

The convolution sum $\sum_{m < n/8} \sigma_3(m) \sigma_3(n - 8m)$

Aeran Kim

Department of Mathematics,
Chonbuk National University,
Chonju,
Chonbuk 561-756,
South Korea.

Type of Article

Received: XX December 20XX

Accepted: XX December 20XX

Online Ready: XX December 20XX

Abstract

The convolution sum $\sum_{m < n/8} \sigma_3(m) \sigma_3(n - 8m)$ is evaluated for all $n \in \mathbb{N}$. This evaluation is used to determine the formulae of the convolution sums $\sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2) \sigma_{3,0}(n - 8m; 2)$, $\sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_{3,1}(n - 8m; 2)$, etc., and the number of representations of n through the sum of eight triangular numbers.

Keywords: Divisor functions : Convolution sum

2010 Mathematics Subject Classification: 11A05

1 Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan.

For $n \in \mathbb{N}$, $s, r \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define some necessary divisor functions for later use, which also appear in many areas of number theory:

$$\begin{aligned} \sigma_s(n) &= \sum_{d|n} d^s, & \sigma_{s,r}(n; m) &= \sum_{\substack{d|n \\ d \equiv r \pmod{m}}} d^s, \\ \Delta(q) &:= \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \\ B(q) &:= \sum_{n=1}^{\infty} b(n) q^n = (\Delta(q) \Delta(q^2))^{\frac{1}{3}} = q \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{2n})^8. \end{aligned} \tag{1.1}$$

In general, it is satisfied that

$$b(n) = -8b\left(\frac{n}{2}\right) \tag{1.2}$$

for even n (see ([4], Remark 4.3)). We note that

$$\sigma_s(n) = \sigma_{s,1}(n; 2) + \sigma_{s,0}(n; 2) \quad (1.3)$$

and

$$\sigma_{s,1}(2n; 2) = \sigma_{s,1}(n; 2), \quad \sigma_{s,0}(2n; 2) = 2^s \sigma_s(n). \quad (1.4)$$

Let $q \in \mathbb{C}$ be such that $|q| < 1$. The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad (1.5)$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad (1.6)$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \quad (1.7)$$

see ([1], p. 318). It was shown that

$$\Delta(q) = \frac{1}{1728} (M(q)^3 - N(q)^2) \quad (1.8)$$

by Ramanujan. And he gave in his notebook the following formulae, which are proved in ([2], p. 126–129) :

$$L(q) = (1 - 5x)w^2 + 12x(1 - x)w \frac{dw}{dx}, \quad (1.9)$$

$$M(q) = (1 + 14x + x^2)w^4, \quad (1.10)$$

$$N(q) = (1 + x)(1 - 34x + x^2)w^6, \quad (1.11)$$

where w is defined by

$$w = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n} x^n.$$

From (1.8), (1.10), and (1.11), we obtain

$$\Delta(q) = \frac{x(1-x)^4 w^{12}}{2^4}. \quad (1.12)$$

Applying the principle of duplication (see ([2], p. 125))

$$q \rightarrow q^2, \quad x \rightarrow \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2, \quad w \rightarrow \left(\frac{1 + \sqrt{1-x}}{2} \right) w$$

to (1.10) and (1.12), we deduce that

$$M(q^2) = (1 - x + x^2)w^4, \quad (1.13)$$

$$\Delta(q^2) = \frac{x^2(1-x)^2 w^{12}}{2^8}. \quad (1.14)$$

Again applying the principle of duplication to (1.13) and (1.14), we obtain

$$M(q^4) = (1 - x + \frac{1}{16}x^2)w^4, \quad (1.15)$$

$$\Delta(q^4) = \frac{x^4(1-x)w^{12}}{2^{16}}. \quad (1.16)$$

Lastly, applying the principle of duplication to (1.15), we have

$$\begin{aligned} M(q^8) &= (\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x})w^4 \\ &= -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4. \end{aligned} \quad (1.17)$$

And from the above information K. S. Williams finds that

$$w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4), \quad (1.18)$$

$$xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2), \quad (1.19)$$

$$x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4), \quad (1.20)$$

in ([5], p. 392). In this paper, we obtain the convolution sums

$$\begin{aligned} \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_3(n-8m), \quad \sum_{m < \frac{n}{8}} \sigma_{3,0}(m;2)\sigma_{3,0}(n-8m;2), \\ \sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_{3,1}(n-8m;2), \quad \sum_{m < \frac{n}{8}} \sigma_{3,1}(m;2)\sigma_{3,0}(n-8m;2) \end{aligned}$$

for $n \in \mathbb{N}$, using the Ramanujan and Williams' results. Furthermore, as an application of $\sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$, we evaluate in Section 4 the number

$$\begin{aligned} T(n) = \text{card} \left\{ (x_1, \dots, x_{16}) \in \mathbb{N}_0^{16} \mid n = \frac{1}{2}x_1(x_1+1) + \dots + \frac{1}{2}x_8(x_8+1) \right. \\ \left. + 4 \left(\frac{1}{2}x_9(x_9+1) + \dots + \frac{1}{2}x_{16}(x_{16}+1) \right) \right\}, \end{aligned}$$

for $n \in \mathbb{N}_0$ (see Theorem 4.1).

