THE CORE ERROR PROBLEM IN VARIABLES FOR STATISTICAL LEAST SQUARES WITH INTERVAL DATA

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Abstract: The paper discusses the core problem error in variables for statistical Least Squares which have noise in the data. Via Mobius transformation for complex disk in the sense of Petkovic, the exact inversion of a disk was described, wherein, Modified Certifylss relaxed refinement Cholesky decomposition in the sense of Rump comes in handy, as a useful tool for the solution of resulting interval linear system with guaranteed error bounds . The well known Oettli-Prager theorem was used as an alternative measure of performance to the described method. It was discovered that the proposed technique performed far better with high yield of certainty than the Oettli-Prager theorem.

Keywords: least squares, interval relaxed iterative refinement, mobius transformation, modified Certifylss relaxed refinement Cholesky decomposition, Oetlli-Prager theorem, ball arithmetic

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1. Introduction

Consider statistical measurements without noise which is described by the equation

$$Y = f(\beta) \qquad (1.1)$$

where, $Y \in R^m$ is a measurement vector, and $\beta \in R^n$ is a parameter vector respectively to be determined. It is assumed that function f has at least $C^1[a,b],\ a \le \beta \le b$. In real world experiments, the measured parameters are corrupted by noise, thus applying traditional Statistical methods to solve this will undoubtedly fail. There is thus, a need to overcome this setback, by applying the interval arithmetic.

Such situation often arise when the number of equations is more than the number of unknowns which is called an over determined system. Similarly, the case of under determined system wherein, the number of equations is less than the number of unknowns can also be found in the literatures.

The least squares problem using 2-norm is described in the form

$$Y(\beta) = \min_{\beta \in IR^n} \left\| f(\beta) - \hat{Y} \right\|_2^2 = \min_{\beta \in IR^n} \left\| A\beta - \hat{Y} \right\|_2^2 = \left(A\beta - \hat{Y} \right)^T \left(A\beta - \hat{Y} \right)$$
$$= \beta^T A^T A\beta - 2\beta^T A^T \hat{Y} + \hat{Y}^T A^T A \hat{Y} \qquad (1.2)$$

where $f(\beta) = A\beta$ and A is a real $m \times n$ rectangular matrix with m>n.

We then minimize β as the least squares solution in the form

$$\beta = \left(A^T A\right)^{-1} A^T \hat{Y} \qquad (1.3)$$

The statistical meaning [1] of the matrix $(A^T A)^{-1}$ could be deduced by considering the components $Y_i : i = 1,2,...,m$ which are independently normally distributed random variables having μ_i as a mean and, variance δ^2 such that:

$$E(Y_i) = \mu_i \text{ ; and } E\left(Y_i - \mu_i\right)\left(Y_k - \mu_k\right) = \begin{cases} \delta^2, i = k \\ 0 \quad otherwise \end{cases} \qquad (1.4)$$

Letting $\mu = \left(\mu_1, \mu_2, ..., \mu_m\right)^T$ it can be proved [1] that

$$E(Y) = \mu,\ E(Y - \mu)(Y - \mu)^T = \delta^2 I .$$

It follows immediately that

$$E[(\beta - E(\beta))(\beta - E(\beta))^T] = E\left[\left(A^T A\right)^{-1} A^T (y-\mu)(y-\mu)^T A\left(A^T A\right)^{-1}\right]$$

$$= \left(A^T A\right)^{-1} A^T E\left[(y-\mu)(y-\mu)^T A\left(A^T A\right)^{-1}\right] = \delta^2 \left(A^T A\right)^{-1} = \delta^2 \sum_{j=1}^{n} \frac{1}{\lambda_j},$$

$\lambda_j : j = 1,2,...,p$ are eigenvalues of $A^T A$.

As a result, it holds that $E[\left(A^T A\right)^{-1} A^T y] = \left(A^T A\right)^{-1} A^T E(y) = \left(A^T A\right)^{-1} A^T \mu$.

