

Positive Solutions for Systems of Singular Higher-Order Multi-Point Boundary Value Problems

Johnny Henderson¹, Rodica Luca^{2*}

¹Baylor University, Department of Mathematics, Waco, Texas, 76798-7328 USA

²Gh. Asachi Technical University, Department of Mathematics, Iasi 700506, Romania

ABSTRACT

We study the existence of positive solutions of a system of higher-order nonlinear differential equations subject to multi-point boundary conditions, where the nonlinearities do not possess any sublinear or superlinear growth conditions and may be singular. In the proof of the main results, we use the Guo-Krasnosel'skii fixed point theorem.

Keywords: higher-order differential system, singular equations, multi-point boundary conditions, positive solutions.

2010 Mathematics Subject Classification: 34B10, 34B18.

1. INTRODUCTION

We consider the system of higher-order singular ordinary differential equations

$$(S): \begin{cases} u^{(n)}(t) + f(t, v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + g(t, u(t)) = 0, & t \in (0, T), \end{cases}$$

with the multi-point boundary conditions

$$(BC): \begin{cases} u(0) = \sum_{i=1}^p a_i u(\xi_i), & u'(0) = \dots = u^{(n-2)}(0) = 0, & u(T) = \sum_{i=1}^q b_i u(\eta_i), \\ v(0) = \sum_{i=1}^r c_i v(\zeta_i), & v'(0) = \dots = v^{(m-2)}(0) = 0, & v(T) = \sum_{i=1}^l d_i v(\rho_i), \end{cases}$$

where $n, m \in \mathbb{N}$, $n, m \geq 2$, $p, q, r, l \in \mathbb{N}$. In the case $n = 2$ or $m = 2$ the above conditions are of the form $u(0) = \sum_{i=1}^p a_i u(\xi_i)$, $u(T) = \sum_{i=1}^q b_i u(\eta_i)$, or $v(0) = \sum_{i=1}^r c_i v(\zeta_i)$, $v(T) = \sum_{i=1}^l d_i v(\rho_i)$, respectively, that is without conditions on the derivatives of u and v in the point 0.

We present some weaker assumptions on f and g , which do not possess any sublinear or superlinear growth conditions and may be singular at $t = 0$ and/or $t = T$, such that positive solutions for problem $(S) - (BC)$ exist. By a positive solution of $(S) - (BC)$, we understand a pair of functions $(u, v) \in (C([0, T]; \mathbb{R}_+) \cap C^n((0, T))) \times (C([0, T]; \mathbb{R}_+) \cap C^m((0, T)))$ satisfying (S) and (BC) with $\sup_{t \in [0, T]} u(t) > 0$, $\sup_{t \in [0, T]} v(t) > 0$. This problem is a generalization of the problem studied in [1], where in (BC) we have $a_i = 0$ for all $i = 1, \dots, p$ and $c_i = 0$ for all $i = 1, \dots, r$ (denoted by $(\widetilde{BC})$).

The system (S) with $n = m = 2$ and the boundary conditions $u(0) = 0$, $u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, $v(0) = 0$, $v(T) = \sum_{i=1}^{n-2} c_i v(\eta_i)$ has been investigated in [2]. In [3], the authors studied the

* Corresponding author: E-mail address: rluca@math.tuiasi.ro; rlucaudor@yahoo.com.

existence of positive solutions for system (S) with $n = m = 2$ and the boundary conditions $u(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = 0$, $v(1) = \alpha v(\eta)$ with $\eta \in (0,1)$, $0 < \alpha\eta < 1$ ($T = 1$). In [4], we investigated the existence and multiplicity of positive solutions for system (S) where f and g are nonsingular functions and the boundary conditions $(\widetilde{BC})$. The particular case of (S) with $n = m = 2$, $T = 1$ and boundary conditions which contain only one intermediate point has been studied in [5]. We also mention the paper [6], where the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) with $f(t, v(t))$ and $g(t, u(t))$ replaced by $c(t)\tilde{f}(u(t), v(t))$ and $d(t)\tilde{g}(u(t), v(t))$, respectively, (with $\tilde{f}$ and $\tilde{g}$ singular functions) and $(\widetilde{BC})$ where $\frac{1}{2} \leq \eta_1 < \eta_2 < \dots < \eta_q < 1$, $\frac{1}{2} \leq \rho_1 < \rho_2 < \dots < \rho_l < 1$ ($T = 1$). Some multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by using the Guo-Krasnosel'skii fixed point theorem. Namely, in [7], the authors give sufficient conditions for λ, μ, f and g (f, g nonsingular functions) such that the system

$$(S_1): \begin{cases} u^{(n)}(t) + \lambda a(t)f(u(t), v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + \mu b(t)g(u(t), v(t)) = 0, & t \in (0, T), \end{cases}$$

with the boundary conditions (BC) has positive solutions. The system (S_1) with the boundary conditions $(\widetilde{BC})$ has been studied in [8]. The system (S_1) with $n = m = 2$ and the multi-point boundary conditions $\alpha u(0) - \beta u'(0) = 0$, $u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i)$, $\gamma v(0) - \delta v'(0) = 0$, $v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i)$, has been investigated in [9]. Some particular cases of the above problems have been studied in [10-14].

In the last decades, nonlocal boundary value problems (including multi-point boundary value problems) for ordinary differential or difference equations/systems have become a rapidly growing area of research. Several phenomena in engineering, physics and life sciences can be modeled in this way. These problems have been studied by many authors by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In Section 2, we present some auxiliary results which investigate two boundary value problems for higher-order equations. In Section 3, we prove two existence results for the positive solutions with respect to a cone for our problem $(S) - (BC)$, which are based on the Guo-Krasnosel'skii fixed point theorem (see [15]) which is presented below.

