

A new iterative scheme for common solution of equilibrium problems, variational inequalities and fixed point of k -strictly pseudo-contractive mappings in Hilbert spaces

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Abstract

In this paper, we present a new iterative scheme for finding a common point among the set of solution of equilibrium problems, the set of solution to a variational inequality problem and the fixed point set of $\tilde{k}$ -strictly pseudo-contractive mappings in a real Hilbert space. We then prove that the proposed scheme converges strongly to a common element which is the solution of a variational inequality problem, system of equilibrium problems, and a fixed point of $\tilde{k}$ -strictly pseudo-contractive mappings. These results improve and generalize recent works in this direction.

Keywords: Fixed point; Nonexpansive mapping; $\tilde{k}$ -strictly pseudo-contractive mappings; System of equilibrium problems; Variational inequality problem.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H and $\{F_k\}_{k \in \Delta}$ a countable family of bifunctions from $C \times C$ to $\mathbb{R}$. The equilibrium problem associated with the family $\{F_k\}_{k \in \Delta}$ where Δ is an arbitrary index set, is to find $x \in C$ such that

$$F_k(x, y) \geq 0, \forall k \in \Delta, \forall y \in C. \quad (1)$$

Assume Δ is singleton, we have that (1) becomes the equilibrium problem of finding $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \quad (2)$$

We denote the set of solutions of (2) by $EP(F)$.

Combettes and Hirstoaga [7] in 2005, proved a strong convergence theorem for an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty, see also [12].

Given a map $T : C \rightarrow H$, let $F(x, z) = \langle Tx, z - x \rangle$, $\forall x, z \in C$. Therefore problem (2) becomes a variational inequality problem of finding $x \in C$ such that

$$F(x, z) = \langle Tx, z - x \rangle \geq 0, \forall z \in C \quad (3)$$

The set of solution of (3) is denoted by $\text{VI}(C, A)$.

Problem (1) is general since numerous problems in optimization, physics, economics, variational inequalities and minimax problems are special cases; see [2, 6, 21, 26, 30, 19].

Definition 1.1 *Let C be a nonempty closed and convex subset of a real Hilbert space H . A map $T : C \rightarrow H$ is said to be nonexpansive if for all $x, z \in C$ we have*

$$\|Tx - Tz\| \leq \|x - z\|$$

We denote the fixed point set of T by $\text{Fix}(T)$.

Definition 1.2 *Let C be a nonempty closed and convex subset of a real Hilbert space H . A map $T : C \rightarrow H$ is said to be k -strictly pseudo-contractive if there exists a constant $0 \leq k < 1$ such that for all $x, z \in C$*

$$\|Tx - Tz\|^2 \leq \|x - z\|^2 + k\|(I - T)x - (I - T)z\|^2. \quad (4)$$

In a real Hilbert space it follows that (4) is equivalent to

$$\langle Tx - Tz, x - z \rangle \leq \|x - z\|^2 - \frac{1 - k}{2} \|(I - T)x - (I - T)z\|^2 \quad (5)$$

Definition 1.3 *For any $x \in H$, we define the map $P_C : H \rightarrow C$ satisfying*

$$\|x - P_C x\| \leq \|x - z\| \quad \forall z \in C$$

P_C is called the metric projection of H onto C . It is well known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - z \rangle \geq 0 \quad \forall x \in H \text{ and } \forall z \in C. \quad (6)$$

Clearly (6) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2 \quad \forall x \in H \text{ and } \forall z \in C. \quad (7)$$

For details see [24, 27].

Definition 1.4 *A mapping A of C into H is called monotone if*

$$\langle Ax - Az, x - z \rangle \geq 0 \quad \forall x, z \in C,$$

A is called α -inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - z, Ax - Az \rangle \geq \alpha \|Ax - Az\|^2 \quad \forall x, z \in C,$$

also A is L -Lipschitz-continuous if there exists $L > 0$ such that for all $x, z \in C$

$$\|Ax - Az\| \leq L \|x - z\|.$$

Given a monotone mapping A of C into H , (6) implies the following:

$$x \in \text{VI}(C, A) \Rightarrow x = P_C(x - \lambda Ax), \forall \lambda > 0,$$

and

$$x = P_C(x - \lambda Ax), \text{ for some } \lambda > 0 \Rightarrow x \in \text{VI}(C, A).$$

It is well known that H satisfies the Opial's condition [18], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - z\| \quad \forall z \in H \text{ with } x \neq z.$$

Observe that the class of k -strictly pseudo-contractive mappings includes as a subclass of the class of nonexpansive mappings i.e., when $k = 0$. The mapping T is as well said to be pseudo-contractive if $k = 1$, and T is said to be strongly pseudo-contractive if there exists $k \in (0, 1)$ such that $T - kI$ is pseudo-contractive.

Definition 1.5 A set valued Mapping $T : H \rightarrow 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply that $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \rightarrow 2^H$ is said to be maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping, and we say T is maximal monotone.

It is well known that a mapping $T : H \rightarrow 2^H$ is maximal monotone if and only if for any $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ imply that $f \in Tx$.

Given A a monotone mapping of C into H and $N_C w$ the normal cone to C at $w \in C$, i.e., $N_C w = \{x \in H : \langle w - y, x \rangle \geq 0, \forall y \in C\}$ and define

$$\begin{cases} Aw + N_C w, & w \in C, \\ \emptyset, & w \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tw$ if and only if $w \in \text{VI}(C, A)$; see [22].