The remaining part of the paper is categorized as follows: Section 2 in the paper discusses ball arithmetic, in section 3 we consider the differentiability as well as backward perturbation error as it relates the condition number of a differentiable function. In section 5 regularity of interval Jacobian matrix is explained while section 5 in the paper discusses the numerical experiment for the considered method. Section 6 gives the conclusion of our findings.

2. The Ball arithmetic

Because this paper deals on interval arithmetic, we mention basic properties of interval arithmetic which are well documented in the standard text books by [2], [3] and [4].The case of floating point arithmetic counterpart can be found in [5] as well.

The ball arithmetic was incorporated into interval arithmetic computation for the first time by [6]) and also [4] had gained prominence amongst interval researchers. Fundamental to this underlining work is [7]. We will describe briefly basic properties of interval arithmetic.

Let $*$ represent an operation from the set $\{+,-,\cdot,/\}$ .Then assuming $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ are two real intervals as in [2], we have:

$[a].[b] = \{a * b \mid a \in [a], b \in [b]\}$, the following four operations for intervals are defined:

$$[a] + [b] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$

$$[a] - [b] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}],$$

$$[a] * [b] = [\min(\underline{a}\,\underline{b}, \underline{a}\,\overline{b}, \overline{a}\,\underline{b}, \overline{a}\,\overline{b}), \max\left(\underline{a}\,\underline{b}, \underline{a}\,\overline{b}, \overline{a}\,\underline{b}, \overline{a}\,\overline{b}\right)] \quad . \quad (2.1)$$

The extended multiplication for intervals [a] and [b] was further studied in [8] , see also, [9] in the form

$$
[a].[b] = \begin{cases}
\left[\underline{a}\,\underline{b}, \overline{a}\,\overline{b}\right] & if\ \ \underline{a} \geq 0\ and\ \ \underline{b} \geq 0 \\
\left[\overline{a}\,\underline{b}, \overline{a}\,\overline{b}\right] & if\ \ \underline{a} \geq 0\, and\ \ \underline{a} < 0 < \overline{b} \\
\left[\overline{a}\,\underline{b}, \underline{a}\,\overline{b}\right] & if\ \ \underline{a} \geq 0\ and\ \ \overline{b} \leq 0 \\
\left[\underline{a}\,\overline{b}, \overline{a}\,\overline{b}\right] & if\ \ \underline{a} < 0 < \overline{a}\ and\ \ \underline{b} \geq 0 \\
\left[\underline{a}\,\overline{b}, \overline{a}\,\underline{b}\right] & if\ \ \underline{a} < 0 < \overline{a}\ and\ \overline{b} \leq 0 \\
\left[\underline{a}\,\overline{b}, \overline{a}\,\underline{b}\right] & if\ \ \overline{a} \leq 0\ and\ \ \underline{b} \geq 0 \\
\left[\underline{a}\,\overline{b}, \underline{a}\,\underline{b}\right] & if\ \ \overline{a} \leq 0\ \ and\ \ \underline{b} < 0 < \overline{b} \\
\left[\overline{a}\,\overline{b}, \underline{a}\,\underline{b}\right] & if\ \ \overline{a} \leq 0\ \ and\ \ \overline{b} \leq 0 \\
\left[\min\left(\overline{a}\,\underline{b}, \underline{a}\,\overline{b}\right), \max\left(\underline{a}\,\underline{b}, \overline{a}\,\overline{b}\right)\right] & if\ \ \underline{a} < 0 < \overline{a}\ and\ \ \underline{b} < 0 < \overline{b}
\end{cases} \tag{2.2}
$$

The operation for division is defined as $\frac{1}{[b]} = \left[\frac{1}{\overline{b}}, \frac{1}{\underline{b}}\right], 0 \notin [b]$ and $\frac{[a]}{[b]} = [a].[b]^{-1}$.