2 Preparations to find $\sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$

In order to calculate the convolution sum $\sum_{m < \frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$ we introduce the constants $g(n)$ and $h(n)$ for $n \in \mathbb{N}$ defined by

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4 \left(\frac{\Delta(q^2)^{11}}{\Delta(q^4)^3 \Delta(q)^4} \right)^{\frac{1}{6}} = x\sqrt{1-x}w^8 \quad (2.1)$$

and

$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12} \left(\frac{\Delta(q^4)^5}{\Delta(q^2)} \right)^{\frac{1}{6}} = x^3\sqrt{1-x}w^8. \quad (2.2)$$

Corollary 2.1. We have

(a)

$$G(q) = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}}.$$

(b)

$$H(q) = 2^{12} q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4 (1-q^{4n})^{16}.$$

Proof. (a) By (1.1), (1.12), (1.14), (2.1), and (1.16), we obtain that

$$\begin{aligned} G(q) &= x\sqrt{1-x}w^8 = 2^4 \left(\frac{\Delta(q^2)^{11}}{\Delta(q^4)^3 \Delta(q)^4} \right)^{\frac{1}{6}} \\ &= 2^4 \left(\frac{(q^2 \prod_{n=1}^{\infty} (1-q^{2n})^{24})^{11}}{(q^4 \prod_{n=1}^{\infty} (1-q^{4n})^{24})^3 (q \prod_{n=1}^{\infty} (1-q^n)^{24})^4} \right)^{\frac{1}{6}} \\ &= 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{44}}{(1-q^{4n})^{12} (1-q^n)^{16}} = 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{44}}{(1-q^{2n})^{12} (1+q^{2n})^{12} (1-q^n)^{16}} \\ &= 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{32}}{(1+q^{2n})^{12} (1-q^n)^{16}} = 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^n)^{32} (1+q^n)^{32}}{(1+q^{2n})^{12} (1-q^n)^{16}} \\ &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32} (1-q^n)^{16}}{(1+q^{2n})^{12}}. \end{aligned}$$

(b) It is similar to Corollary 2.1 (a). □

The first eighteen values of $b(n)$ are given in the Table 1

n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	7	1016	13	1382
2	-8	8	-512	14	-8128
3	12	9	-2043	15	-2520
4	64	10	1680	16	4096
5	-210	11	1092	17	14706
6	-96	12	768	18	16344

TABLE 1. $b(n)$ for n ($1 \leq n \leq 18$)

and $g(n)$ are in the Table 2

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	16	7	-25728	13	233056
2	256	8	-49152	14	260096
3	1728	9	-44976	15	398976
4	6144	10	-53760	16	393216
5	10976	11	-55744	17	-301280
6	3072	12	73728	18	-523008

TABLE 2. $g(n)$ for n ($1 \leq n \leq 18$)

and $h(n)$ are given in the Table 3.

n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	7	−24576	13	540672
2	0	8	0	14	0
3	4096	9	−163840	15	385024
4	0	10	0	16	0
5	16384	11	−20480	17	−163840
6	0	12	0	18	0

TABLE 3. $h(n)$ for n ($1 \leq n \leq 18$)

We can see Proposition 2.1 in the wide area of [3].

Proposition 2.1. *We obtain*

(a) (See ([3], (2.5)))

$$M^2(q) = 1 + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_7(n) q^n.$$

(b) (See ([3], Theorem 5.1 (ii)))

$$M(q)M(q^2) = \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q).$$

(c) (See ([3], Theorem 5.1 (v)))

$$M(q)M(q^4) = \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2).$$

By (1.1) and (1.12) we note that

$$B(q) = \frac{x(1-x)^2w^8}{2^4}. \quad (2.3)$$

Lemma 2.2. *We have*

(a)

$$w^8 = \frac{1}{255}M^2(q) - \frac{2}{255}M^2(q^2) + \frac{256}{255}M^2(q^4) + \frac{512}{17}B(q) + \frac{8192}{17}B(q^2).$$

(b)

$$xw^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) + \frac{240}{17}B(q) + 256B(q^2).$$

(c)

$$x^2w^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q).$$

(d)

$$x^3w^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q) - 256B(q^2).$$

Proof. The proofs are similar, so we prove only Lemma 2.2 (a) and (b).