Many studies have been done on iterative methods for nonexpansive mappings in the literature, see [4, 14, 15, 35], but that of strictly pseudo-contractive maps are far less developed because the second term appearing in the right hand side of (4) poses a lot of trouble in computations. However, in 1967, Browder and Petryshyn initiated the study of fixed point of strictly pseudo-contractive maps in their work. Since strictly pseudo-contractive maps is one of the most important class of mappings in nonlinear mappings, and has more interesting and powerful applications in solving inverse problems see Scherzer [23], it is of high importance to develop iterative methods for strictly pseudo-contractive maps. Recently, see [1, 8, 16, 17], many authors have devoted time in developing schemes for finding fixed points for strictly pseudo-contractive maps.

Some methods, see [2, 3, 21, 26, 30], have been proposed by many authors to solve the problem (2). Also, some authors, see [34, 35], have proposed iterative methods for finding common element of fixed point set of nonexpansive mappings and the set of solutions to the variational inequality for a monotone, Lipschitz continuous mappings, the set of solution to a system of equilibrium problems.

Combining the Mann iteration technique, the extragradient methods for variational inequality and system of equilibrium problems proposed by Korpelevich in [12], Jian-Wen Peng, Soon-Yi Wu, Gang-Lun Fan in [19] as well as Yonghong Yao, Yeong-Cheng Liou, Jen-Chih Yao in [32]. We consider a new iterative scheme for finding a common element

of the set of solution to a system of equilibrium problems, the fixed point set of a k -strictly pseudo-contractive map and the set of solutions to the variational inequality for a monotone, Lipschitz continuous mappings. We obtain a strong convergence result for the sequence generated by our scheme. The results in this paper generalize and improve so many well known results in the literature.

2 Preliminaries

We present, in this section, some useful lemmas that will be used to prove our main results.

Lemma 2.1 *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \quad (8)$$

for all $x, y \in H$.

Lemma 2.2 *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad (9)$$

for all $x, y \in H$.

Lemma 2.3 [11] *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \quad (10)$$

for all $x, y \in H$, $\lambda \in [0, 1]$.

Lemma 2.4 [28, 24] *Let $\{a_n\}$ be a sequence of non negative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n\eta_n + \delta_n, \quad n \geq 1 \quad (11)$$

where

$$(i) \ \{\sigma_n\} \subset [0, 1], \quad \sum_{n=1}^{\infty} \sigma_n = \infty;$$

$$(ii) \ \limsup_{n \rightarrow \infty} \eta_n \leq 0;$$

$$(iii) \ \delta_n \geq 0, \ n \geq 1, \ \sum_{n=0}^{\infty} \delta_n < \infty.$$

Then,

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Lemma 2.5 [20] *Let X be a Banach space, $\{x_n\}, \{y_n\}$ be two bounded sequences in X and $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying*

$$0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$$

Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$, $\forall n \geq 1$ and

$$\limsup_{n \rightarrow \infty} \{\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\} \leq 0,$$

then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

In order to solve the equilibrium problem, we assume that the bifunction F satisfies the following conditions imposed in [2]:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$,
- (A3) For each $x, y, z \in C$,

$$\lim_{t \rightarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

- (A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.6 [2] *Let C be a nonempty, closed and convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $x \in H$ and $r > 0$. Then there exists $c \in C$ such that*

$$F(c, y) + \frac{1}{r} \langle y - c, c - x \rangle \geq 0$$

for all $y \in C$.

Lemma 2.7 [7] *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $x \in H$ and $r > 0$, define a mapping $T_r^F : H \rightarrow C$ as follows:*

$$T_r^F(x) = \left\{ c \in C : F(c, y) + \frac{1}{r} \langle y - c, c - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. We then have that the following statements hold:

- (1) T_r^F is singled-valued;
- (2) T_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y \rangle;$$

- (3) $\text{Fix}(T_r^F) = EP(F)$;
- (4) $EP(F)$ is closed and convex.

Lemma 2.8 [13] *Let $T : C \rightarrow H$ be $\tilde{k}$ -strictly pseudo-contractive mapping. Define $S : C \rightarrow H$ by*

$$Sx = \alpha x + (1 - \alpha)Tx$$

for each $x \in C$.

Then, as $\alpha \in [\tilde{k}, 1)$, S is nonexpansive such that $\text{Fix}(S) = \text{Fix}(T)$. We call S the S -mapping generated by T .

Lemma 2.9 (Demi-closed principle) *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a λ -strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at 0, i.e., if $x_n \rightarrow x \in C$ and $x_n - Tx_n \rightarrow 0$, then $x = Tx$.*

3 Main Results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H . For each $k = 1, 2, \dots, m$, let F_k be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and A be a strongly monotone and L -Lipschitz continuous mapping of C into H .*

Let $T : C \longrightarrow C$ be a $\tilde{k}$ -strictly pseudo-contractive mapping and S be the S -mapping generated by T , such that $\Omega := \text{Fix}(T) \cap \text{VI}(A, C) \cap (\bigcap_{k=1}^m EP(F_k)) \neq \emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,

$$\begin{cases} u_n = T_{r_m,n}^{F_m} T_{r_{m-1},n}^{F_{m-1}} \cdots T_{r_2,n}^{F_2} T_{r_1,n}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ q_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = (1 - \beta_n) x_n + \beta_n S q_n - \alpha_n (x_n - u) \end{cases} \quad (12)$$

for all $n = 1, 2, \dots$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_{k,n}\}$, $k \in \{1, 2, \dots, m\}$ are sequences of real numbers satisfying the following conditions:

(B1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(B2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;

(B3) $(\lambda_n)_n \subset (0, \frac{1}{L})$, $\lim_{n \rightarrow \infty} \lambda_n = 0$;

(B4) $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ and $\lim_{n \rightarrow \infty} |r_{k,n+1} - r_{k,n}| = 0$ for each $k \in \{1, 2, \dots, m\}$.