More fundamentally, is the case when [b] contains the origin, i.e., $0 \in [b]$.

Therefore, Kahan arithmetic in the sense of [10] could be defined in the form:

$$\frac{[a]}{[b]}=\begin{cases}[\frac{\bar{a}}{\underline{b}},\infty] & if\ \bar{a}\le 0\ and\ \bar{b}=0\\ (-\infty,\frac{\bar{a}}{\bar{b}}]\cup[\frac{\bar{a}}{\underline{b}},\infty) & if\ \bar{a}\le 0\, and\, \underline{b}<0<\bar{b}\\ (-\infty,\frac{\bar{a}}{\bar{b}}] & if\ \ \bar{a}\le 0\, and\ \underline{b}=0\\ (-\infty,\infty) & if\ \underline{a}<0<\bar{a}\\ (-\infty,\frac{a}{\bar{b}}) & if\ \ \underline{a}\ge 0\ and\ \bar{b}=0\\ (-\infty,\frac{a}{\underline{b}}]\cup[\frac{a}{\bar{b}},\infty) & if\ \ \underline{a}\ge 0\ and\ \underline{b}<0<\bar{b}\\ [\frac{a}{\bar{b}},\infty) & if\ \bar{a}\ge 0\ and\ \underline{b}=0\end{cases} \qquad (2.3)$$

Besides, the interval operation due to [2] mentioned above, we also have the circular complex interval arithmetic with centre $a_c=\frac{1}{2}\left(\underline{a}+\bar{a}\right)$ and radius $\Delta a=\frac{1}{2}[\bar{a}-\underline{a}]$ and is by the parametric notation defined in the form

$$[a]=\{a:|a-a_c|<\Delta a\} \qquad (2.4)$$

We need to stress salient concepts in circular interval computation- the negation and relative extent of a disk. The negation of a disk $[a]=-[a]=(-a_c,\Delta a)$, and the relative extent is defined:

$R\mathbf{x}(a)=\frac{\Delta_a}{a_c}, for\ a_c\neq 0.$

Via Mobius transformation in the sense of[11], the exact inversion of a disk could be derived. First observe that the mapping $a\to\frac{1}{a}$ could be defined in the form:

$$[a]^{-1}=\{a_c,\Delta a\}^{-1}=\left\{\frac{\bar{a}_c}{|a_c|^2-\Delta a^2},\frac{\Delta a}{|a_c|^2-\Delta a^2}\right\}=\left\{\frac{1}{a}:a\in\{a_c,\Delta a\},0\notin\{a_c,\Delta a\}\right\}. \quad (2.5)$$

With equation 2.5 introduced as above, the inverse disk can be written in polynomial form:

$$\left\{\frac{\bar{a}_c}{|a_c|^2-\Delta a},\frac{\Delta r}{|a_c|^2-\Delta a^2}\right\}=\left\{\frac{1}{a_c}+\frac{(\Delta a/|a_c|)^2}{a_c}\left(1+(\Delta a/|a_c|)^2-(\Delta a/|a_c|)^4+\ldots\right),\frac{\Delta a}{|a_c|^2-\Delta a^2}\right\} \quad (2.6)$$

The circumference of exact inversion of a disk D_c would be defined as

$$D_c=\left\{a:\left|a-\frac{\bar{a}_c}{|a_c|^2-\Delta a^2}\right|=\frac{\Delta a}{|a|^2-\Delta a^2}\right\}. \quad (2.6)$$

Where from,

$$\{a_c,\Delta a\}^{I_c}=D_c\bigcup \operatorname{int} D_c\supseteq\{a_c,\Delta a\}^{-1} \quad . \quad (2.7)$$

