

Modeling and parameter sensitivity analysis of population dynamics in Uganda

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Author's contribution

The author (JO) conceived the idea, modeling and analysis, read and approved the manuscript.

ABSTRACT

Keeping track of the human population is essential for proper planning for facilities such as healthcare, infrastructure, education, and other essential needs. There are various ways by which the government can ensure that service provision is improved and maintained for its citizens and very often this starts by knowing the changes in demography as a function of time. In this work mathematical modeling and simulations are used to study the population dynamics of Uganda. The models are used for prediction of the country's population dynamics. Parameter sensitivity analysis shows huge influence of variations of the model parameters. The results show that the difference between the per capita birth and death rate parameters is crucial for changes in the country's population. Such findings can also be analogously applied to countries with a similar population structure and economy.

Keywords: population growth; parameter sensitivity analysis; modeling; birth rate; death rate.

1. INTRODUCTION

Currently Uganda has one of the highest population growth rates in the world [1][2]. Already the provision of essential social (and basic and needs) services such as water, housing, education, hospitals is a challenge to not only the developed countries but even more so to the developing countries such as Uganda. It is often poorly understood how changes in certain crucial parameters that influence the total population of people in a country. Gaining insight into how small and large variations in parameters impact on the national population is useful not only for the government planning but also for ensuring that in the event of sudden outbreak of epidemics, the communities are well prepared to cope.

One of the tools that have been effectively used in tackling such challenges is mathematical and statistical modeling and simulation. However, this process requires good quality data that is collected from population census. Such data acquisition processes are very expensive and time-consuming. Many third world countries are not equipped both financially and technologically to cope with such challenges alongside the already existing ones. Therefore, by using existing data from previous studies or information available from government archives, mathematical simulations can be used to make predictions of population in the future, and the associated implications of such predictions on the daily lives

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38 of the people (as demonstrated and discussed in [3]). We live in a universe of limited
39 resources. With the ever-increasing demand for food and fuels, it has become very crucial
40 for every community to plan for its population.

41 In this paper mathematical modeling and simulations are performed using prior knowledge on
42 parameters that are considered crucial for understanding the population dynamics of
43 Uganda. In literature, little attention has been devoted to unraveling the complexity in the
44 dynamics of population growth models for a third world country such as Uganda, e.g. [4] and
45 yet in a country like Uganda adequate provision of many basic needs such as shelter, water
46 and other socio-economic needs lags behind. There is also a problem of insufficient
47 communal facilities such as public toilets and or latrines for hygiene, good sanitation,
48 infrastructure and general welfare (see [5] for detailed insight). Adequate provision for most
49 if not all these elements relies heavily on the government having up to date knowledge –
50 not only on the number of people living on the country but also those that immigrate,
51 emigrate and or die. First, an analytic assessment of the model is given. This is then followed
52 by an assessment of the implications of the results from the computer simulation. Already,
53 the country's population has nearly doubled in just about two decades. Such a rapid
54 population growth can be argued to be essential for the country's economic growth. It still
55 remains a challenge for Uganda to cope with its high birth rate which by 2006 was reported
56 to be at about 3.1% with a total birth rate of about 6 to 7 children per woman.

57 Recently an article published in the New Vision newspaper Uganda on December 14th,
58 2012 highlighted the potential benefits and challenges that might arise from the young and
59 rapidly growing population of Uganda. They noted that a total of 78% of Ugandans were
60 below the age of 30 years and 52% were below the age of 15 years. They also observed
61 that 6.5 million Ugandans in the age group 18-30 years constituted 21.3% of the national
62 population – an age group that is projected to grow to about 7.7 million by 2015

63 2. METHODOLOGY

64 2.1 Population growth models

65 There are many models that have been proposed in literature as a state-of-the-art approach
66 for modeling population growth. These models include: the Exponential population growth
67 model by Malthus [6]; the Logistic growth model (e.g. [7]) – which is a modified version of
68 Malthus' model, the Gompertz population growth model (e.g. [8]). Details of the theory and
69 underlying principles are not discussed here since they can be found in literature. Many
70 variants of these models have been very useful in making predictions of various populations
71 of species e.g. Fish, human and even microbial populations. These predictions are
72 necessary since they can be used for monitoring populations and ensuring that specific
73 species of animals already well protected for instance in national parks or large water
74 reserves where fishing is not allowed. However, the need to use such mathematical models
75 to make predictions of population growth is essential for every society and nation –
76 particularly for those with rapidly changing demography in third world countries e.g. Uganda.
77 In this work, an illustration of how mathematical models can be used to identify sensitive
78 parameters is demonstrated.