(a) From (1.18) we can calculate

$$\begin{aligned}
 w^8 &= (w^4)^2 = \left(\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4) \right)^2 \\
 &= \left(\frac{1}{15} \right)^2 \{ M^2(q) + 4M^2(q^2) + 256M^2(q^4) - 4M(q)M(q^2) \\
 &\quad - 64M(q^2)M(q^4) + 32M(q)M(q^4) \} \\
 &= \left(\frac{1}{15} \right)^2 [M^2(q) + 4M^2(q^2) + 256M^2(q^4) \\
 &\quad - 4 \left\{ \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q) \right\} \\
 &\quad - 64 \left\{ \frac{1}{17}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{3600}{17}B(q^2) \right\} \\
 &\quad + 32 \left\{ \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2) \right\}],
 \end{aligned}$$

where we insert Proposition 2.1 (b) and (c). So we obtain

$$w^8 = \frac{1}{255}M^2(q) - \frac{2}{255}M^2(q^2) + \frac{256}{255}M^2(q^4) + \frac{512}{17}B(q) + \frac{8192}{17}B(q^2).$$

(b) From (1.18) and (1.19), we can observe that

$$\begin{aligned}
 xw^8 &= w^4 \cdot xw^4 \\
 &= \left(\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4) \right) \left(\frac{1}{15}M(q) - \frac{1}{15}M(q^2) \right) \\
 &= \left(\frac{1}{15} \right)^2 \{ M^2(q) - 3M(q)M(q^2) + 2M^2(q^2) + 16M(q)M(q^4) - 16M(q^2)M(q^4) \} \\
 &= \left(\frac{1}{15} \right)^2 \left[M^2(q) - 3 \left\{ \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q) \right\} + 2M^2(q^2) \right. \\
 &\quad + 16 \left\{ \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2) \right\} \\
 &\quad \left. - 16 \left\{ \frac{1}{17}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{3600}{17}B(q^2) \right\} \right] \\
 &= \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) + \frac{240}{17}B(q) + 256B(q^2),
 \end{aligned}$$

where we use Proposition 2.1 (b) and (c).

□

Lemma 2.3. *We have*

(a)

$$x^2\sqrt{1-xw^8} = 8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q).$$

(b)

$$\begin{aligned}\sqrt{1-xw^8} &= \frac{1}{255}M^2(q^2) - \frac{86}{85}M^2(q^4) + \frac{512}{255}M^2(q^8) - \frac{2208}{17}B(q^2) \\ &\quad - \frac{70656}{17}B(q^4) + \frac{3}{2}G(q) - \frac{1}{4}H(q).\end{aligned}$$

Proof. (a) First we apply the principle of duplication to $B(q)$ in (2.3) :

$$B(q^2) = \sum_{n=1}^{\infty} b(n)q^{2n} = \sum_{n=1}^{\infty} b\left(\frac{n}{2}\right)q^n = (\Delta(q^2)\Delta(q^4))^{1/3} = \frac{x^2(1-x)w^8}{2^8}. \quad (2.4)$$

Again applying the principle of duplication to (2.4), we have

$$\begin{aligned}B(q^4) &= \sum_{n=1}^{\infty} b(n)q^{4n} = \sum_{n=1}^{\infty} b\left(\frac{n}{4}\right)q^n = (\Delta(q^4)\Delta(q^8))^{1/3} \\ &= \frac{-2x^2w^8 + 2x^3w^8 + 2x^2\sqrt{1-xw^8} - x^3\sqrt{1-xw^8}}{2^{14}}.\end{aligned} \quad (2.5)$$

This leads to

$$\begin{aligned}x^2\sqrt{1-xw^8} &= 2^{13}B(q^4) + x^2w^8 - x^3w^8 + \frac{1}{2}x^3\sqrt{1-xw^8} \\ &= 8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q),\end{aligned}$$

where we use Lemma 2.2 (c), (d), and the definition of $H(q)$ in (2.2).