Then the sequences $\{x_n\}$, $\{u_n\}$, $\{q_n\}$ and $\{y_n\}$ converge strongly to the common point $x^* \in \Omega$, given by $x^* = P_\Omega(u)$.

Proof. We shall divide the proof into 8 steps as follows:

Step 1: We show that the sequence $\{x_n\}$ is bounded.

Let $p \in F(T)$. We take $G_n^k = T_{r_k,n}^{F_k} \cdots T_{r_2,n}^{F_2} T_{r_1,n}^{F_1}$ for each $k \in \{1, 2, \dots, m\}$ and $G_n^0 = I$ for all n , hence $u_n = G_n^m x_n$.

Let $p \in \Omega$. By (3) of lemma(2.7) for each $k \in \{1, 2, \dots, m\}$ $T_{r_k,n}^{F_k}$ is nonexpansive and p is a fixed point of $T_{r_k,n}^{F_k}$, we have that

$$\|u_n - p\| = \|G_n^m x_n - G_n^m p\| \leq \|x_n - p\|, \quad \forall n \in \mathbb{N} \quad (13)$$

By using (7), the fact that A is monotone and that $p \in \text{VI}(C, A)$, we have the following

$$\begin{aligned} \|q_n - p\|^2 &\leq \|u_n - \lambda_n A y_n - p\|^2 - \|u_n - \lambda_n A y_n - q_n\|^2 \\ &= \|u_n - p\|^2 - \|q_n - u_n\|^2 + 2\lambda_n \langle A y_n, p - q_n \rangle \\ &= \|u_n - p\|^2 - \|q_n - u_n\|^2 + 2\lambda_n (\langle A y_n - A u, p - y_n \rangle + \langle A p, p - y_n \rangle \\ &\quad + \langle A y_n, y_n - q_n \rangle) \\ &\leq \|u_n - p\|^2 - \|q_n - u_n\|^2 + 2\lambda_n \langle A y_n, y_n - q_n \rangle \\ &\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - q_n \rangle - \|y_n - q_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, y_n - q_n \rangle \\ &= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\langle u_n - \lambda_n A y_n - y_n, q_n - y_n \rangle. \end{aligned}$$

Now, $y_n = P_C(u_n - \lambda_n A u_n)$ and A L -Lipschitz continuous gives that

$$\begin{aligned} \langle u_n - \lambda_n A y_n - y_n, q_n - y_n \rangle &= \langle u_n - \lambda_n A u_n - y_n, q_n - y_n \rangle + \langle \lambda_n A u_n - \lambda_n A y_n, q_n - y_n \rangle \\ &\leq \langle \lambda_n A u_n - \lambda_n A y_n, q_n - y_n \rangle \\ &\leq \lambda_n L \|u_n - y_n\| \|q_n - y_n\|. \end{aligned}$$

Therefore

$$\begin{aligned}
\|q_n - p\|^2 &\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\lambda_n L \|u_n - y_n\| \|q_n - y_n\| \\
&\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - q_n\|^2 + \lambda_n^2 L^2 \|u_n - y_n\|^2 + \|q_n - y_n\|^2 \\
&= \|u_n - p\|^2 + (\lambda_n^2 L^2 - 1) \|u_n - y_n\|^2 \\
&\leq \|u_n - p\|^2
\end{aligned} \tag{14}$$

Now using the fact that A is L - Lipschitz continuous and monotone, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(p - \lambda_n A p)\|^2 \\
&\leq \|u_n - p - \lambda_n (A p - A u_n)\|^2 \\
&\leq \|u_n - p\|^2 + L^2 \lambda_n^2 \|u_n - p\|^2 + 2L \lambda_n \|u_n - p\|^2 \\
&= (1 + L \lambda_n)^2 \|u_n - p\|^2
\end{aligned}$$

Hence by (13) we have

$$\|y_n - p\| \leq (1 + L \lambda_n) \|x_n - p\| \quad \forall n \geq 1 \tag{15}$$

We have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \beta_n)x_n + \beta_n S q_n - \alpha_n(x_n - u) - p\| \\
&= \|(1 - \beta_n)(x_n - p) + \beta_n(S q_n - p) - \alpha_n(x_n - u)\| \\
&= \|(1 - \beta_n)(x_n - p) + \beta_n(S q_n - p) - \alpha_n(x_n - p) + \alpha_n(u - p)\| \\
&= \|(1 - \beta_n - \alpha_n)(x_n - p) + \beta_n(S q_n - p) + \alpha_n(u - p)\|
\end{aligned}$$

Using (13), (14) and lemma(2.8) we have

$$\|p - S q_n\|^2 = \|S p - S q_n\|^2 \leq \|p - q_n\|^2 \leq \|x_n - p\|^2 \tag{16}$$

Hence

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \beta_n - \alpha_n) \|x_n - p\| + \beta_n \|x_n - p\| + \alpha_n \|u - p\| \\
&= (1 - \alpha_n) \|x_n - p\| + \alpha_n \|u - p\| \\
&\leq \max\{\|x_n - p\|, \|u - p\|\}
\end{aligned} \tag{17}$$

Inductively, we get

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \|u - p\|\}.$$

Hence, $\{x_n\}$ is bounded. From (13), (14) and (15) we as well obtain that $\{u_n\}, \{q_n\}$ and $\{y_n\}$ are bounded.

Since the mapping A is Lipschitz continuous, we also obtain the boundedness of the sequences $\{A x_n\}, \{A u_n\}, \{A y_n\}$.