79 The logistic population growth model has for long been used for making prediction and
80 studying the population dynamics of different species of animals (see e.g. [7][9]). In some
81 cases they are applied to a specific population. Consider the model for the population growth
82 as described by Wali et al. [4] which is given as:

$$\frac{d}{dt}P(t) = \alpha P(t) \left(1 - \frac{\beta}{\alpha} P(t)\right); \quad P(0) > 0 \quad (1)$$

where $P(t)$ is the population in the country at any given point in time t . Here α and β are referred to as the vital parameters of the population; α is often referred to as the Malthusian parameter, and it is basically the difference between the per-capita birth and death rate in a given population. The parameter β is the amount by which the difference in per-capita birth and death rate changes in response to the addition of one individual to the population. It is important to note that the logistic population growth model is a modification of the Malthusian model – which is in principle a simple exponential growth model.

The model in eq. (1) is much more realistic compared to the model proposed by Malthus [6] in which it was assumed that the population grows at a rate proportional to the size of the population of species under consideration. Malthus model assumption is only reasonable for a population living under ideal conditions of adequate nutrition, vast and unlimited environment in which the species lives, absence of any predators and disease among other things.

Let the population at some arbitrary defined starting time zero be $P(0)$. Let the parameters be denoted as $\vartheta_r \in \{\alpha, \beta\}$. The differential equation given in eq. (1) has the solution:

$$P(t) = \frac{\beta}{\alpha} \left(P(0) / \left(P(0) + \left(\frac{\beta}{\alpha} - P(0) \right) e^{-\alpha t} \right) \right) \quad (2)$$

This expression can be used to estimate the population of a given population at a specific time instant given that some information is available about the parameters and initial conditions. In the limiting time ($t \rightarrow \infty$), the population carrying capacity is obtained as $P(\infty) = \alpha/\beta$ for $\beta \neq 0$. At steady state $dP/dt = 0$, and from here-on the notation $\bar{P}$ is used to denote the steady state population value. By setting the left hand side of eq. (1) to zero and solving for $\bar{P}(0)$ on the right hand side, clearly, $\bar{P}(0) \gg 0$ (i.e. $\bar{P}(0) \neq 0$ the population at "time zero" is far from zero, in this case it is in millions of people), which implies that $\bar{P} = \alpha/\beta$. Clearly, if $\alpha \gg \beta$ then the maximum population attained in the distant future becomes extremely large. This would imply too many people sharing limited resources. In the above model formulation the (per capita) birth and death parameters are coalesced together into a single parameter. In principle the value of this parameter can vary depending on changes in demographics and social economic factors in any given population.

2.2 Parameter sensitivity analysis

To study the sensitivity of the parameter in the population growth model, the expression in eq. (1), is differentiated partially with respect to each parameter in the model. First let us define the local sensitivity function in order to be able to analyze the model for the population growth. Generally the local sensitivity function takes the form:

$$dS_P(t, \vartheta_r)/dt = (\partial \phi / \partial P) S_P(t, \vartheta_r) + (\partial \phi / \partial \vartheta_r) \quad (3)$$

Let us define the sensitivity matrix by: $S_P(t, \vartheta_r) := \partial P(t) / \partial \vartheta_r \in \mathbb{R}^{N \times q}$ where N is the length of time instants in years for which the population is enumerated, q – number of parameters of interest. Values of $N = 1000$ and $q = 2$ are used in this paper, where ϕ represents the right hand side of eq. (1) and $\partial \phi / \partial P$ is a diagonal system Jacobian matrix. The theory and details of applications of sensitivity analysis can be found in the work of Gunawan et al. [10] and Keesman [11]. The sensitivity functions are obtained by computing the partial derivatives with respect to each of the parameters in eq. (1). The time-dependent parameter sensitivities are computed using the expressions given in eqs. (1) and (3) for each parameter. The

126 resulting sensitivities are obtained by substituting the computed partial derivatives into the
127 expression in eq. (3). This leads to the expressions:

$$128 \quad dS_P(t, \alpha)/dt = (\alpha - 2\beta P)S_P(t, \alpha) + P \quad (4)$$

$$129 \quad dS_P(t, \beta)/dt = (\alpha - 2\beta P)S_P(t, \beta) - P^2 \quad (5)$$

130 with the initial conditions (i.e. $t = 0$) satisfying the condition that $S_P(0, \alpha) = S_P(0, \beta) = 0$. The
131 eqs. (4) and (5) are then solved to obtain the sensitivity functions $S_P(t, \alpha)$ and $S_P(t, \beta)$.