(b) By Lemma 2.2 (a) and (b), we can calculate $(1-x)w^8$ as follows :

$$(1-x)w^8 = w^8 - xw^8 = -\frac{1}{255}M^2(q^2) + \frac{256}{255}M^2(q^4) + 16B(q) + \frac{3840}{17}B(q^2). \quad (2.6)$$

Applying the principle of duplication to (2.6), we have

$$\begin{aligned}\frac{1}{2}w^8 + \frac{1}{2}\sqrt{1-xw^8} - xw^8 &- \frac{3}{4}x\sqrt{1-xw^8} + \frac{19}{32}x^2w^8 \\ &+ \frac{9}{32}x^2\sqrt{1-xw^8} - \frac{3}{32}x^3w^8 - \frac{1}{64}x^3\sqrt{1-xw^8} \\ &= -\frac{1}{255}M^2(q^4) + \frac{256}{255}M^2(q^8) + 16B(q^2) + \frac{3840}{17}B(q^4).\end{aligned} \quad (2.7)$$

Using Lemma 2.2 (a), (b), (c), (d) and Lemma 2.3 (a) into (2.7), we obtain

$$\begin{aligned}\frac{1}{2}\sqrt{1-xw^8} - \frac{3}{4}x\sqrt{1-xw^8} &= \frac{1}{510}M^2(q^2) - \frac{43}{85}M^2(q^4) + \frac{256}{255}M^2(q^8) \\ &\quad - \frac{1104}{17}B(q^2) - \frac{35328}{17}B(q^4) - \frac{1}{8}x^3\sqrt{1-xw^8}.\end{aligned}$$

Inserting the definitions of $G(q)$ and $H(q)$ into the above equation, we conclude that

$$\begin{aligned}\sqrt{1-xw^8} &= \frac{1}{255}M^2(q^2) - \frac{86}{85}M^2(q^4) + \frac{512}{255}M^2(q^8) - \frac{2208}{17}B(q^2) \\ &\quad - \frac{70656}{17}B(q^4) + \frac{3}{2}G(q) - \frac{1}{4}H(q).\end{aligned}$$

□

3 The formula of $\sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_3(n - 8m)$

Theorem 3.1. *We have*

$$\begin{aligned} M(q)M(q^8) &= \frac{1}{4352}M^2(q) + \frac{15}{4352}M^2(q^2) + \frac{15}{272}M^2(q^4) + \frac{16}{17}M^2(q^8) + \frac{17325}{136}B(q) \\ &\quad + \frac{23175}{17}B(q^2) - \frac{424800}{17}B(q^4) + \frac{225}{32}G(q) - \frac{225}{128}H(q). \end{aligned}$$

Proof. By (1.17), we observe that

$$\begin{aligned} M(q)M(q^8) &= M(q) \left\{ -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-xw^4} - \frac{15}{64}x\sqrt{1-xw^4} \right\} \\ &= -\frac{1}{32}M(q)M(q^2) + \frac{9}{16}M(q)M(q^4) \\ &\quad + M(q) \left(\frac{15}{32}\sqrt{1-xw^4} - \frac{15}{64}x\sqrt{1-xw^4} \right). \end{aligned} \tag{3.1}$$

Then, by (1.10), the third term of (3.1) can be written as

$$\begin{aligned} &M(q) \left(\frac{15}{32}\sqrt{1-xw^4} - \frac{15}{64}x\sqrt{1-xw^4} \right) \\ &= (1 + 14x + x^2)w^4 \left(\frac{15}{32}\sqrt{1-xw^4} - \frac{15}{64}x\sqrt{1-xw^4} \right) \\ &= \frac{15}{32}\sqrt{1-xw^8} + \frac{405}{64}x\sqrt{1-xw^8} - \frac{45}{16}x^2\sqrt{1-xw^8} - \frac{15}{64}x^3\sqrt{1-xw^8} \\ &= \frac{1}{544}M^2(q^2) - \frac{129}{272}M^2(q^4) + \frac{16}{17}M^2(q^8) - \frac{13275}{17}B(q^2) - \frac{424800}{17}B(q^4) \\ &\quad + \frac{225}{32}G(q) - \frac{225}{128}H(q), \end{aligned}$$

where we use Lemma 2.3 (a), (b) and the definitions of $G(q)$ and $H(q)$. Therefore, by the help of Proposition 2.1 (b) and (c), Eq. (3.1) is

$$\begin{aligned} M(q)M(q^8) &= \frac{1}{4352}M^2(q) + \frac{15}{4352}M^2(q^2) + \frac{15}{272}M^2(q^4) + \frac{16}{17}M^2(q^8) + \frac{17325}{136}B(q) \\ &\quad + \frac{23175}{17}B(q^2) - \frac{424800}{17}B(q^4) + \frac{225}{32}G(q) - \frac{225}{128}H(q). \end{aligned}$$