As a result, an inversion of a disk will be $\{a_c,\Delta a\}^{I_c}=\left\{\frac{1}{a_c},\rho\right\}$, it can be derived [11] that

$$\rho=rad\,(a_c,\Delta a)^{I_c}=\max_{\tau\in\{a_c,\Delta a\}^{-1}}\left|\frac{1}{a_c}-\tau\right|=\max_{\theta\in[0,2\pi]}\left|\frac{1}{a_c}-\frac{a_c^{-1}.\Delta a\exp(i\theta)}{|a_c|^2-\Delta a^2}\right|$$

$$=\frac{\Delta a}{|a_c|}\max_{\theta\in[0,2\pi]}\left|\frac{\Delta a+a_c\exp(i\theta)}{|a_c^2|-\Delta a^2}\right|=\frac{\Delta a}{|a_c|}\max_{\tau\in[0,2\pi]}\left|\frac{\Delta a+|a_c|\exp(i\tau)}{|a_c|^2-\Delta a^2}\right|=\frac{\Delta a}{|a_c|(|a_c|-\Delta a)} \quad , \quad (2.8)$$

The centred inversion of a disk in our work would be adopted to be in the form:

$$\{a_c,\Delta a\}^{I_c}=\left\{\frac{1}{a_c},\frac{\Delta a}{|a_c|(|a_c|-\Delta a)}\right\}\supseteq\{a_c,\Delta a\}^{-1},\ 0\notin\{a_c,\Delta a\} \ . \quad (2.9)$$

In what follows we introduce multiplication of two disks $a=\{a_c,\Delta a\},\ b=\{b_c,\Delta b\}$ where $|a_c|>\Delta a, and\ |b_c|>\Delta b$ in the form:

$$a.b=\{a_c.b_c,|a_c|\Delta b+|b_c|\Delta a+\Delta a\Delta b\}, \quad (2.10)$$

Further applications of equation 2.10 were used in [12, 13] in the form:

$$a.b\subseteq\{a_c.b_c+\eta a.\eta b,|a_c|.\Delta b+\Delta a(|b_c|+\Delta b)-|\eta a|.|\eta b|\} \quad (2.11)$$

where,

$$\eta a=sign(a_c)*\min(|a_c|,\Delta a) \quad . \quad (2.12)$$

3. Differentiability of a continuous map of a function and optimal condition number

Differentiability would imply that $Y(\beta_0) = \|Df(\beta_0)\|$ where the operator norm is the one induced from the norms for R^m and R^n.

Definition 1.1 [5, 14] The Frechet -derivative of f at β_0 is the unique linear operator $l = Df(\beta_0)$ for which the following (deleted) limit vanishes,

$$\lim_{\beta \to \beta_0} \frac{\| f(\beta) - f(\beta_0) - l(\beta - \beta_0)\|}{\|\beta - \beta_0\|} = 0 \qquad (3.1)$$

Definition 1.2 [3]. A real function $f : IR^n \to IR^n$ is called a Lipschitz continuous in $\beta_0 \in IR^n$ if $f(\beta_0) < \infty$ and there is a row vector $l_0 \in R^{n\times 1}$ such that

$q(f(\beta_1), f(\beta_2)) \le l_0 q(\beta_1, \beta_2)$ for all $\beta_1, \beta_2 \subseteq \beta_0$

The following expositions hold for the function F.

(i) Let $F : ID \subseteq IR^n \to IR^n$ be F-differentiable on the convex set $ID_0 \subseteq ID$, and if $\|F^{/}(\beta)\| \le K < \infty$, for all $\beta \in ID_0$, then F is Lipschitz continuous on ID_0.