Theorem 1. *Let X be a Banach space and let $C \subset X$ be a cone in X . Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A}: C \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow C$ be a completely continuous operator such that, either*

- i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, or*
- ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.*

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2. AUXILIARY RESULTS

In this section, we present some auxiliary results from [7] and [16] related to the following n -order differential equation with multi-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0, T), \quad (2.1)$$

$$u(0) = \sum_{i=1}^p a_i u(\xi_i), \quad u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u(T) = \sum_{i=1}^q b_i u(\eta_i), \quad (2.2)$$

where $n \in \mathbb{N}$, $n \geq 2$, $p, q \in \mathbb{N}$. If $n = 2$, the condition (2.2) has the form $u(0) = \sum_{i=1}^p a_i u(\xi_i)$, $u(T) = \sum_{i=1}^q b_i u(\eta_i)$.

91 **Lemma 2.1** ([7]) If $\Delta_1 = (1 - \sum_{i=1}^q b_i) \sum_{i=1}^p a_i \xi_i^{n-1} + (1 - \sum_{i=1}^p a_i)(T^{n-1} - \sum_{i=1}^q b_i \eta_i^{n-1}) \neq 0$,
 92 $0 < \xi_1 < \dots < \xi_p < T$, $0 < \eta_1 < \dots < \eta_q < T$ and $y \in C([0, T])$, then the solution of (2.1) –
 93 (2.2) is given by

$$\begin{aligned} u(t) = & - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) ds + \frac{t^{n-1}}{\Delta_1} \left\{ \left(1 - \sum_{i=1}^q b_i\right) \sum_{i=1}^p a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s) ds \right. \\ & + \left(1 - \sum_{i=1}^p a_i\right) \frac{1}{(n-1)!} \left[\int_0^T (T-s)^{n-1} y(s) ds - \sum_{i=1}^q b_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s) ds \right] \\ & + \frac{1}{\Delta_1} \left\{ \left(\sum_{i=1}^p a_i \xi_i^{n-1}\right) \frac{1}{(n-1)!} \left[\int_0^T (T-s)^{n-1} y(s) ds - \sum_{i=1}^q b_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s) ds \right] \right. \\ & \left. \left. - \left(T^{n-1} - \sum_{i=1}^q b_i \eta_i^{n-1}\right) \sum_{i=1}^p a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s) ds \right\} \right\}. \end{aligned} \quad (2.3)$$

94
 95 **Lemma 2.2** ([7]) Under the assumptions of Lemma 2.1, the Green's function for the
 96 boundary value problem (2.1) – (2.2) is given by

$$\begin{aligned} G_1(t, s) = & g_1(t, s) \\ & + \frac{1}{\Delta_1} \left[(T^{n-1} - t^{n-1}) \left(1 - \sum_{i=1}^q b_i\right) + \sum_{i=1}^q b_i (T^{n-1} - \eta_i^{n-1}) \right] \sum_{i=1}^p a_i g_1(\xi_i, s) \\ & + \frac{1}{\Delta_1} \left[t^{n-1} \left(1 - \sum_{i=1}^p a_i\right) + \sum_{i=1}^p a_i \xi_i^{n-1} \right] \sum_{i=1}^q b_i g_1(\eta_i, s), \quad (t, s) \in [0, T] \times [0, T], \end{aligned} \quad (2.4)$$

97 where

$$g_1(t, s) = \frac{1}{(n-1)! T^{n-1}} \begin{cases} t^{n-1} (T-s)^{n-1} - T^{n-1} (t-s)^{n-1}, & 0 \leq s \leq t \leq T, \\ t^{n-1} (T-s)^{n-1}, & 0 \leq t \leq s \leq T. \end{cases} \quad (2.5)$$

98
 99 By using G_1 , the solution u of problem (2.1) – (2.2) given by (2.3) can be written as $u(t) =$
 100 $\int_0^T G_1(t, s) y(s) ds$.

101
 102 **Lemma 2.3** ([16]; see also [4]) The function g_1 given by (2.5) has the properties

103 a) $g_1: [0, T] \times [0, T] \rightarrow \mathbb{R}_+$ is a continuous function and $g_1(t, s) \geq 0$ for all $(t, s) \in [0, T] \times$
 104 $[0, T]$.

105 b) $g_1(t, s) \leq g_1(\theta_1(s), s)$, for all $(t, s) \in [0, T] \times [0, T]$.

106 c) For any $c \in (0, T/2)$, we have $\min_{t \in [c, T-c]} g_1(t, s) \geq \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s), s)$, for all $s \in [0, T]$,

107 where $\theta_1(s) = s$ if $n = 2$ and $\theta_1(s) = \begin{cases} \frac{s}{1 - \left(1 - \frac{s}{T}\right)^{n-2}}, & s \in (0, T], \\ \frac{T(n-2)}{n-1}, & s = 0, \end{cases}$ if $n \geq 3$.

108
 109 **Lemma 2.4** ([7]) If $a_i \geq 0$ for all $i = 1, \dots, p$, $\sum_{i=1}^p a_i < 1$, and $b_i \geq 0$ for all $i = 1, \dots, q$,
 110 $\sum_{i=1}^q b_i < 1$, $0 < \xi_1 < \dots < \xi_p < T$, $0 < \eta_1 < \dots < \eta_q < T$, then the Green's function G_1 of the
 111 problem (2.1) – (2.2) (given by (2.4)) is continuous on $[0, T] \times [0, T]$ and satisfies $G_1(t, s) \geq$
 112 0 for all $(t, s) \in [0, T] \times [0, T]$. Moreover, if $y \in C([0, T])$ satisfies $y(t) \geq 0$ for all $t \in [0, T]$,
 113 then the unique solution u of problem (2.1) – (2.2) satisfies $u(t) \geq 0$ for all $t \in [0, T]$.