□

Theorem 3.2. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_3(n - 8m) \\ &= \frac{1}{8355840} \left\{ 16\sigma_7(n) + 240\sigma_7\left(\frac{n}{2}\right) + 3840\sigma_7\left(\frac{n}{4}\right) + 65536\sigma_7\left(\frac{n}{8}\right) \right. \\ &\quad - 34816\sigma_3(n) - 34816\sigma_3\left(\frac{n}{8}\right) + 18480b(n) + 197760b\left(\frac{n}{2}\right) \\ &\quad \left. - 3624960b\left(\frac{n}{4}\right) + 1020g(n) - 255h(n) \right\}. \end{aligned}$$

Proof. To prove we expand $M(q)M(q^8)$ in Theorem 3.1 with (1.6) :

$$\begin{aligned} M(q)M(q^8) &= \left\{ 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right\} \left\{ 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{8n} \right\} \\ &= \left\{ 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right\} \left\{ 1 + 240 \sum_{n=1}^{\infty} \sigma_3\left(\frac{n}{8}\right)q^n \right\} \\ &= 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m + 240 \sum_{n=1}^{\infty} \sigma_3\left(\frac{n}{8}\right)q^n \\ &\quad + 240^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sigma_3(m)\sigma_3\left(\frac{n}{8}\right)q^{m+n}. \end{aligned}$$

This leads to

$$\begin{aligned} &\sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{8}} \sigma_3(m)\sigma_3(N-8m) \right) q^N \\ &= \frac{1}{240^2} \left\{ M(q)M(q^8) - 1 - 240 \sum_{N=1}^{\infty} \left(\sigma_3(N) + \sigma_3\left(\frac{N}{8}\right) \right) q^N \right\}, \end{aligned}$$

since

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sigma_3(m)\sigma_3\left(\frac{n}{8}\right)q^{m+n} &= \sum_{N=2}^{\infty} \sum_{m < N} \sigma_3(m)\sigma_3\left(\frac{N-m}{8}\right)q^N \\ &= \sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{8}} \sigma_3(N-8m)\sigma_3(m) \right) q^N \end{aligned}$$

by letting $m+n := N$. Finally, by using Proposition 2.1 (a), the proof is complete. $\square$

Now we will find some convolution formulae based on Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned} &\sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2)\sigma_{3,0}(n-8m; 2) \\ &= \frac{1}{130560} \left\{ 16\sigma_7\left(\frac{n}{2}\right) + 240\sigma_7\left(\frac{n}{4}\right) + 3840\sigma_7\left(\frac{n}{8}\right) + 65536\sigma_7\left(\frac{n}{16}\right) \right. \\ &\quad - 34816\sigma_3\left(\frac{n}{2}\right) - 34816\sigma_3\left(\frac{n}{16}\right) + 18480b\left(\frac{n}{2}\right) + 197760b\left(\frac{n}{4}\right) \\ &\quad \left. - 3624960b\left(\frac{n}{8}\right) + 1020g\left(\frac{n}{2}\right) - 255h\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_{3,1}(n-8m; 2) \\ &= \frac{1}{8355840} \left\{ 16\sigma_7(n) - 1808\sigma_7\left(\frac{n}{2}\right) - 26880\sigma_7\left(\frac{n}{4}\right) - 458752\sigma_7\left(\frac{n}{8}\right) \right. \\ & \quad - 34816\sigma_3(n) + 278528\sigma_3\left(\frac{n}{2}\right) + 243712\sigma_3\left(\frac{n}{8}\right) + 18480b(n) \\ & \quad \left. - 78720b\left(\frac{n}{2}\right) - 8048640b\left(\frac{n}{4}\right) + 1020g(n) - 255h(n) \right\}. \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_{3,1}(m; 2) \sigma_{3,0}(n-8m; 2) \\ &= \frac{1}{130560} \left\{ 16\sigma_7\left(\frac{n}{2}\right) + 240\sigma_7\left(\frac{n}{4}\right) + 4352\sigma_7\left(\frac{n}{8}\right) - 65536\sigma_7\left(\frac{n}{16}\right) \right. \\ & \quad + 30464\sigma_3\left(\frac{n}{2}\right) - 4352\sigma_3\left(\frac{n}{8}\right) + 34816\sigma_3\left(\frac{n}{16}\right) - 14160b\left(\frac{n}{2}\right) \\ & \quad \left. - 128640b\left(\frac{n}{4}\right) + 3624960b\left(\frac{n}{8}\right) - 1020g\left(\frac{n}{2}\right) + 255h\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

Proof. (a) By (1.4), we note that

$$\begin{aligned} \sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2) \sigma_{3,0}(n-8m; 2) &= \sum_{m < \frac{n}{8}} 2^3 \sigma_3\left(\frac{m}{2}\right) \cdot 2^3 \sigma_3\left(\frac{n}{2} - 4m\right) \\ &= 2^6 \sum_{m < \frac{n}{16}} \sigma_3(m) \sigma_3\left(\frac{n}{2} - 8m\right). \end{aligned}$$

Thus we refer to Theorem 3.2.