(ii) If $F : ID \subseteq IR^n \to IR^n$ is F-differentiable on the convex set $ID_0 \subset ID$, then for any $\beta_0, \beta_1 \in ID_0$,

$$\|F(\beta_1) - F(\beta_0) - F^{/}(\beta)(\beta_1 - \beta_0)\| \le$$
$$\sup\left\|F^{/}(\beta_0 + t(\beta - \beta_0)) - F^{/}(\beta_0)\right\| \|\beta_1 - \beta_0\| \ , 0 \le t \le 1 \ .$$

The Lipschitz continuity of $F^{/}(\beta)$ holds from the following convexity that, for arbitrary vectors, $v, z \in \bar{S}(\beta, \delta)$, *and* $t \in [0,1]$, we have that

$$\|z + t(v - z) - v\| = \|t(v - \beta) + (1 - t)(z - \beta\| < t\delta + (1 - t)\delta = \delta \ .$$

and the estimate

$$\|F(v) - F(z) - F^{/}(\beta)(\beta - z)\| \le \sup_{0 \le t \le 1} \|F^{/}(z + t(\beta - z)) - F^{/}(\beta)\| \|v - z\| \le \varepsilon \|v - z\| \qquad (1.4)$$

remain valid.

As a result, Hemi-continuity of F is implied by the fact that Jacobian matrix of F as well as Riemann integrability of F will be given in the form:

$$\left\|F(v)-F(\beta)-F^{/}(\beta)(v-\beta)\right\| = \left\|\int_0^1 (F^{/}(\beta+t(v-\beta))-F^{/}(\beta)(v-\beta)dt\right\| \leq$$

$$\int_0^1 \left\|(F^{/}(\beta+t(v-\beta))-F^{/}(\beta))(v-\beta)dt\right\| \leq \int_0^1 \left\|F^{/}(\beta+t(v-\beta))-F^{/}(\beta)\right\|\left\|v-\beta\right\|dt$$

$$\leq \left\|v-\beta\right\|^{p+1}\int_0^1 t^p dt \leq \left(\frac{\alpha}{p+1}\right)\left\|v-\beta\right\|^{p+1}$$

Therefore, it follows from the reasoning given in [5] that F has a local homeomorphism at β_0 and $F^{/}(\beta_0)$ is also non singular.

In what follows we relate condition number to backward perturbation error. This will help in understanding optimal sensitivity analysis for the perturbed system. Assuming that β_0 solves the Least squares problem of equation 1.1, and assuming further that $\beta_0 = \beta_0 + \Delta\beta$ is the perturbation of β_0 which solves the perturbed matrix A as $\beta \to \beta_0$. It was showed [14] that optimal backward error asymptotically is

$$\min_{\Delta A}\left\|\Delta A\right\|_F = \left\|\left(\left\|r_0\right\|_2^2 I + \left\|\beta_0\right\|_2^2 A^T A\right)^{-\frac{1}{2}} A^T r\right\|_2 . \qquad (3.2)$$

Note that $r = Y - A\beta$ is the least squares residual and that $r_0 = Y - A\beta$ is the approximate residual. As a result, it was proved [14] that equation 3.2 is the asymptotic backward error estimate for the perturbed system. The over estimation factor for the spectral norm relative condition number is at most the factor of $\sqrt{2}$, and hence

$$\beta_2^{(Ls,rel)}(A) \approx \left(\frac{\left\|r_0\right\|_2}{\left\|\beta_0\right\|_2 \delta_{\min}} + 1\right)^{\frac{1}{2}} K_2 \qquad (3.3)$$

In equation 3.3 , K_2 is the spectral matrix condition number and is defined as $\frac{\delta_{\max}}{\delta_{\min}}$, where $\delta_{\max}$ is the largest singular value of A.

4. The Regularity question of an interval Jacobian matrix

In what follows, we denote R as approximate inverse of midpoint interval matrix A, and [15] proved that the bound

$\left|\mathbf{A}^{-1} - R\right| \leq \left(\left(I - \left|R\right|\Delta A\right)^{-1} - I\right)\left|R\right|$ is valid. The information we shall need for certifying the regularity of interval matrices are well documented in [16] and [17] in the form:

(i) If A_c is regular and $\max_j \left(\Delta A|R|\right)_{jj} \geq 1$ then A is regular.