114

115 **Lemma 2.5** ([7]) Assume that $a_i \geq 0$ for all $i = 1, \dots, p$, $\sum_{i=1}^p a_i < 1$, and $b_i \geq 0$ for all $i =$
 116 $1, \dots, q$, $\sum_{i=1}^q b_i < 1$, $0 < \xi_1 < \dots < \xi_p < T$, $0 < \eta_1 < \dots < \eta_q < T$. Then the Green's function G_1
 117 of the problem (2.1) – (2.2) satisfies the inequalities

118 a) $G_1(t, s) \leq J_1(s)$, $\forall (t, s) \in [0, T] \times [0, T]$, where

$$J_1(s) = g_1(\theta_1(s), s) + \frac{1}{\Delta_1} \left[T^{n-1} \left(1 - \sum_{j=1}^q b_j\right) + \sum_{i=1}^q b_i (T^{n-1} - \eta_i^{n-1}) \right] \times$$

$$\times \sum_{i=1}^p a_i g_1(\xi_i, s) + \frac{1}{\Delta_1} \left[T^{n-1} \left(1 - \sum_{j=1}^p a_j \right) + \sum_{i=1}^p a_i \xi_i^{n-1} \right] \sum_{i=1}^q b_i g_1(\eta_i, s). \quad (2.6)$$

119 b) For every $c \in (0, T/2)$, we have

$$\min_{t \in [c, T-c]} G_1(t, s) \geq \gamma_1 J_1(s) \geq \gamma_1 G_1(t', s), \quad \forall t', s \in [0, T], \quad (2.7)$$

120 where $\gamma_1 = c^{n-1}/T^{n-1}$.

121

122 **Lemma 2.6** ([7]) Assume that $a_i \geq 0$ for all $i = 1, \dots, p$, $\sum_{i=1}^p a_i < 1$, and $b_i \geq 0$ for all $i =$
 123 $1, \dots, q$, $\sum_{i=1}^q b_i < 1$, $0 < \xi_1 < \dots < \xi_p < T$, $0 < \eta_1 < \dots < \eta_q < T$, $c \in (0, T/2)$ and $y \in C([0, T])$,
 124 $y(t) \geq 0$ for all $t \in [0, T]$. Then the solution $u(t)$, $t \in [0, T]$ of problem (2.1) – (2.2) satisfies
 125 the inequality $\min_{t \in [c, T-c]} u(t) \geq \gamma_1 \max_{t' \in [0, T]} u(t')$.

126

127 We can also formulate similar results as Lemmas 2.1-2.6 above for the boundary value
 128 problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0, T), \quad (2.8)$$

$$v(0) = \sum_{i=1}^r c_i v(\zeta_i), \quad v'(0) = \dots = v^{(m-2)}(0) = 0, \quad v(T) = \sum_{i=1}^l d_i v(\rho_i), \quad (2.9)$$

130 where $0 < \zeta_1 < \dots < \zeta_r < T$, $c_i \geq 0$ for all $i = 1, \dots, r$, $0 < \rho_1 < \dots < \rho_l < T$, $d_i \geq 0$ for all
 131 $i = 1, \dots, l$ and $h \in C([0, T])$. We denote by $\Delta_2, \gamma_2, g_2, \theta_2, G_2$ and J_2 the corresponding
 132 constants and functions for the problem (2.8) – (2.9) defined in a similar manner as
 133 $\Delta_1, \gamma_1, g_1, \theta_1, G_1$ and J_1 , respectively.

134

135 3. MAIN RESULTS

136

137 In this section, we investigate the existence of positive solutions for our problem (S) – (BC),
 138 under various assumptions on singular functions f and g .

139

140 We present the assumptions that we shall use in the sequel

141

142 (H_1) $0 < \xi_1 < \dots < \xi_p < T$, $a_i \geq 0$ for all $i = 1, \dots, p$, $\sum_{i=1}^p a_i < 1$, $0 < \eta_1 < \dots < \eta_q < T$, $b_i \geq 0$
 143 for all $i = 1, \dots, q$, $\sum_{i=1}^q b_i < 1$, $0 < \zeta_1 < \dots < \zeta_r < T$, $c_i \geq 0$ for all $i = 1, \dots, r$, $\sum_{i=1}^r c_i < 1$,
 144 $0 < \rho_1 < \dots < \rho_l < T$, $d_i \geq 0$ for all $i = 1, \dots, l$, $\sum_{i=1}^l d_i < 1$.

145 (H_2) The functions $f, g \in C((0, T) \times \mathbb{R}_+, \mathbb{R}_+)$ and there exist $p_i \in C((0, T), \mathbb{R}_+)$, $q_i \in$
 146 $C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, with $0 < \int_0^T p_i(t) dt < \infty$, $i = 1, 2$, $q_1(0) = 0$, $q_2(0) = 0$ such that

$$f(t, x) \leq p_1(t) q_1(x), \quad g(t, x) \leq p_2(t) q_2(x), \quad \forall t \in (0, T), \quad x \in \mathbb{R}_+.$$

147 (H_3) There exist $r_1, r_2 \in (0, \infty)$ with $r_1 r_2 \geq 1$ such that

$$i) \quad q_{10}^s = \limsup_{x \rightarrow 0^+} \frac{q_1(x)}{x^{r_1}} \in [0, \infty); \quad ii) \quad q_{20}^s = \limsup_{x \rightarrow 0^+} \frac{q_2(x)}{x^{r_2}} = 0.$$

148 (H_4) There exist $l_1, l_2 \in (0, \infty)$ with $l_1 l_2 \geq 1$ and $c \in (0, T/2)$ such that

$$i) \quad f_{\infty}^i = \liminf_{x \rightarrow \infty} \inf_{t \in [c, T-c]} \frac{f(t, x)}{x^{l_1}} \in (0, \infty]; \quad ii) \quad g_{\infty}^i = \liminf_{x \rightarrow \infty} \inf_{t \in [c, T-c]} \frac{g(t, x)}{x^{l_2}} = \infty.$$