132 2.3 Evaluation of parameter sensitivities

133 Since the parameters and state variables of a given population model may take on a wide
134 range of values, it is important to assess how these parameters influence the
135 population. Here the parameter and state variables are $\vartheta_r \in \{\alpha, \beta\}$ and $P(t)$, respectively.
136 The ranking metric is computed from the area under the parameter sensitivity function using:

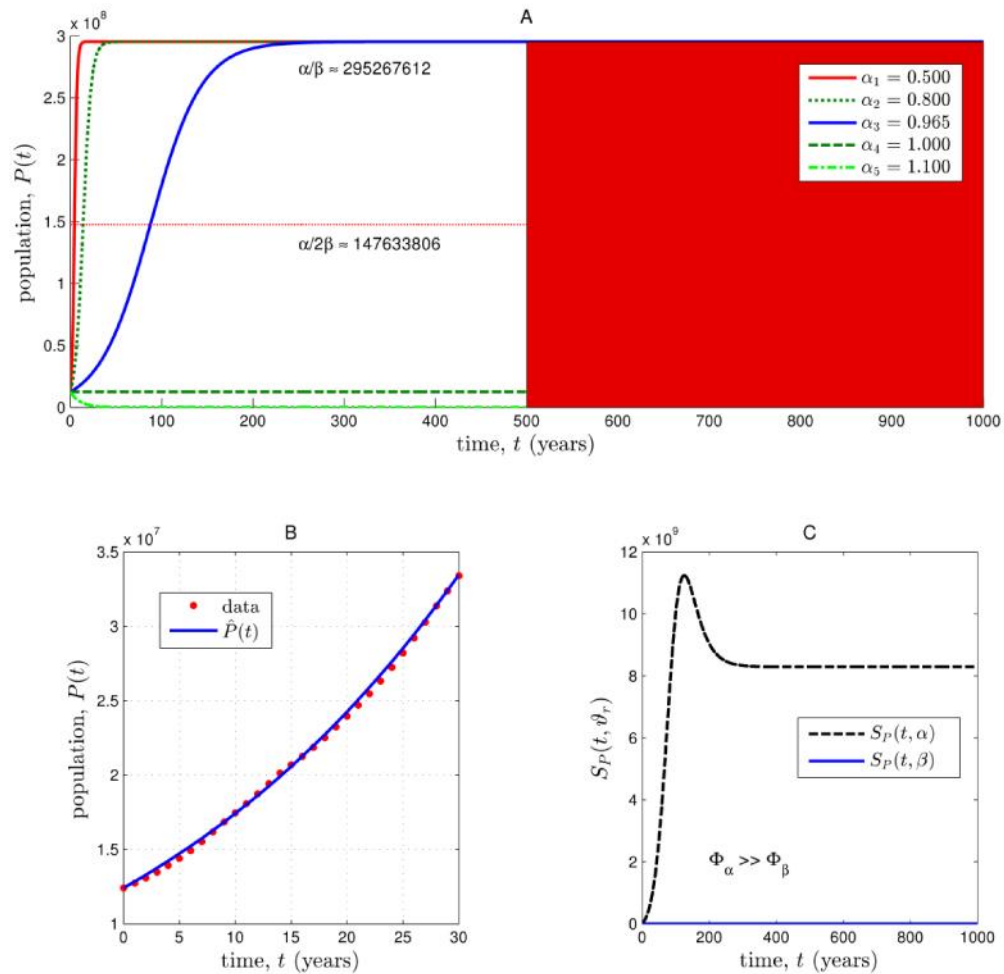
$$137 \quad \Phi_{\vartheta_r} = \int_{t(0)}^{t(f)} \|S_P(t, \vartheta_r)\|_1 dt \quad (6)$$

138 where $t(0)$ and $t(f)$ are the start and final time instants. This integral sensitivity metric is
139 based on time integrals that are analogous to the criterion described in [12]. The metric
140 specified in eq. (6) was used to compute the parameter sensitivities from the expressions
141 given in eq. (4) and (5). All the analysis and simulation in this paper was performed in the
142 MATLAB[®] Version 7.12.0 (R2011a) software under the Windows[®] 7 environment.