Also, since S is nonexpansive, we obtain that $\{S x_n\}$ and $\{S q_n\}$ are bounded, using lemma (2.8).

Indeed,

$$\|S x_n - p\| = \|S x_n - S p\| \leq \|x_n - p\| \tag{18}$$

and

$$\|Sq_n - p\| = \|Sq_n - Sp\| \leq \|q_n - p\|. \quad (19)$$

Hence, boundedness of $\{Sx_n\}$ and $\{Sq_n\}$ follows from the boundedness of $\{x_n\}$ and $\{q_n\}$ respectively.

Step 2

Let $\{s_n\}$ be a bounded sequence in C . We shall show that

$$\lim_{n \rightarrow \infty} \|G_n^m s_n - G_{n+1}^m s_n\| = 0. \quad (20)$$

By step 2 of the proof of Theorem 3.1 in [5], it follows that for any $k \in \{1, 2, \dots, m\}$,

$$\lim_{n \rightarrow \infty} \|T_{r_k, n+1}^{F_k} s_n - T_{r_k, n}^{F_k} s_n\| = 0. \quad (21)$$

Using condition 2 of lemma (2.7) ($T_{r_k, n}^{F_k}$ is nonexpansive) and the definition of G_n^m , we have

$$\begin{aligned} \|G_n^m s_n - G_{n+1}^m s_n\| &= \|T_{r_m, n}^{F_m} G_n^{m-1} s_n - T_{r_m, n+1}^{F_m} G_{n+1}^{m-1} s_n\| \\ &\leq \|T_{r_m, n}^{F_m} G_n^{m-1} s_n - T_{r_m, n+1}^{F_m} G_n^{m-1} s_n\| + \|T_{r_m, n+1}^{F_m} G_n^{m-1} s_n - T_{r_m, n+1}^{F_m} G_{n+1}^{m-1} s_n\| \\ &\leq \|T_{r_m, n}^{F_m} G_n^{m-1} s_n - T_{r_m, n+1}^{F_m} G_n^{m-1} s_n\| + \|G_n^{m-1} s_n - G_{n+1}^{m-1} s_n\| \\ &\leq \|T_{r_m, n}^{F_m} G_n^{m-1} s_n - T_{r_m, n+1}^{F_m} G_n^{m-1} s_n\| + \|T_{r_{m-1}, n}^{F_{m-1}} G_n^{m-2} s_n - T_{r_{m-1}, n+1}^{F_{m-1}} G_n^{m-2} s_n\| \\ &\quad + \|G_n^{m-2} s_n - G_{n+1}^{m-2} s_n\| \\ &\leq \|T_{r_m, n}^{F_m} G_n^{m-1} s_n - T_{r_m, n+1}^{F_m} G_n^{m-1} s_n\| + \|T_{r_{m-1}, n}^{F_{m-1}} G_n^{m-2} s_n - T_{r_{m-1}, n+1}^{F_{m-1}} G_n^{m-2} s_n\| \\ &\quad + \dots + \|T_{r_2, n}^{F_2} G_n^1 s_n - T_{r_2, n+1}^{F_2} G_n^1 s_n\| + \|T_{r_1, n}^{F_1} s_n - T_{r_1, n+1}^{F_1} s_n\|. \end{aligned}$$

from which (20) follows by (21).

Step 3: $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

We know that $u_n = G_n^m x_n$ and $u_{n+1} = G_{n+1}^m x_{n+1}$. We then have that

$$\begin{aligned} \|u_n - u_{n+1}\| &= \|G_n^m x_n - G_{n+1}^m x_{n+1}\| \\ &\leq \|G_n^m x_n - G_{n+1}^m x_n\| + \|G_{n+1}^m x_n - G_{n+1}^m x_{n+1}\| \\ &\leq \|G_n^m x_n - G_{n+1}^m x_{n+1}\| + \|x_n - x_{n+1}\|. \end{aligned} \quad (22)$$

Now observe that

$$\begin{aligned} \|q_{n+1} - q_n\| &= \|P_C(u_{n+1} - \lambda_{n+1} A y_{n+1}) - P_C(u_n - \lambda_n A y_n)\| \\ &\leq \|(u_{n+1} - \lambda_{n+1} A y_{n+1}) - (u_n - \lambda_n A y_n)\| \\ &= \|u_{n+1} - u_n - \lambda_{n+1} (A u_{n+1} - A u_n) + \lambda_{n+1} (A u_{n+1} - A y_{n+1} - A u_n) + \lambda_n A y_n\| \\ &\leq \|u_{n+1} - u_n\| + \lambda_{n+1} \|A u_{n+1} - A u_n\| + \lambda_{n+1} \|A u_{n+1} - A y_{n+1} - A u_n\| + \lambda_n \|A y_n\| \\ &\leq \|u_{n+1} - u_n\| + L \lambda_{n+1} \|A u_{n+1} - A u_n\| + \lambda_{n+1} \|A u_{n+1} - A y_{n+1} - A u_n\| + \lambda_n \|A y_n\| \\ &\leq \|u_{n+1} - u_n\| + (\lambda_{n+1} + \lambda_n) M, \end{aligned} \quad (23)$$

where M is a constant such that

$$M \geq \sup_{n \geq 1} \{k \|Au_{n+1} - Au_n\| + \|Au_{n+1} - Ay_{n+1} - Au_n\| + \|Ay_n\|\}.$$

Hence, from (22) and (23) we have

$$\|q_{n+1} - q_n\| = \|G_n^m x_n - G_{n+1}^m x_n\| + \|x_n - x_{n+1}\| + (\lambda_{n+1} + \lambda_n)M.$$

