities in locally convex topological
531 vector spaces. J Math Anal Appl. 1985;108(2): 333–343.
- 532 25. Takahashi W. Nonlinear variational inequalities and fixed-point theorems, J Math Soc,
533 Japan. 1976; 28(1): 168–181.
- 534 26. Tarafdar E. A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-
535 Mazurkiewicz theorem. J Math Anal Appl. 1987; 128(2): 475–479.
- 536 27. Yang XQ, Chen GY. A class of non-convex functions and pre-variational inequalities. J
537 Math Anal Appl. 1992; 169(2): 359–373.
- 538 28. Yen CL. A minimax inequality and its application to variational inequalities. Pacific J
539 Math. 1981; (2):477–481.