149 (H_5) There exist $\alpha_1, \alpha_2 \in (0, \infty)$ with $\alpha_1 \alpha_2 \leq 1$ such that

$$i) \quad q_{1\infty}^s = \limsup_{x \rightarrow \infty} \frac{q_1(x)}{x^{\alpha_1}} \in [0, \infty); \quad ii) \quad q_{2\infty}^s = \limsup_{x \rightarrow \infty} \frac{q_2(x)}{x^{\alpha_2}} = 0.$$

150 (H_6) There exist $\beta_1, \beta_2 \in (0, \infty)$ with $\beta_1 \beta_2 \leq 1$ and $c \in (0, T/2)$ such that

$$i) \quad f_0^i = \liminf_{x \rightarrow 0^+} \inf_{t \in [c, T-c]} \frac{f(t, x)}{x^{\beta_1}} \in (0, \infty]; \quad ii) \quad g_0^i = \liminf_{x \rightarrow 0^+} \inf_{t \in [c, T-c]} \frac{g(t, x)}{x^{\beta_2}} = \infty.$$

151

152 The pair of functions $(u, v) \in (C([0, T]) \cap C^n((0, T))) \times (C([0, T]) \cap C^m((0, T)))$ is a solution
 153 for our problem (S) – (BC) if and only if $(u, v) \in C([0, T]) \times C([0, T])$ is a solution for the
 154 nonlinear integral equations
 155

$$\begin{cases} u(t) = \int_0^T G_1(t,s)f(s,v(s))ds, & t \in [0,T], \\ v(t) = \int_0^T G_2(t,s)g(s,u(s))ds, & t \in [0,T]. \end{cases} \quad (3.1)$$

160 The system (3.1) can be written as the nonlinear integral system

$$\begin{cases} u(t) = \int_0^T G_1(t,s)f\left(s, \int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau\right)ds, & t \in [0,T], \\ v(t) = \int_0^T G_2(t,s)g(s,u(s))ds, & t \in [0,T]. \end{cases} \quad (3.2)$$

161 We consider the Banach space $X = C([0,T])$ with supremum norm $\|u\| = \sup_{t \in [0,T]} |u(t)|$
 162 and define the cone $P \subset X$ by $P = \{u \in X, u(t) \geq 0, \forall t \in [0,T]\}$. For any $r > 0$, let $B_r =$
 163 $\{u \in C([0,T]), \|u\| < r\}$ and $\partial B_r = \{u \in C([0,T]), \|u\| = r\}$.

164 We also define the operator $\mathcal{A}: P \rightarrow X$ by

$$(\mathcal{A}u)(t) = \int_0^T G_1(t,s)f\left(s, \int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau\right)ds, \quad t \in [0,T]. \quad (3.3)$$

166

167 **Lemma 3.1** Assume that $(H_1) - (H_2)$ hold. Then $\mathcal{A}: P \rightarrow P$ is completely continuous.

168

169 **Proof.** We denote $\alpha = \int_0^T J_1(s)p_1(s)ds$ and $\beta = \int_0^T J_2(s)p_2(s)ds$. Using (H_2) , we deduce that
 170 $0 < \alpha < \infty$ and $0 < \beta < \infty$. By Lemma 2.4 and the corresponding lemma for G_2 , we get that
 171 $\mathcal{A}$ maps P into P .

172

173 We shall prove that $\mathcal{A}$ maps bounded sets into relatively compact sets. Suppose $D \subset P$ is an
 174 arbitrary bounded set. First we prove that $\mathcal{A}(D)$ is a bounded set. Because D is bounded,
 175 then there exists $M_1 > 0$ such that $\|u\| < M_1$ for all $u \in D$. By the continuity of q_2 , there exists
 176 $M_2 > 0$ such that $M_2 = \sup_{x \in [0,M_1]} q_2(x)$. By using Lemma 2.5 for G_2 , for any $u \in D$ and
 177 $s \in [0,T]$, we obtain

$$\int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau \leq \int_0^T G_2(s,\tau)p_2(\tau)q_2(u(\tau))d\tau \leq \beta M_2. \quad (3.4)$$

178 Because q_1 is continuous, there exists $M_3 > 0$ such that $M_3 = \sup_{x \in [0,\beta M_2]} q_1(x)$. Therefore,
 179 from (3.4), (H_2) and Lemma 2.5, we deduce

$$\begin{aligned} (\mathcal{A}u)(t) &\leq \int_0^T G_1(t,s)p_1(s)q_1\left(\int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau\right)ds \\ &\leq M_3 \int_0^T J_1(s)p_1(s)ds = \alpha M_3, \quad \forall t \in [0,T]. \end{aligned} \quad (3.5)$$

180 So, $\|\mathcal{A}u\| \leq \alpha M_3$ for all $u \in D$. Therefore $\mathcal{A}(D)$ is bounded.

181

182 In what follows, we shall prove that $\mathcal{A}(D)$ is equicontinuous. By using Lemma 2.2, for all
 183 $t \in [0,T]$, we have

$$\begin{aligned} (\mathcal{A}u)(t) &= \int_0^T \left\{ g_1(t,s) + \frac{1}{\Delta_1} \left[(T^{n-1} - t^{n-1}) \left(1 - \sum_{i=1}^q b_i \right) + \sum_{i=1}^q b_i (T^{n-1} - \eta_i^{n-1}) \right] \right. \\ &\quad \times \sum_{i=1}^p a_i g_1(\xi_i, s) + \frac{1}{\Delta_1} \left[t^{n-1} \left(1 - \sum_{i=1}^p a_i \right) + \sum_{i=1}^p a_i \xi_i^{n-1} \right] \sum_{i=1}^q b_i g_1(\eta_i, s) \Big\} \times \\ &\quad \times f\left(s, \int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau\right)ds = \frac{1}{(n-1)!T^{n-1}} \int_0^T [t^{n-1}(T-s)^{n-1} \\ &\quad - T^{n-1}(t-s)^{n-1}] f\left(s, \int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau\right)ds + \frac{1}{(n-1)!T^{n-1}} \int_t^T t^{n-1}(T-s)^{n-1} \times \end{aligned}$$

$$\begin{aligned}
& \times f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds + \frac{1}{\Delta_1} \left[(T^{n-1} - t^{n-1}) \left(1 - \sum_{i=1}^q b_i\right) \right. \\
& + \sum_{i=1}^q b_i (T^{n-1} - \eta_i^{n-1}) \left. \right] \sum_{i=1}^p a_i \int_0^T g_1(\xi_i, s) f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
& + \frac{1}{\Delta_1} \left[t^{n-1} \left(1 - \sum_{i=1}^p a_i\right) + \sum_{i=1}^p a_i \xi_i^{n-1} \right] \times \\
& \times \sum_{i=1}^q b_i \int_0^T g_1(\eta_i, s) f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds. \tag{3.6}
\end{aligned}$$