We define the sequence $\{z_n\}$ to be such that for any $n \geq 1$,

$$x_{n+1} = \beta_n z_n + (1 - \beta_n)x_n. \quad (24)$$

Therefore

$$\begin{aligned} \|z_{n+1} - z_n\| &= \left\| \frac{x_{n+2} - (1 - \beta_{n+1})x_{n+1}}{\beta_{n+1}} - \frac{x_{n+1} - (1 - \beta_n)x_n}{\beta_n} \right\| \\ &= \left\| \frac{(1 - \beta_{n+1})x_{n+1} + \beta_{n+1}Sq_{n+1} - \alpha_{n+1}(x_{n+1} - u) - (1 - \beta_{n+1})x_{n+1}}{\beta_{n+1}} \right. \\ &\quad \left. - \frac{(1 - \beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u) - (1 - \beta_n)x_n}{\beta_n} \right\| \\ &= \left\| Sq_{n+1} - Sq_n - \frac{\alpha_{n+1}}{\beta_{n+1}}(x_{n+1} - u) + \frac{\alpha_n}{\beta_n}(x_n - u) \right\| \\ &\leq \|q_{n+1} - q_n\| + \frac{\alpha_{n+1}}{\beta_{n+1}}\|u - x_{n+1}\| + \frac{\alpha_n}{\beta_n}\|u - x_n\|. \end{aligned} \quad (25)$$

Hence, by (24) and (25), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_n - x_{n+1}\| &\leq \|G_n^m x_n - G_{n+1}^m x_n\| + \frac{\alpha_{n+1}}{\beta_{n+1}}\|u - x_{n+1}\| + \frac{\alpha_n}{\beta_n}\|u - x_n\| \\ &\quad + (\lambda_{n+1} - \lambda_n)M. \end{aligned}$$

Therefore, by conditions (B1)-(B3) we have

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_n - x_{n+1}\|) \leq 0.$$

By lemma (2.5), we then have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0,$$

from which it follows that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \beta_n \|z_n - x_n\| = 0.$$

We obtain also from (20), (22) and (23) that

$$\lim_{n \rightarrow \infty} \|q_{n+1} - q_n\| = 0$$

and

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0.$$

Step 4: $\lim_{n \rightarrow \infty} \|u_n - q_n\| = 0$.

We know that

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u). \quad (26)$$

Observe that

$$\begin{aligned} \|x_n - Sq_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - Sq_n\| \\ &= \|x_{n+1} - x_n\| + \|(1 - \beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u) - Sq_n\| \\ &\leq \|x_{n+1} - x_n\| + (1 - \beta_n)\|x_n - Sq_n\| + \alpha_n\|x_n - u\|. \end{aligned} \quad (27)$$

Hence,

$$\beta_n\|x_n - Sq_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|x_n - u\|. \quad (28)$$

It follows from conditions (B1) and (B2) that

$$\lim_{n \rightarrow \infty} \|x_n - Sq_n\| = 0. \quad (29)$$

Now

$$\begin{aligned} \|y_n - q_n\| &\leq \|P_C(u_n - \lambda_n Ay_n) - P_C(u_n - \lambda_n Au_n)\| \\ &\leq \|u_n - \lambda_n Ay_n - u_n - \lambda_n Au_n\| \\ &\leq \lambda_n L \|y_n - u_n\| \\ &\leq \lambda_n L M_1, \quad \text{for some } M_1 > 0. \end{aligned} \quad (30)$$

Therefore by condition (B3), we have

$$\lim_{n \rightarrow \infty} \|y_n - q_n\| = 0. \quad (31)$$

Now, let $p \in \Omega$, using lemma (2.8), (13) and (14), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|(1 - \beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u) - p\|^2 \\ &\leq \|(1 - \beta_n)x_n + \beta_n Sq_n - p\|^2 + 2\alpha_n\|x_n - u\|\|(1 - \beta_n)x_n + \beta_n Sq_n - p\| \\ &\quad + \alpha_n^2\|x_n - u\|^2 \\ &\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|Sq_n - p\|^2 \\ &\quad + 2\alpha_n\|x_n - u\|\|(1 - \beta_n)x_n + \beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2 \\ &\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|q_n - p\|^2 + 2\alpha_n\|x_n - u\|\|(1 - \beta_n)x_n \\ &\quad + \beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2 \\ &\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n [\|x_n - p\|^2 - (1 - \lambda_n^2 L^2)\|u_n - y_n\|^2] \\ &\quad + 2\alpha_n\|x_n - u\|\|(1 - \beta_n)x_n + \beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2. \end{aligned} \quad (32)$$

It follows that

$$\begin{aligned} \|u_n - y_n\|^2 &\leq \frac{1}{1 - \lambda_n^2 L^2} [(\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| \\ &\quad + \alpha_n^2\|x_n - u\|\|(1 - \beta_n)x_n + \beta_n Sq_n - p\| + \alpha_n^2\|x_n - u\|^2]. \end{aligned} \quad (33)$$

By condition (B1) and step 3, we have

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (34)$$

Hence, since

$$\|u_n - q_n\| \leq \|u_n - y_n\| + \|y_n - q_n\|, \quad (35)$$

we have

$$\lim_{n \rightarrow \infty} \|u_n - q_n\| = 0. \quad (36)$$

Step 5: $\lim_{n \rightarrow \infty} \|G_n^k x_n - G_n^{k-1} x_n\| = 0$, $k = 1, 2, \dots, m$.