(ii) If $\lambda_{\max}\left((\Delta A)^T \Delta A\right) < \lambda_{\max}\left(A_c^T A_c\right)$ holds, then A is regular whenever λ is the eigenvalue of the matrix.

(iii) If there exists a matrix R such that $\left(I + |I - A_c R|\right)_j \leq \left(\Delta A|R|\right)_j$ for some $j \in (1,2,\ldots, n)$ then A is singular.

(iv) If A_c is regular and $\rho\left(|R|\Delta A\right) < 1$ then A is regular.

Let us take note that verification for positive definiteness of an interval matrix A can be obtained if

$A_c^T A_c - \left\|\Delta A^T \Delta A\right\| > 0$.

The above named properties of interval arithmetic will be used in the solution to system (1.1)

5. Iterative Methods for the problem

Following [7], Cholesky decomposition was adopted for our purpose . Computation of Cholesky decomposition was executed to run to completion with $\bar{R}$ as approximate inverse to midpoint interval matrix A_c such that, $\bar{R}^T \bar{R} = A_c + \Delta A$, and the residual satisfies the error estimates, [7] in the form:

$$\left| a_{ij} - \sum_{k=1}^{i} \bar{r}_{ki}\, \bar{r}_{kj} \right| \leq \eta_i \sum \left|\bar{r}_{ki}\right| \left|\bar{r}_{kj}\right| \quad for\ i < j\ and$$

$$\left| a_{jj} - \sum_{k=1}^{j} \bar{r}_{kj}^2 \right| \leq \eta_{j+1} \sum_{k=1}^{j} \bar{r}_{kj}^2$$

where $\eta_k = \dfrac{ku}{(1-ku)};\ u = 2^{-53}\ and\ \eta^{-1074}$

The operator at our disposal is the modified certified linear interval system solver for Cholesky decomposition (Certifylss relaxed Cholesky decomposition) that takes into consideration the reach-ability theory in a manner likened to [18] in the algorithmic form describe below.

Algorithm (Modified Certifylss relaxed via Cholesky decomposition)

Input: matrix $A \in IR^{n\times n}, b \in IR^n$.

Compute $\bar{\beta} = A_c \backslash b_c$ (through Cholesky decomposition)

While (not converged)

$r = (b_c - A\bar{\beta})$
$[e] = A \backslash r$ (Through method 2.9)

$\beta = \bar{\beta} + [e]$

In what follows we introduce the Oettli-Prager Theorem for a useful purpose.

Theorem 5.1 [3]. Let Let $\bar{\beta}$ denote element of an interval vector β. Then

$$\bar{\beta} \in \sum(A,b) \Leftrightarrow \left| A_c\,\bar{\beta} - b_c \right| \le rad(A)\left|\bar{\beta}\right| + rad(b) \quad (5.1)$$

Thoerem 5.2 [3]. Let $A \in IR^{m\times n}, b \in IR^m$. Then

$$\sum(A,b) = \left\{ \bar{\beta} \in R^n \;\middle|\; A\bar{\beta} \cap b \ne 0 \right\} = \left\{ \bar{\beta} \in R^n \;\middle|\; 0 \in A\bar{\beta} - b \right\}.$$

In [19] , Oettli-Prager formula for the Linear Least Squares was used in the form

$$\beta = \left(\underline{\beta}\right)_i = \sum_{j=1}^{n} \min\left\{ (Q)_{ij}(\underline{Y})_j, (Q)_{ij}\left(\bar{Y}\right)_j \right\} \le (\beta)_i \le \sum_{j=1}^{n} \max\left\{ (Q)_{ij}(Y)_j, (Q)_{ij}\left(\bar{Y}\right)_j \right\} = \left(\bar{\beta}\right)_i \quad (5.2)$$

In equation 3.4, [20] gave hints of how to compute the solution in the following way where it is assumed that there exists an orthant