184

185

Therefore, we obtain

$$\begin{aligned}
(\mathcal{A}u)'(t) &= \int_0^t \frac{t^{n-2}(T-s)^{n-1} - T^{n-1}(t-s)^{n-2}}{(n-2)! T^{n-1}} f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
&+ \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)! T^{n-1}} f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds + \frac{1}{\Delta_1} \left[-(n-1)t^{n-2} \left(1 - \sum_{i=1}^q b_i\right) \right] \times \\
&\times \sum_{i=1}^p a_i \int_0^T g_1(\xi_i, s) f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds + \frac{1}{\Delta_1} \left[(n-1)t^{n-2} \left(1 - \sum_{i=1}^p a_i\right) \right] \times \\
&\times \sum_{i=1}^q b_i \int_0^T g_1(\eta_i, s) f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds, \quad \forall t \in (0, T). \tag{3.7}
\end{aligned}$$

186

187

So, for any $t \in (0, T)$, we deduce

$$\begin{aligned}
|(\mathcal{A}u)'(t)| &\leq \int_0^t \frac{t^{n-2}(T-s)^{n-1} + T^{n-1}(t-s)^{n-2}}{(n-2)! T^{n-1}} p_1(s) q_1\left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
&+ \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)! T^{n-1}} p_1(s) q_1\left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
&+ \frac{(n-1)t^{n-2}}{\Delta_1} \left(1 - \sum_{i=1}^q b_i\right) \sum_{i=1}^p a_i \int_0^T g_1(\xi_i, s) p_1(s) q_1\left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
&+ \frac{(n-1)t^{n-2}}{\Delta_1} \left(1 - \sum_{i=1}^p a_i\right) \sum_{i=1}^q b_i \int_0^T g_1(\eta_i, s) p_1(s) q_1\left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
&\leq M_3 \left(\int_0^t \frac{t^{n-2}(T-s)^{n-1} + T^{n-1}(t-s)^{n-2}}{(n-2)! T^{n-1}} p_1(s) ds + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)! T^{n-1}} p_1(s) ds \right. \\
&+ \frac{(n-1)t^{n-2}}{\Delta_1} \left(1 - \sum_{i=1}^q b_i\right) \sum_{i=1}^p a_i \int_0^T g_1(\xi_i, s) p_1(s) ds + \frac{(n-1)t^{n-2}}{\Delta_1} \left(1 - \sum_{i=1}^p a_i\right) \times \\
&\times \sum_{i=1}^q b_i \int_0^T g_1(\eta_i, s) p_1(s) ds \left. \right). \tag{3.8}
\end{aligned}$$

188

We denote

$$\begin{aligned}
h(t) &= \int_0^t \frac{t^{n-2}(T-s)^{n-1} + T^{n-1}(t-s)^{n-2}}{(n-2)! T^{n-1}} p_1(s) ds + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)! T^{n-1}} p_1(s) ds, \tag{3.9} \\
\mu(t) &= h(t) + \frac{(n-1)t^{n-2}}{\Delta_1} \left(1 - \sum_{i=1}^q b_i\right) \sum_{i=1}^p a_i \int_0^T g_1(\xi_i, s) p_1(s) ds \\
&+ \frac{(n-1)t^{n-2}}{\Delta_1} \left(1 - \sum_{i=1}^p a_i\right) \sum_{i=1}^q b_i \int_0^T g_1(\eta_i, s) p_1(s) ds, \quad t \in (0, T). \tag{3.10}
\end{aligned}$$

189

190

191

For the integral of the function h , by exchanging the order of integration, we obtain after some computations

$$\int_0^T h(t) dt = \int_0^T \frac{(T-s)^{n-1}}{(n-2)! T^{n-1}} \left(\frac{T^{n-1} - s^{n-1}}{n-1} \right) p_1(s) ds + \int_0^T \frac{p_1(s)(T-s)^{n-1}}{(n-1)!} ds$$

192

$$+ \int_0^T \frac{(T-s)^{n-1} s^{n-1}}{(n-1)! T^{n-1}} p_1(s) ds = \frac{2}{(n-1)!} \int_0^T (T-s)^{n-1} p_1(s) ds < \infty. \quad (3.11)$$

193

194 For the integral of the function μ , we have

$$\begin{aligned} \int_0^T \mu(t) dt &\leq \frac{2}{(n-1)!} \int_0^T (T-s)^{n-1} p_1(s) ds + \frac{T^{n-1}}{\Delta_1} \left(1 - \sum_{i=1}^q b_i\right) \times \\ &\times \sum_{i=1}^p a_i \int_0^T g_1(\theta_1(s), s) p_1(s) ds + \frac{T^{n-1}}{\Delta_1} \left(1 - \sum_{i=1}^p a_i\right) \sum_{i=1}^q b_i \int_0^T g_1(\theta_1(s), s) p_1(s) ds \\ &\leq \frac{1}{(n-1)!} \left(2 + \frac{T^{n-1}}{\Delta_1} \left(1 - \sum_{i=1}^q b_i\right) \left(\sum_{i=1}^p a_i\right) + \frac{T^{n-1}}{\Delta_1} \left(1 - \sum_{i=1}^p a_i\right) \left(\sum_{i=1}^q b_i\right)\right) \times \\ &\times \int_0^T (T-s)^{n-1} p_1(s) ds < \infty. \end{aligned} \quad (3.12)$$