Let $p \in \Omega$. Firmly nonexpansiveness of $T_{r_k, n}^{F_k}$ for each $k = 1, 2, \dots, m$ gives

$$\begin{aligned} \|G_n^k x_n - p\|^2 &= \|T_{r_k, n}^{F_k} G_n^{k-1} x_n - T_{r_k, n}^{F_k} p\|^2 \\ &\leq \langle G_n^k x_n - p, G_n^{k-1} x_n - p \rangle \\ &= \frac{1}{2} (\|G_n^k x_n - p\| + \|G_n^{k-1} x_n - p\|^2 - \|G_n^k x_n - G_n^{k-1} x_n\|^2). \end{aligned} \quad (37)$$

Therefore, we obtain that

$$\|G_n^k x_n - p\|^2 \leq \|G_n^{k-1} x_n - p\|^2 - \|G_n^k x_n - G_n^{k-1} x_n\|^2, \quad \text{for } k = 1, 2, \dots, m, \quad (38)$$

which implies that for each $k \in \{1, 2, \dots, m\}$,

$$\begin{aligned} \|G_n^k x_n - p\|^2 &\leq \|G_n^0 x_n - p\|^2 - \|G_n^k x_n - G_n^{k-1} x_n\|^2 - \|G_n^{k-1} x_n - G_n^{k-2} x_n\|^2 \\ &\quad - \dots - \|G_n^2 x_n - G_n^1 x_n\|^2 - \|G_n^1 x_n - G_n^0 x_n\|^2 \\ &\leq \|x_n - p\|^2 - \|G_n^k x_n - G_n^{k-1} x_n\|^2. \end{aligned} \quad (39)$$

Now, using the fact that $u_n = G_n^m x_n$ and (14), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \beta_n)x_n + \beta_n S q_n - \alpha_n(x_n - u) - p\|^2 \\ &\leq \|(1 - \beta_n - \alpha_n)(x_n - p) + \beta_n(S q_n - p)\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &= \|[1 - (\beta_n + \alpha_n)](x_n - p) + (\alpha_n + \beta_n)(S q_n - p) - \alpha_n(S q_n - p)\|^2 \\ &\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq [1 - (\beta_n + \alpha_n)]\|x_n - p\|^2 + (\alpha_n + \beta_n)\|u_n - p\|^2 \\ &\quad + 2\alpha_n \langle p - S q_n, x_{n+1} - p \rangle + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq [1 - (\beta_n + \alpha_n)]\|x_n - p\|^2 + (\alpha_n + \beta_n)\|G_n^k x_n - p\|^2 \\ &\quad + 2\alpha_n \langle p - S q_n, x_{n+1} - p \rangle + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \end{aligned}$$

Therefore,

$$\begin{aligned} (\alpha_n + \beta_n)\|G_n^k x_n - G_n^{k-1} x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n[\|x_{n+1} - p\|\|x_n - p\| + \|u - p\|\|x_{n+1} - p\|] \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| \\ &\quad + 2\alpha_n[\|x_{n+1} - p\|\|x_n - p\| + \|u - p\|\|x_{n+1} - p\|] \end{aligned}$$

Using condition (B1) and that $\|x_{n+1} - x_n\| \rightarrow 0$, we obtain that

$$\lim_{n \rightarrow \infty} \|G_n^k x_n - G_n^{k-1} x_n\| = 0. \quad (40)$$

Step 6: $\lim_{n \rightarrow \infty} \|S y_n - y_n\| = 0$.

Observe that

$$\begin{aligned} \|S y_n - y_n\| &\leq \|S y_n - S q_n\| + \|S q_n - x_n\| + \|G_n^0 x_n - G_n^1 x_n\| + \|G_n^1 x_n - G_n^2 x_n\| \\ &\quad + \dots + \|G_n^{m-1} x_n - G_n^m x_n\| + \|u_n - y_n\| \\ &\leq \|y_n - q_n\| + \|S q_n - x_n\| + \|G_n^0 x_n - G_n^1 x_n\| + \|G_n^1 x_n - G_n^2 x_n\| \\ &\quad + \dots + \|G_n^{m-1} x_n - G_n^m x_n\| + \|u_n - y_n\|. \end{aligned} \quad (41)$$

Therefore,

$$\lim_{n \rightarrow \infty} \|Sy_n - y_n\| = 0. \quad (42)$$

Step 7: $\limsup_{n \rightarrow \infty} \langle u - x^*, x_n - x^* \rangle \leq 0$ where $x^* = P_\Omega(u)$,
i.e., $\langle x^* - u, z - x^* \rangle \geq 0 \quad \forall z \in \Omega$.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle u - x^*, x_{n_j} - x^* \rangle = \limsup_{n \rightarrow \infty} \langle u - x^*, x_n - x^* \rangle. \quad (43)$$

Using the boundedness of $\{x_n\}$, there exist a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_k}} \rightharpoonup w$.

Without loss of generality, we assume that $x_{n_j} \rightharpoonup w$.

We show that $w \in \Omega$.

By step 5,

$$\|G_n^k x_n - G_n^{k-1} x_n\| \rightarrow 0 \quad \text{for each } k = 1, 2, \dots, m,$$

which implies that $G_n^k x_{n_j} \rightharpoonup w$ for each $k = 1, 2, \dots, m$.

Since $\|u_n - y_n\| \rightarrow 0$ and $\|u_n - q_n\| \rightarrow 0$, we have that $u_{n_j} \rightharpoonup w$, $y_{n_j} \rightharpoonup w$ and $q_{n_j} \rightharpoonup w$.

Also, $\{u_{n_j}\} \subset C$ and C is closed and convex implies that $w \in C$.