$IR_z^n = \{\beta; T_z\beta \ge 0\}$, (T_z is a diagonal matrix with sign vector z on the diagonal) such that $z_i \in \{-1,1\}$ and where from,

$$\beta \cap IR^n = \{\beta, (A_c + \Delta T_z)z \ge b_c - \delta,\ (A_c - \Delta T_z)z \le b_c + \delta,\ T_z\beta \ge 0\} \quad (5.3)$$

The matrices $Q, Q^{/}$ appearing in the inequality 5.2 solves the system of linear equation

$$QA_cT_z + |Q|\Delta = E \qquad (5.4)$$
$$Q^{/}A_cT_z - |Q^{/}|\Delta = E \qquad (5.5)$$

with unique solution .

The solution bounds being provided by the numerical methods to the given problem should fulfil in the sense of [21], the estimate

$$\left\|\hat{\beta}-A^{-1}b\right\|_{\infty}=\left\|\left(1-\left(I-RA\right)\right)^{-1}R\left(A\hat{\beta}-b\right)\right\|_{\infty}\leq\frac{\left\|R\left(A\hat{\beta}-b\right)\right\|_{\infty}}{1-\left\|I-RA\right\|_{\infty}} \quad (5.6)$$

6. NUMERICAL EXAMPLE

Consider fitting of polynomial of degree 3: $\hat{y}(\hat{\beta})=\hat{\beta}_0+\hat{\beta}_1 t+\hat{\beta}_2 t^2+\hat{\beta}_3 t^3$ for the sample data taken from [19], as given, in Table 1

Table 1: the set of Statistical data for experiments

T	**y**
2	**[1.5,2.5]**
-1	**[-1.5,-0.5]**
0	**[-0.5,0.5]**
1	**[3.5,4.5]**
2	**[3.5,4.5]**
3	**[5.5,6.5]**
4	**[8.5,9.5]**
5	**[6.5,7.5]**
6	**[10.5,11.5]**
7	**[10.5,11.5]**
8	**[9.5,10.5]**

In floating point arithmetic we have the result using MATLAB 2007 $\beta=\left(0.4465,1.5357,0.1595,-0.0246\right)^T$ **. If we use the Oettli-Prager formula given in equation 3.4 , as well as Modified Certylss Cholesky method inclusion intervals were obtained as showed in Tables 2-3.**

Table 2: Results for original set of data from Table 1

Modified certylss relaxed Cholesky method	Oetlli-Prager formula r
Values of interval $[\hat{\beta}]$	Values of interval $[\hat{\beta}]$
[0.4465, 0.4465]	[-0.0535,0.9465]
[1.5357,1.5357]	[1.5357,1.5357]
[0.1595, 0.1595]	[0.1595,0.1595]
[-0.0246, -0.0246]	[-0.0246,-0.0246]

Table 3: Results for the variable t with 0.5 tolerance from Table 1

Modified Certylss relaxed Cholesky method in midpoint radius vector	Oetlli-Prager formula

$[0.4465,\ 0.3977\times10^{-14}]$ $[1.5357,-0.0859\times10^{-14}]$ $[0.1595,-0.0161\times10^{-14}]$ $[-0.0246,0.0028\times10^{-14}]$	[0.1416,0.7033] [1.4623 , 1.6311] [0.1053,0.2271 [-0.0285, -0.0196]

4. Conclusion

We have discussed the core problem error in variables for the Statistical Least squares using Oetlli-Prager formula and the modified Certylss Cholesky decomposition via Mobius transformation of ball arithmetic. The procedure was tested using polynomial of degree 3 for the interval data set where the parameters appearing as coefficients were obtained with high degree accuracy, as can be seen in Tables 2 and 3. The two methods performed optimally fast to the solution but with modified Certylss Cholesky decomposition preferred to Oetlli-Prager formula. This will provide further insight in the study of Least squares using circular interval arithmetic.

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