(b) First we can expand (1.6) as

$$\begin{aligned} M(q) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240 \sum_{n=1}^{\infty} (\sigma_{3,1}(n; 2) + \sigma_{3,0}(n; 2)) q^n \\ &= 1 + 240 \sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n + 240 \sum_{n=1}^{\infty} 8\sigma_3\left(\frac{n}{2}\right) q^n \\ &= 1 + 240 \sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n + 8 \cdot 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \\ &= 8 \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) - 7 + 240 \sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n \\ &= 8M(q^2) - 7 + 240 \sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n. \end{aligned}$$

This shows that

$$\sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n = \frac{M(q) - 8M(q^2) + 7}{240}. \quad (3.2)$$

Second, by (1.6), we have

$$\sum_{n=1}^{\infty} \sigma_3(n) q^{8n} = \frac{M(q^8) - 1}{240}. \quad (3.3)$$

Multiplying (3.2) by (3.3) we obtain

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_3(m) q^{8m} \right) \\ &= \frac{1}{240^2} (M(q) - 8M(q^2) + 7) (M(q^8) - 1) \\ &= \frac{1}{240^2} \{M(q)M(q^8) - 8M(q^2)M(q^8) + 7M(q^8) - M(q) + 8M(q^2) - 7\}. \end{aligned} \quad (3.4)$$

Then the left hand side of (3.4) is

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} \sigma_{3,1}(n; 2) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_3(m) q^{8m} \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_{3,1}(n; 2) \sigma_3(m) q^{n+8m} \\ &= \sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{8}} \sigma_{3,1}(N - 8m; 2) \sigma_3(m) \right) q^N \end{aligned}$$

by letting $n + 8m := N$. And for the right hand side of (3.4) we use (1.6), Proposition 2.1 (a), (c), and Theorem 3.1.

(c) We note that

$$\begin{aligned} & \sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_3(n - 8m) \\ &= \sum_{m < \frac{n}{8}} \{ \sigma_{3,1}(m; 2) + \sigma_{3,0}(m; 2) \} \{ \sigma_{3,1}(n - 8m; 2) + \sigma_{3,0}(n - 8m; 2) \} \\ &= \sum_{m < \frac{n}{8}} \sigma_{3,1}(m; 2) \sigma_{3,1}(n - 8m; 2) + \sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2) \sigma_{3,1}(n - 8m; 2) \\ & \quad + \sum_{m < \frac{n}{8}} \sigma_{3,1}(m; 2) \sigma_{3,0}(n - 8m; 2) + \sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2) \sigma_{3,0}(n - 8m; 2) \\ &= \sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_{3,1}(n - 8m; 2) + \sum_{m < \frac{n}{8}} \sigma_{3,1}(m; 2) \sigma_{3,0}(n - 8m; 2) \\ & \quad + \sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2) \sigma_{3,0}(n - 8m; 2). \end{aligned}$$

Thus we refer to Theorem 3.2, Theorem 3.3 (a), and (b).

□

4 The number of representations of n through the sum of eight triangular numbers

The triangular numbers are the nonnegative integers

$$T_k = \frac{1}{2}k(k+1), \quad k \in \mathbb{N}_0,$$

so that

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 3, \quad T_3 = 6, \quad T_4 = 10, \quad T_5 = 15, \dots$$

Let k be a positive integer. We denote the number of representations of n ($\in \mathbb{N}_0$) as a sum of k triangular numbers by $t_k(n)$, that is,

$$t_k(n) := \text{card} \left\{ (m_1, \dots, m_k) \in \mathbb{N}_0^k \mid n = \frac{1}{2}m_1(m_1+1) + \dots + \frac{1}{2}m_k(m_k+1) \right\}.$$

Obviously, $t_k(0) = 1$ for all $k \in \mathbb{N}$.