195 We deduce that $\mu \in L^1(0, T)$. Thus for any given $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$ and $u \in D$, by
196 (3.8), we obtain

$$|(\mathcal{A}u)(t_1) - (\mathcal{A}u)(t_2)| = \left| \int_{t_1}^{t_2} (\mathcal{A}u)'(t) dt \right| \leq M_3 \int_{t_1}^{t_2} \mu(t) dt. \quad (3.13)$$

197

198 From (3.12), (3.13) and absolute continuity of the integral function, we obtain that $\mathcal{A}(D)$ is
199 equicontinuous. This conclusion together with (3.5) and Ascoli-Arzelà theorem yields that
200 $\mathcal{A}(D)$ is relatively compact. Therefore $\mathcal{A}$ is a compact operator.

201

202 By using similar arguments as those used in the proof of Lemma 2.4 from [3], we can show
203 that $\mathcal{A}$ is continuous on P . Therefore $\mathcal{A}: P \rightarrow P$ is completely continuous. ■

204

205 For $c \in (0, T/2)$, we define the cone

$$P_0 = \left\{ u \in X, u(t) \geq 0, \forall t \in [0, T], \min_{t \in [c, T-c]} u(t) \geq \gamma \|u\| \right\}, \quad (3.14)$$

206 where $\gamma = \min\{\gamma_1, \gamma_2\}$, γ_1 and γ_2 are given in Section 2. Under the assumptions $(H_1), (H_2)$,
207 we have $\mathcal{A}(P) \subset P_0$. Indeed, for $u \in P$, let $v = \mathcal{A}(u)$. By Lemma 2.6, we have
208 $\min_{t \in [c, T-c]} v(t) \geq \gamma_1 \|v\| \geq \gamma \|v\|$, that is $v \in P_0$.

209

210 **Theorem 3.1** Assume that $(H_1) - (H_4)$ hold. Then the problem (S) – (BC) has at least one
211 positive solution $(u(t), v(t))$, $t \in [0, T]$.

212

213 **Proof.** We consider the cone P_0 with c given in (H_4) . From (H_3) i) and (H_2) , we deduce that
214 there exists $C_1 > 0$ such that

$$q_1(x) \leq C_1 x^{r_1}, \quad \forall x \in [0, 1]. \quad (3.15)$$

215

216 From (H_3) ii) and (H_2) , for $C_2 = \min\left\{\left(1/(C_1 \alpha \beta^{r_1})\right)^{1/r_1}, 1/\beta\right\} > 0$ with α, β defined in the proof
217 of Lemma 3.1, we conclude that there exists $\delta_1 \in (0, 1)$ such that

$$q_2(x) \leq C_2 x^{r_2}, \quad \forall x \in [0, \delta_1]. \quad (3.16)$$

218

219 From (3.16), (H_2) and Lemma 2.5, for any $u \in \partial B_{\delta_1} \cap P_0$ and $s \in [0, T]$, we obtain

$$\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \leq C_2 \int_0^T J_2(\tau) p_2(\tau) d\tau \cdot \|u\|^{r_2} = C_2 \beta \delta_1^{r_2} \leq \delta_1^{r_2} < 1. \quad (3.17)$$

220

221 By using (3.15) – (3.17) and (H_2) , for any $u \in \partial B_{\delta_1} \cap P_0$ and $t \in [0, T]$, we get

$$(\mathcal{A}u)(t) \leq C_1 \int_0^T G_1(t, s) p_1(s) \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right)^{r_1} ds$$

$$\begin{aligned}
&\leq C_1 \int_0^T G_1(t, s) p_1(s) \left(C_2 \int_0^T G_2(s, \tau) p_2(\tau) (u(\tau))^{r_2} d\tau \right)^{r_1} ds \\
&\leq C_1 \int_0^T J_1(s) p_1(s) ds \cdot \left(C_2 \int_0^T J_2(\tau) p_2(\tau) d\tau \right)^{r_1} \cdot \|u\|^{r_1 r_2} \leq \|u\|.
\end{aligned} \tag{3.18}$$

222

223 Therefore

$$\|\mathcal{A}u\| \leq \|u\|, \quad \forall u \in \partial B_{\delta_1} \cap P_0. \tag{3.19}$$

224

225 From (H_4) i), we deduce that there exist $C_3 > 0$ and $x_1 > 0$ such that

$$f(t, x) \geq C_3 x^{l_1}, \quad \forall x \geq x_1, \quad \forall t \in [c, T - c]. \tag{3.20}$$

226

227 We consider now $C_4 = \max \left\{ (\gamma_2 \gamma^{l_2} \theta_2)^{-1}, (C_3 \gamma_1 \gamma_2^{l_1} \gamma^{l_1 l_2} \theta_1 \theta_2^{l_1})^{-1/l_1} \right\} > 0$, where $\theta_1 =$
 228 $\int_c^{T-c} J_1(s) ds > 0$ and $\theta_2 = \int_c^{T-c} J_2(s) ds > 0$. From (H_4) ii), we conclude that there exists
 229 $x_2 \geq 1$ such that

$$g(t, x) \geq C_4 x^{l_2}, \quad \forall x \geq x_2, \quad \forall t \in [c, T - c]. \tag{3.21}$$

230

231 Now we choose $R_0 = \max\{x_1, x_2\}$ and $R > \max\{R_0/\gamma, R_0^{1/l_2}\}$. Then for any $u \in \partial B_R \cap P_0$, we
 232 have $\min_{t \in [c, T-c]} u(t) \geq \gamma \|u\| = \gamma R > R_0$.