(i) We show that $w \in \text{Fix}(S) = \text{Fix}(T)$.

By lemma (2.9) and step 6, we obtain that $w \in F(S) = F(T)$.

(ii) We show that $w \in \bigcap_{k=1}^m EP(F_k)$.

By lemma (2.6), for each $k = 1, 2, \dots, m$, we have

$$F_k(G_n^k x_n, y) + \frac{1}{r_n} \langle y - G_n^k x_n, G_n^k x_n - G_n^{k-1} x_n \rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\frac{1}{r_n} \langle y - G_n^k x_n, G_n^k x_n - G_n^{k-1} x_n \rangle \geq F_k(y, G_n^k x_n), \quad \forall y \in C.$$

Therefore, for each $k = 1, 2, \dots, m$, we have

$$\left\langle y - G_{n_j}^k x_{n_j}, \frac{G_{n_j}^k x_{n_j} - G_{n_j}^{k-1} x_{n_j}}{r_{n_j}} \right\rangle \geq F_k(y, G_{n_j}^k x_{n_j}), \quad \forall y \in C.$$

By (A4), we have

$$\frac{G_{n_j}^k x_{n_j} - G_{n_j}^{k-1} x_{n_j}}{r_{n_j}} \rightarrow 0 \quad \text{and} \quad G_{n_j}^k x_{n_j} \rightharpoonup w.$$

Hence, for each $k = 1, 2, \dots, m$,

$$F_k(y, w) \leq 0, \quad \forall y \in C.$$

Define

$$y_s = (1 - s)w + sy, \quad \forall y \in C \quad \text{and} \quad s \in (0, 1].$$

By the convexity of C , we have that $y_s \in C$ from which it follows that

$$F_k(y_s, w) \leq 0, \quad \text{for each } k = 1, 2, \dots, m.$$

Using (A4), we have

$$\begin{aligned} 0 = F_k(y_s, y_s) &\leq sF_k(y_s, y) + (1-s)F_k(y_s, w) \\ &\leq sF_k(y_s, y). \end{aligned}$$

Since $s \neq 0$, we obtain that

$$F_k(y_s, y) \geq 0.$$

By (A3), for any $y \in C$, $F_k(w, y) = \lim_{s \rightarrow \infty} F_k(y_s, y) \geq 0$, for each $k = 1, 2, \dots, m$.

Therefore for each $k = 1, 2, \dots, m$, $w \in \text{EP}(F_k)$, which gives that

$$w \in \bigcap_{k=1}^m \text{EP}(F_k).$$

(iii) We show that $w \in \text{VI}(C, A)$.

Define the set-valued mapping T from H to 2^H i.e., $T : H \rightarrow 2^H$ by

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C, \\ \emptyset, & w_1 \notin C. \end{cases}$$

Where N_C is the normal cone to C at $w_1 \in C$.

The mapping T in this case is maximal monotone, and $0 \in Tw_1$ if and only if $w_1 \in \text{VI}(C, A)$, see[22].

Let $(w_1, h) \in G(T)$. It follows that $Tw_1 = Aw_1 + N_C w_1$, therefore $h - Aw_1 \in N_C w_1$. Hence we obtain that $\langle w_1 - s, h - Aw_1 \rangle \geq 0$ for any $s \in C$. Since $q_n = P_C(u_n - \lambda_n A y_n)$ and $w_1 \in C$, we have

$$\langle u_n - \lambda_n A y_n - q_n, q_n - w_1 \rangle \geq 0.$$

Therefore

$$\langle w_1 - q_n, \frac{q_n - u_n}{\lambda_n} + A y_n \rangle \geq 0 \text{ for each } n \geq 1.$$

Hence

$$\begin{aligned} \langle q_{n_j} - w_1, h \rangle &\leq \langle q_{n_j} - w_1, A w_1 \rangle \\ &\leq \langle q_{n_j} - w_1, A w_1 \rangle - \langle q_{n_j} - w_1, \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}} + A y_{n_j} \rangle \\ &= \langle q_{n_j} - w_1, A w_1 - A y_{n_j} - \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}} \rangle \\ &= \langle q_{n_j} - w_1, A w_1 - A q_{n_j} \rangle + \langle q_{n_j} - w_1, A q_{n_j} - A y_{n_j} \rangle \\ &\quad - \langle q_{n_j} - w_1, \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}} \rangle \end{aligned}$$

So

$$\langle q_{n_j} - w_1, h \rangle \leq \langle q_{n_j} - w_1, A q_{n_j} - A y_{n_j} \rangle - \langle q_{n_j} - w_1, \frac{q_{n_j} - u_{n_j}}{\lambda_{n_j}} \rangle$$

Hence it follows that

$$\langle w_1 - w, h \rangle \geq 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$, and it follows that $w \in \text{VI}(C, A)$.
(i),(ii) and (iii) give that $w \in \Omega$.

Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - x^*, x_n - x^* \rangle &= \lim_{j \rightarrow \infty} \langle u - x^*, x_{n_j} - x^* \rangle \\ &= \langle u - x^*, w - x^* \rangle \leq 0. \end{aligned} \quad (44)$$

Step 8: $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$.

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)x_n + \beta_n Sq_n - \alpha_n(x_n - u) - x^*\|^2 \\ &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(Sq_n - x^*) - \alpha_n(x_n - x^*) + \alpha_n(u - x^*)\|^2 \\ &= \|(1 - \beta_n - \alpha_n)(x_n - x^*) + \beta_n(Sq_n - x^*) + \alpha_n(u - x^*)\|^2 \\ &\leq \|(1 - \beta_n - \alpha_n)(x_n - x^*) + \beta_n(Sq_n - x^*)\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &= (1 - 2\alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \left[\frac{\alpha_n}{2} \|x_n - x^*\|^2 + \langle u - x^*, x_{n+1} - x^* \rangle \right]. \end{aligned}$$

Using lemma(2.4) and (44), it follows that $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$.