Proposition 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

(a) (See ([6], Theorem 16.1))

$$t_2(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid (4n+1)}} \left(\frac{-4}{d} \right).$$

(b) (See ([6], Theorem 16.7))

$$t_4(n) = \sigma_1(2n+1).$$

(c) (See ([6], Theorem 16.12))

$$t_6(n) = -\frac{1}{8} \sum_{\substack{d \in \mathbb{N} \\ d \mid (4n+3)}} \left(\frac{-4}{d} \right) d^2.$$

(d) (See ([6], Theorem 16.13))

$$t_8(n) = \sigma_3(n+1) - \sigma_3\left(\frac{n+1}{2}\right).$$

Theorem 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} T(n) = \frac{1}{557056} \left\{ 16\sigma_7(n+5) - 16\sigma_7\left(\frac{n+5}{2}\right) + 1072b(n+5) + 23936b\left(\frac{n+5}{2}\right) \right. \\ \left. + 278528b\left(\frac{n+5}{4}\right) - 68g(n+5) + 17h(n+5) \right\}. \end{aligned}$$

Proof. First we note that

$$T(n) = \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+4m=n}} t_8(l)t_8(m) = \sum_{0 \leq m \leq \frac{n}{4}} t_8(n-4m)t_8(m).$$

By Proposition 4.1 (d), the above equation can be written as

$$\begin{aligned}
 & \sum_{0 \leq m \leq \frac{n}{4}} t_8(m) t_8(n-4m) \\
 &= \sum_{0 \leq m \leq \frac{n}{4}} \left\{ \sigma_3(m+1) - \sigma_3\left(\frac{m+1}{2}\right) \right\} \left\{ \sigma_3(n-4m+1) - \sigma_3\left(\frac{n-4m+1}{2}\right) \right\} \\
 &= \sum_{0 \leq m \leq \frac{n}{4}} \sigma_3(m+1) \sigma_3(n-4m+1) - \sum_{0 \leq m \leq \frac{n}{4}} \sigma_3(m+1) \sigma_3\left(\frac{n+1}{2} - 2m\right) \\
 &\quad - \sum_{0 \leq m \leq \frac{n}{4}} \sigma_3\left(\frac{m+1}{2}\right) \sigma_3(n-4m+1) + \sum_{0 \leq m \leq \frac{n}{4}} \sigma_3\left(\frac{m+1}{2}\right) \sigma_3\left(\frac{n+1}{2} - 2m\right) \\
 &= \sum_{1 \leq m \leq \frac{n}{4}+1} \sigma_3(m) \sigma_3(n-4(m-1)+1) - \sum_{1 \leq m \leq \frac{n}{4}+1} \sigma_3(m) \sigma_3\left(\frac{n+1}{2} - 2(m-1)\right) \\
 &\quad - \sum_{1 \leq m \leq \frac{n}{4}+1} \sigma_3\left(\frac{m}{2}\right) \sigma_3(n-4(m-1)+1) \\
 &\quad + \sum_{1 \leq m \leq \frac{n}{4}+1} \sigma_3\left(\frac{m}{2}\right) \sigma_3\left(\frac{n+1}{2} - 2(m-1)\right) \\
 &= \sum_{1 \leq m \leq \frac{n}{4}+1} \sigma_3(m) \sigma_3(n+5-4m) - \sum_{1 \leq m \leq \frac{n+3}{4}} \sigma_3(m) \sigma_3\left(\frac{n+5}{2} - 2m\right) \\
 &\quad - \sum_{1 \leq m \leq \frac{n+4}{8}} \sigma_3(m) \sigma_3(n+5-8m) + \sum_{1 \leq m \leq \frac{n+3}{8}} \sigma_3(m) \sigma_3\left(\frac{n+5}{2} - 4m\right).
 \end{aligned}$$

Therefore, we refer to

$$\begin{aligned}
 & \sum_{m < \frac{n}{2}} \sigma_3(m) \sigma_3(n-2m) \\
 &= \frac{1}{2040} \sigma_7(n) + \frac{2}{255} \sigma_7\left(\frac{n}{2}\right) - \frac{1}{240} \sigma_3(n) - \frac{1}{240} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{272} b(n),
 \end{aligned}$$

in ([3], Theorem 5.2 (ii)),

$$\begin{aligned}
 \sum_{m < \frac{n}{4}} \sigma_3(m) \sigma_3(n-4m) &= \frac{1}{32640} \sigma_7(n) + \frac{1}{2176} \sigma_7\left(\frac{n}{2}\right) + \frac{2}{255} \sigma_7\left(\frac{n}{4}\right) \\
 &\quad - \frac{1}{240} \sigma_3(n) - \frac{1}{240} \sigma_3\left(\frac{n}{4}\right) + \frac{9}{2176} b(n) + \frac{9}{136} b\left(\frac{n}{2}\right),
 \end{aligned}$$

in ([3], Theorem 5.2 (v)), and Theorem 3.2.