233

234 By using (3.20) and (3.21), for any $u \in \partial B_R \cap P_0$ and $s \in [c, T - c]$, we obtain

$$\begin{aligned}
&\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \geq \gamma_2 C_4 \int_c^{T-c} J_2(\tau) (u(\tau))^{l_2} d\tau \\
&\geq \gamma_2 C_4 \gamma^{l_2} \int_c^{T-c} J_2(\tau) d\tau \cdot \|u\|^{l_2} \geq \|u\|^{l_2} = R^{l_2} > R_0.
\end{aligned} \tag{3.22}$$

235

236 Then for any $u \in \partial B_R \cap P_0$ and $t \in [c, T - c]$, we have

$$\begin{aligned}
(\mathcal{A}u)(t) &\geq \int_c^{T-c} G_1(t, s) f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
&\geq C_3 \int_c^{T-c} G_1(t, s) \left(\gamma_2 \int_c^{T-c} J_2(\tau) C_4 (u(\tau))^{l_2} d\tau \right)^{l_1} ds \\
&\geq C_3 C_4^{l_1} \gamma_2^{l_1} \int_c^{T-c} G_1(t, s) \gamma^{l_1 l_2} \|u\|^{l_1 l_2} \left(\int_c^{T-c} J_2(\tau) d\tau \right)^{l_1} ds \\
&\geq C_3 C_4^{l_1} \gamma_2^{l_1} \gamma_1 \gamma^{l_1 l_2} \left(\int_c^{T-c} J_1(s) ds \right) \left(\int_c^{T-c} J_2(\tau) d\tau \right)^{l_1} \|u\|^{l_1 l_2} \geq \|u\|.
\end{aligned} \tag{3.23}$$

237

238 Therefore we deduce

$$\|\mathcal{A}u\| \geq \|u\|, \quad \forall u \in \partial B_R \cap P_0. \tag{3.24}$$

239

240 By (3.19), (3.24) and Theorem 1.1 i), we obtain that $\mathcal{A}$ has a fixed point $u_1 \in (\bar{B}_R \setminus B_{\delta_1}) \cap P_0$,
 241 that is $\delta_1 < \|u_1\| < R$. Let $v_1(t) = \int_0^T G_2(t, s) g(s, u_1(s)) ds$. Then $(u_1, v_1) \in P_0 \times P_0$ is a solution
 242 of $(S) - (BC)$. In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$ for all $t \in [0, T]$, then
 243 by using (H_2) we have $f(s, v_1(s)) = f(s, 0) = 0$ for all $s \in [0, T]$. This implies $u_1(t) = 0$ for all
 244 $t \in [0, T]$, which contradicts $\|u_1\| > 0$. By using Theorem 1.1 from [17] (see also [18]), we
 245 obtain $u_1(t) > 0$ and $v_1(t) > 0$ for all $t \in (0, T - c]$. The proof of Theorem 3.1 is completed.

246 ■

247

248 **Theorem 3.2** Assume that (H_1) , (H_2) , (H_5) and (H_6) hold. Then the problem $(S) - (BC)$ has
 249 at least one positive solution $(u(t), v(t))$, $t \in [0, T]$.

250

251 **Proof.** We consider the cone P_0 with c given in (H_6) . By (H_5) i) we deduce that there exist
 252 $C_5 > 0$ and $C_6 > 0$ such that

$$q_1(x) \leq C_5 x^{\alpha_1} + C_6, \quad \forall x \in [0, \infty). \quad (3.25)$$

253

254 From (H_5) ii), for $\epsilon_0 > 0$, $\epsilon_0 < (2^{\alpha_1} C_5 \alpha \beta^{\alpha_1})^{-1/\alpha_1}$, we conclude that there exists $C_7 > 0$ such
 255 that

$$q_2(x) \leq \epsilon_0 x^{\alpha_2} + C_7, \quad \forall x \in [0, \infty). \quad (3.26)$$

256

257 By using (3.25), (3.26) and (H_2) , for any $u \in P_0$, we obtain

$$\begin{aligned} (\mathcal{A}u)(t) &\leq \int_0^T G_1(t, s) p_1(s) q_1 \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds \\ &\leq C_5 \int_0^T G_1(t, s) p_1(s) \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right)^{\alpha_1} ds + C_6 \int_0^T J_1(s) p_1(s) ds \\ &\leq C_5 \int_0^T J_1(s) p_1(s) ds \left(\int_0^T J_2(\tau) p_2(\tau) d\tau \right)^{\alpha_1} (\epsilon_0 \|u\|^{\alpha_2} + C_7)^{\alpha_1} + \alpha C_6 \\ &\leq C_5 2^{\alpha_1} \epsilon_0^{\alpha_1} \alpha \beta^{\alpha_1} \|u\|^{\alpha_1 \alpha_2} + C_5 2^{\alpha_1} \alpha \beta^{\alpha_1} C_7^{\alpha_1} + \alpha C_6, \quad \forall t \in [0, T]. \end{aligned} \quad (3.27)$$

258

259 By definition of ϵ_0 , we can choose sufficiently large $R_1 > 0$ such that

$$\|\mathcal{A}u\| \leq \|u\|, \quad \forall u \in \partial B_{R_1} \cap P_0. \quad (3.28)$$

260

261 From (H_6) i), we deduce that there exist positive constants $C_8 > 0$ and $x_3 > 0$ such that
 262 $f(t, x) \geq C_8 x^{\beta_1}$, for all $x \in [0, x_3]$ and $t \in [c, T - c]$. From (H_6) ii), for
 263 $\epsilon_1 = (C_8 \gamma_1 \gamma_2^{\beta_1} \gamma^{\beta_1 \beta_2} \theta_1 \theta_2^{\beta_1})^{-1/\beta_1} > 0$, we conclude that there exists $x_4 > 0$ such that $g(t, x) \geq$
 264 $\epsilon_1 x^{\beta_2}$ for all $x \in [0, x_4]$ and $t \in [c, T - c]$.