143 3. RESULTS AND DISCUSSION

144 In the simulation, we used nominal parameter values of $\hat{\alpha} = \bar{\alpha} \approx 0.0356$ and $\hat{\beta} = \bar{\beta} \approx$
145 $1.20569e - 10$, respectively. In this case the nominal values are considered to be equal to
146 the estimated parameter values from [4]. By considering the population data provided in
147 Table 1 (Appendix), simulations were performed and the results are provided in Figure 1.
148 The simulations also yielded the value $\Phi_{\alpha} \approx 8.061e + 12$ and $\Phi_{\beta} \approx 0$, respectively. In the
149 simulation the point $t(0)$ is considered to be at 1980. It is important to note that the point of
150 inflection, which was computed to be at $t^* = 88$ years (at the year 2068, see [4]; with a small
151 change of notation). This means that in less than a century half of the resources that can
152 support the population will have been consumed. The population is sensitive to a change in
153 α (Figure 1). This implies that a drastic or sudden variation in the difference between the
154 birth and death rates is likely to result in a rapid population change (Figure 1: C).

155 As indicated in the results, assuming the population dynamics and demography continues in
156 the projected path, then it is very likely that around in about 120 years into the future
157 (counting from the year 1980, the time for the $P(0)$), there is likely to be a significant
158 influence on the demography. Of course these predictions are all based on the assumption
159 that little will change in the rates of parameters such as immigration and emigration. This
160 calls for perhaps even better models to that with the aid of demographic data in the future
161 can help improve the predictions. This will aid improved planning for social and economic
162 services for the population. Compared to decades ago, today Ugandan has a longer life
163 expectancy and lower death rates (death as a result of all atrocities combined) – improved
164 health care and education. This simply ensures that α increases and so does the overall
165 population.



166

167 **Fig. 1. Population growth and results of parameter sensitivity analysis.** A: projected
 168 population growth. The shaded region indicates the time frame at which the population
 169 carrying capacity is reached (assuming the population keeps growing). B: Plot showing the
 170 normalized parameter sensitivities. C: A plot of sensitivity function which shows that the
 171 change in sensitivities for α and β significantly differ with α being the more sensitive of the
 172 two.

173 It is recommendable that for more accurate predictions the parameters α and β have to
 174 be estimated perhaps every decade. This is because of variations that can arise from, e.g.
 175 inaccurate accountability of the number of people in the country Uganda, porous borders,
 176 non-registered and non-counted individuals and those entering the country illegally. All these
 177 are factors that can largely influence mathematical predictions, thereby, hugely biasing the
 178 resulting inferences and recommendations that arise from such analysis.

179 One challenge that remains of interest to scientists is to understand the influx of people from
 180 rural into urban areas – thereby, putting a lot of pressure onto the already limited resources
 181 in the cities. This is a problem that is not only faced by Uganda but also other countries all

182 over the world. It would be interesting to see more advancement into the development of
183 mathematical models that help in the identification of sensitive (key) parameters that
184 influence the drive for people to move from rural to urban areas. The work in this paper
185 already illustrates how hugely the parameter α affects the population in a given country – in
186 this case Uganda.

187 4. CONCLUSION

188 The results in this work have demonstrated the impact of changes in parameters on the
189 overall population of a community – as illustrated for the case of Uganda. Changes in such
190 parameters need to be closely monitored by ensuring that quality demographic data is
191 collected and used for better predictions and planning to improve the livelihoods of the
192 citizens. The use of parameter sensitivity analysis in identifying crucial parameters and how
193 they are likely to impact on the population growth has been demonstrated. Although not
194 considered or discussed here, in real life there are surely other factors that influence on the
195 population dynamics. Results from the mathematical modeling and simulations based on
196 prior knowledge and demographic data is useful especially in cases when population growth
197 needs to be accurately predicted for proper planning purposes. This work shows that the
198 Logistic growth model is sufficiently powerful to accurately predict the population growth in
199 time.

200 ACKNOWLEDGEMENTS

201 The study had no other sponsors, i.e. it was self-sponsored.

202 COMPETING INTERESTS

203 The author declares no competing interests.

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230 APPENDIX

231 **Table 1.** Actual values of population as specified in the manuscript by Wali et al. [4].
232

Year	Actual Population	Year	Actual Population	Year	Actual Population
1980	12414719	1991	18082137	2002	25469579
1981	12725252	1992	18729453	2003	26321962
1982	13078930	1993	19424376	2004	27233661
1983	13470393	1994	20127590	2005	28199390
1984	13919514	1995	20689516	2006	29206503
1985	14391743	1996	21248718	2007	30262610
1986	14910724	1997	21861011	2008	31367972
1987	15520093	1998	22502140	2009	32369558
1988	16176418	1999	23227669	2010	33398682
1989	16832384	2000	23955822		
1990	17455758	2001	24690002		

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