Using (14),(13) and (15) we obtain that the sequences $\{q_n\}, \{u_n\}$ and $\{y_n\}$ converge to x^* . This completes the proof.

4 Applications

The following are direct applications of our main result:

Corollary 4.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H . For each $k = 1, 2, \dots, m$, let F_k be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and A be a strongly monotone and L -Lipschitz continuous mapping of C into H . Let $T : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := \text{Fix}(T) \cap \text{VI}(A, C) \cap (\bigcap_{k=1}^m \text{EP}(F_k)) \neq \emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,*

$$\begin{cases} u_n = T_{r_m, n}^{F_m} T_{r_{m-1}, n}^{F_{m-1}} \dots T_{r_2, n}^{F_2} T_{r_1, n}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ q_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T q_n - \alpha_n(x_n - u) \end{cases} \quad (45)$$

for all $n = 1, 2, \dots$, and $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{r_{k,n}\}, k \in \{1, 2, \dots, m\}$ are sequences of real numbers satisfying the following conditions:

(B1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(B2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;

(B3) $\lim_{n \rightarrow \infty} \lambda_n = 0$;

(B4) $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ and $\lim_{n \rightarrow \infty} |r_{k,n+1} - r_{k,n}| = 0$ for each $k \in \{1, 2, \dots, m\}$.

Then the sequences $\{x_n\}, \{u_n\}, \{q_n\}$ and $\{y_n\}$ converge strongly to the common point $x^* \in \Omega$, given by $x^* = P_\Omega(u)$.

Proof. The conclusion follows immediately by Theorem 3.1, since we have that T is a nonexpansive mapping.

Corollary 4.2 Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let A be a strongly monotone and L -Lipschitz continuous mapping of C into H and $T : C \rightarrow C$ be a nonexpansive mapping such that $\Gamma := \text{Fix}(T) \cap \text{VI}(A, C) \neq \emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T P_C(x_n - \lambda_n A y_n) - \alpha_n(x_n - u) \end{cases} \quad (46)$$

for all $n = 1, 2, \dots$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences of real numbers satisfying conditions B1-B3.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the common point $x^* \in \Gamma$, given by $x^* = P_\Gamma(u)$.

Proof. Let $F_k(x, y) = 0$ for any $k \in \{1, 2, \dots, m\}$, for all $x, y \in C$ and $r_n = 1 \forall n \in \mathbb{N}$ in Theorem 3.1. It follows that $T_{r_m, n}^{F_m} = I$ (the identity mapping) $\forall n, m \in \mathbb{N}$, i.e., $u_n = x_n \forall n \in \mathbb{N}$. Clearly the conditions of Theorem 3.1 hold. Therefore we obtain that the sequence $\{x_n\}$ generated by 46 converges to the point $x^* \in \Gamma$ with $x^* = P_\Gamma(u)$.

Corollary 4.3 Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let A be a strongly monotone and L -Lipschitz continuous mapping of C into H . Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda_n A y_n) - \alpha_n(x_n - u) \end{cases} \quad (47)$$

for all $n = 1, 2, \dots$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences of real numbers satisfying conditions B1-B3.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in C$, given by $x^* = P_{\text{VI}(A, C)}(u)$.

Proof. Let $F_k(x, y) = 0$ for any $k \in \{1, 2, \dots, m\}$, for all $x, y \in C$ and $r_n = 1 \forall n \in \mathbb{N}$ in Theorem 3.1. It follows that $T_{r_m, n}^{F_m} = I$ (the identity mapping) $\forall n, m \in \mathbb{N}$, i.e., $u_n = x_n \forall n \in \mathbb{N}$. Letting $T = I$ (the identity mapping), clearly the conditions of Theorem 3.1 hold. Therefore we obtain that the sequence $\{x_n\}$ generated by 47 converges to the point $x^* \in \text{VI}(A, C)$ with $x^* = P_{\text{VI}(A, C)}(u)$.

Remark 1 Using Corollary 4.3, we have an iterative scheme to obtain the solution of a variational inequality problem involving a monotone and L -Lipschitz continuous mapping A .

We apply Corollary 4.3 directly to the following variational inequality problem.

Proposition 4.1 Let M be a $n \times n$ positive definite matrix, let C be a M -invariant closed subspace of $\mathbb{R}^n$ and $b \in C$. We define $F : C \rightarrow C$ by $F(x) = Mx + b$. Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by $u, x_1 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda_n A y_n) - \alpha_n(x_n - u) \end{cases} \quad (48)$$

for all $n = 1, 2, \dots$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences of real numbers satisfying conditions B1-B3.

Then the sequence $\{x_n\}$ converges strongly to a unique point $x^* \in C$, given by $x^* = P_{VI(A,C)}(u)$.

Proof. Since M is positive definite, we obtain that F is strongly monotone. It is also easy to see that F is a Lipschitzian mapping with Lipschitz constant $\|M\|$. The strong monotonicity of F guarantees a unique solution $x^* \in VI(F, C)$. Therefore, by Corollary 4.3, the sequence $\{x_n\}$ generated by 48 converges to the point $x^* \in VI(A, C)$ with $x^* = P_{VI(A,C)}(u)$.

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Competing Interests

Authors have declared that no competing interests exist.

Authors' Contributions

All authors contributed equally and significantly. All authors read and approved the final manuscript.

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