265

266 We consider $x_5 = \min\{x_3, x_4\}$. So we obtain

$$f(t, x) \geq C_8 x^{\beta_1}, \quad g(t, x) \geq \epsilon_1 x^{\beta_2}, \quad \forall (t, x) \in [c, T - c] \times [0, x_5]. \quad (3.29)$$

267

268 From assumption $q_2(0) = 0$ and the continuity of q_2 , we deduce that there exists sufficiently
 269 small $\epsilon_2 \in (0, \min\{x_5, 1\})$ such that $q_2(x) \leq \beta^{-1} x_5$ for all $x \in [0, \epsilon_2]$.

270

271 Therefore for any $u \in \partial B_{\epsilon_2} \cap P_0$ and $s \in [0, T]$, we have

$$\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \leq \beta^{-1} x_5 \int_0^T J_2(\tau) p_2(\tau) d\tau = x_5. \quad (3.30)$$

272

273 By (3.29), (3.30), Lemma 2.5 and Lemma 2.6, for any $t \in [c, T - c]$, we get

$$\begin{aligned} (\mathcal{A}u)(t) &\geq C_8 \int_c^{T-c} G_1(t, s) \left(\int_c^{T-c} G_2(s, \tau) g(\tau, u(\tau)) d\tau \right)^{\beta_1} ds \\ &\geq C_8 \gamma_1 \int_c^{T-c} J_1(s) \left[(\epsilon_1 \gamma_2)^{\beta_1} \left(\int_c^{T-c} J_2(\tau) (u(\tau))^{\beta_2} d\tau \right)^{\beta_1} \right] ds \\ &\geq C_8 \gamma_1 \gamma_2^{\beta_1} \epsilon_1^{\beta_1} \gamma^{\beta_1 \beta_2} \theta_1 \theta_2^{\beta_1} \|u\|^{\beta_1 \beta_2} \geq \|u\|. \end{aligned} \quad (3.31)$$

274 Therefore

$$\|\mathcal{A}u\| \geq \|u\|, \quad \forall u \in \partial B_{\epsilon_2} \cap P_0. \quad (3.32)$$

275

By (3.28), (3.32) and Theorem 1.1 ii), we deduce that $\mathcal{A}$ has at least one fixed point $u_2 \in (\bar{B}_{R_1} \setminus B_{\epsilon_2}) \cap P_0$. Then our problem $(S) - (BC)$ has at least one positive solution $(u_2, v_2) \in P_0 \times P_0$ where $v_2(t) = \int_0^T G_2(t, s)g(s, u_2(s))ds$. The proof of Theorem 3.2 is completed. ■

4. CONCLUSION

We presented sufficient conditions on the functions (possibly singular) f and g such that the problem $(S) - (BC)$ has positive solutions.

ACKNOWLEDGEMENTS

The authors thank the referees for their valuable comments and suggestions. The work of R. Luca was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0557.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

AUTHORS' CONTRIBUTIONS

The authors contributed equally to this paper. All authors read and approved the final manuscript.

REFERENCES

1. Henderson J, Luca, R. Positive solutions for singular systems of higher-order multi-point boundary value problems. Math Model Anal (in press).
2. Henderson J, Luca R, Positive solutions for singular systems of multi-point boundary value problems. Math Methods Appl Sci. 2013;36:814-828.
3. Liu B, Liu L, Wu Y. Positive solutions for singular systems of three-point boundary value problems. Comput Math Appl. 2007;53:1429-1438.
4. Henderson J, Luca R. Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems. Nonlinear Differ Equ Appl. 2013;20(3): 1035-1054.
5. Zhou Y, Xu Y. Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations. J Math Anal Appl. 2006;320:578-590.
6. Su H, Wei Z, Zhang X, Liu J, Positive solutions of n-order and m-order multi-point singular boundary value system. Appl Math Comput. 2007;188:1234-1243.
7. Henderson J, Luca R. Existence of positive solutions for a system of higher-order multi-point boundary value problems. Appl Math Comput. 2012;219:3709-3720.

- 322 8. Henderson J, Luca R. Positive solutions for a system of higher-order multi-point boundary
323 value problems. *Comput Math Appl.* 2011;62:3920-3932.
- 324 9. Henderson J, Luca R. Positive solutions for a system of second-order multi-point
325 boundary value problems. *Appl Math Comput.* 2012;218:6083-6094.
- 326 10. Henderson J, Luca R. Existence and multiplicity for positive solutions of a multi-point
327 boundary value problem. *Appl Math Comput.* 2012;218:10572-10585.
- 328 11. Henderson J, Ntouyas SK. Positive solutions for systems of n th order three-point
329 nonlocal boundary value problems. *Electron J Qual Theory Differ Equ.* 2007;18:1-12.
- 330 12. Henderson J, Ntouyas SK, Purnaras IK. Positive solutions for systems of m -point
331 nonlinear boundary value problems. *Math Model Anal.* 2008;13:357-370.
- 332 13. Luca R. Positive solutions for a second-order m -point boundary value problem. *Dyn*
333 *Contin Discrete Impuls Syst Ser A Math Anal.* 2011;18:161-176.
- 334 14. Yang Z. Positive solutions to a system of second-order nonlocal boundary value
335 problems. *Nonlinear Anal.* 2005;62:1251-1265.
- 336 15. Guo D, Lakshmikantham V. *Nonlinear problems in abstract cones.* Academic Press: New
337 York; 1988.
- 338 16. Ji Y, Guo Y. The existence of countably many positive solutions for some nonlinear n th
339 order m -point boundary value problems. *J Comput Appl Math.* 2009;232:187-200.
- 340 17. Ji Y, Guo Y, Yu C. Positive solutions to $(n-1, n)$ m -point boundary value problems with
341 dependence on the first order derivative. *Appl Math Mech (English Ed).* 2009;30:527-536.
- 342 18. Eloe PW, Henderson J. Positive solutions for $(n-1, 1)$ conjugate boundary value
343 problems. *Nonlinear Anal.* 1997;28:1669-1680.

344