□

Example 4.2. The values of $T(n)$ for $n = 1, 2, \dots, 10$ are as follows :

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	8	28	64	134	288	568	1024	1793	3024	4868

TABLE 4. $T(n)$ for n ($1 \leq n \leq 10$)

Theorem 4.3. *Let $n \in \mathbb{N}$. Then we have*

$$b(n) \equiv 2\sigma_7(2n) - 2\sigma_7(n) \pmod{17}.$$

In particular, if n is odd, then

$$b(n) \equiv \sigma_7(n) \pmod{17}.$$

Proof. First for $n = 1, 2$, it is obvious that $b(1) \equiv 2\sigma_7(2) - 2\sigma_7(1) \pmod{17}$ and $b(2) \equiv 2\sigma_7(4) - 2\sigma_7(2) \pmod{17}$ by the Table 1.

Second for $n \geq 3$, let us consider Theorem 4.1 as follows :

$$\begin{aligned} 557056T(n) &= 16\sigma_7(n+5) - 16\sigma_7\left(\frac{n+5}{2}\right) + 1072b(n+5) + 23936b\left(\frac{n+5}{2}\right) \\ &\quad + 278528b\left(\frac{n+5}{4}\right) - 68g(n+5) + 17h(n+5). \end{aligned} \quad (4.1)$$

For $N \in \mathbb{N}$, we can put $n = 2N - 1$ in (4.1) so we obtain

$$\begin{aligned} 2^{15} \cdot 17T(2N-1) &= 16\sigma_7(2N+4) - 16\sigma_7(N+2) + 1072b(2N+4) \\ &\quad + 2^7 \cdot 11 \cdot 17b(N+2) + 2^{14} \cdot 17b\left(\frac{N}{2} + 1\right) - 2^2 \cdot 17g(2N+4) \\ &\quad + 17h(2N+4). \end{aligned}$$

This leads us that

$$\begin{aligned} 0 &\equiv -\sigma_7(2N+4) + \sigma_7(N+2) + b(2N+4) \\ &\equiv -\sigma_7(2(N+2)) + \sigma_7(N+2) - 8b(N+2) \pmod{17}, \end{aligned}$$

where we use (1.2). Therefore we have

$$8b(N+2) \equiv -\sigma_7(2(N+2)) + \sigma_7(N+2) \pmod{17}. \quad (4.2)$$

Since $(15, 17) = 1$, (4.2) can be written as

$$\begin{aligned} 15 \cdot 8b(N+2) &\equiv -15\sigma_7(2(N+2)) + 15\sigma_7(N+2) \\ &\equiv 2\sigma_7(2(N+2)) - 2\sigma_7(N+2) \pmod{17} \end{aligned}$$

and so

$$b(N+2) \equiv 2\sigma_7(2(N+2)) - 2\sigma_7(N+2) \pmod{17} \quad \text{for } N \in \mathbb{N}.$$

Thus we conclude that

$$b(n) \equiv 2\sigma_7(2n) - 2\sigma_7(n) \pmod{17} \quad \text{for } n \in \mathbb{N}. \quad (4.3)$$

In particular, if n is odd, then (4.3) becomes

$$b(n) \equiv 2(1 + 2^7)\sigma_7(n) - 2\sigma_7(n) \equiv \sigma_7(n) \pmod{17},$$

and this special case is also seen in ([4], Lemma 4.6).

□

5 CONCLUSIONS

In this paper, we evaluate the convolution sum $\sum_{m < n/8} \sigma_3(m) \sigma_3(n - 8m)$ for all $n \in \mathbb{N}$, which is available us to find the formulae of the convolution sums

$$\sum_{m < \frac{n}{8}} \sigma_{3,0}(m; 2) \sigma_{3,0}(n - 8m; 2), \quad \sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_{3,1}(n - 8m; 2),$$

etc., and the number of representations of n through the sum of eight triangular numbers.

References

- [1] Berndt, B. C., (1989). Ramanujan's Notebooks. Part II. Springer-Verlag, New York.
- [2] Berndt, B. C., (1991). Ramanujan's Notebooks. Part III. Springer-Verlag, New York.
- [3] Cheng, N. and Williams, K. S., (2005). Evaluation of some convolution sums involving the sum of divisors functions. *Yokohama Mathematical J.*, 52, 39–57.
- [4] Kim, A., Kim, D., and Ikkardes, N. Y., (2013). Remarks of congruent arithmetic sums of theta functions derived from divisor functions. *Honam Mathematical J.*, No. 3, 35, 351–372.
- [5] Williams, K. S., (2006). The convolution sum $\sum_{m < n/8} \sigma(m) \sigma(n - 8m)$. *Pacific Journal of Mathematics*, Vol. 228, 2, 387–396.
- [6] Williams, K. S., (2011). *Number Theory in the Spirit of Liouville*. London Mathematical Society, Student Texts 76, Cambridge.

©2013 Kim; This is an Open Access article distributed under the terms of the Creative Commons Attribution License <http://creativecommons.org/licenses/by/3.0>, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.