

A General Inequality Related to Variational Inequalities and its Consequences

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ABSTRACT

ABSTRACT. Hausdorff topological vector space with dual X^* and K a nonempty subset of X . Let $g: K \rightarrow \mathbb{R}$ be any map. In this paper we study the following problem: "Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ ". Some results on this problem have been studied by Takahasi, Park and Kim and Yen. Behera and Panda proved several results on this problem in the setting of a Banach space. This problem includes as special cases, many problems on variational inequalities and generalized variational inequalities studied by many authors. As a consequence of the main results, we also consider the problem, "Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ " where $S: K \rightarrow 2^K$ is any point-to-set map which, includes as a special case, the classical quasi-variational inequality problem. A generalization of Minty's Lemma is also studied.

Keywords: Partition of unity, upper and lower **semicontinuous** point-to setmap, monotone operator

1. INTRODUCTION

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty subset of X . Let the value of $u \in X^*$ at $x \in X$ be denoted by (u, x) . Let $g: K \rightarrow \mathbb{R}$ be a map (possibly nonlinear). The classical minimization problem for the pair (g, K) is to find $x_0 \in K$ such that

$$g(x_0) = \min_{y \in K} g(y).$$

If we define a function $f: K \times K \rightarrow \mathbb{R}$ as $f(x, y) = g(y) - g(x)$ for all $x, y \in K$, then the above problem reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

If $T: K \rightarrow X^*$, then the nonlinear variational inequality problem is to find $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$. If we define $f: K \times K \rightarrow \mathbb{R}$ as $f(x, y) = (Tx, y - x)$, then it reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Motivated by the above facts, Behera and Panda [5] considered the following problem:

P_1 : Given $f: K \times K \rightarrow \mathbb{R}$, find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Certain variation of this problem is known as equilibrium problem. The readers are advised to refer [8], [9] and [17] for a detailed discussion on equilibrium problem and its generalization.

Special cases

1. If $f(x, y) = (Tx, \theta(y, x))$ for some maps $T: K \rightarrow X^*$, and $\theta: K \times K \rightarrow X$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, \theta(y, x_0)) \geq 0$ for all $y \in K$, which is the *variational-like inequality problem* studied by Behera and Panda [2, 4], Dien [11] and Siddiqi et. al. [21–22]. Yang and Chen [26] also studied this problem calling it a *pre-variational inequality problem* because of the relation of this problem with the *pre-invex functions* (also see [6]).

2. If $z \in K$ is fixed and $f(x, y) = (T((z + x)/2), y - x)$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(T((z + x_0)/2), y - x_0) \geq 0$ for all $y \in K$, which is the *variational-type inequality problem* originally introduced by Behera and Panda [3].

3. If $f(x, y) = (Tx - Ax, y - x)$ for some maps $T, A: K \rightarrow X^*$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, y - x_0) \geq (Ax_0, y - x_0)$ for all $y \in K$ which is the *strongly nonlinear variational inequality problem* studied by Nanda [14] and Noor [15].

4. If $f(x, y) = (Tx - Ax, g(y) - g(x))$ for some maps $T, A: K \rightarrow X^*$, $g: K \rightarrow K$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, g(y) - g(x_0)) \geq (Ax_0, g(y) - g(x_0))$ for all $y \in K$, which is the *strongly nonlinear implicit variational inequality problem* introduced and studied by Noor [15] in connection with the solution of the differential equations of odd order.

Some results on Problem P_1 are available in Takahasi [24] and Park and Kim [16]. Behera and Panda [5] proved several results on the existence of solution of this problem in the setting of Banach spaces. Many authors used results on Problem P_1 to prove the existence of solution of variational and generalized variational inequality problems.

In this paper, we prove some results on the existence of solution of Problem P_1 under different assumptions in the setting of Hausdorff topological vector spaces. We also use a result on the existence of solution of Problem P_1 for the study the following problem, which is a generalization of the quasi-variational inequality problem introduced by Benoussan and Lions [7] in connection with impulse control and subsequently studied by Baicchi and Capello [1] and Mosco [13].

P_2 : Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ where $S: K \rightarrow 2^K$ is any point-to-set map.

We also prove a theorem on the uniqueness of solution of Problem P_1 which would serve as a generalization of the well-known Minty's Lemma (see [10], p.6).

2. PRILIMINARIES

In this section we recall some definitions and known results which will be needed in the sequel. Throughout this section, X is a real Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X .

2.1 Definition. A mapping $T: K \rightarrow X^*$ is said to be monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in K$. T is said to be strictly monotone if strict inequality holds whenever $x \neq y$. T is said to be pseudo-monotone if $x, y \in K$ and $(Tx, x - y) \geq 0$ then $(Ty, x - y) \geq 0$.

2.2 Definition. A function $h: K \rightarrow \mathbb{R}$ is said to be upper semi-continuous, if for each real number λ , the set $\{x \in K: h(x) < \lambda\}$ is open; f is said to be lower semi-continuous if $-f$ is upper semi-continuous.

2.3 Definition. A function $h: K \rightarrow \mathbb{R}$ is said to be quasi-convex if for each real number λ , the set $\{x \in K: h(x) < \lambda\}$ is convex; function f is said to be quasi-concave if $-f$ is quasi-convex.

2.4 Definition. Let M and N be two topological spaces, and $h: M \rightarrow 2^N$, a point-to-set map. Then h is said to be upper semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \subset U$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \subset U$; h is said to be lower semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \cap U \neq \emptyset$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \cap U \neq \emptyset$; h is said to be upper semi-continuous (lower semi-continuous) on M if h is upper semi-continuous (lower semi-continuous) at each point of M .

2.5 Theorem. (Isac [12]) Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let f and g be two real valued functions on $K \times K$ having the following properties :

- (a) for all $x, y \in K$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq 0$,
- (b) for each fixed $y \in K$, $f(x, y)$ is a lower semicontinuous function of x on K ,
- (c) for each fixed $x \in K$, $g(x, y)$ is a quasi-concave function of x on K .

Then there exists a point $\hat{x} \in K$ such that $f(\hat{x}, y) \leq 0$ for all $y \in K$.

2.6 Theorem. (Shih and Tan [23]) Let X be a Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X . Let $S: K \rightarrow 2^K$ be an upper semicontinuous point-to-set map such that for each $x \in K$, $S(x)$ is nonempty and bounded. Then for each $p \in X^*$, the map $f_p: K \rightarrow \mathbb{R}$ defined by $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semicontinuous.

2.7 Theorem. (Takahasi [24]) Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X and let $F: K \times K \rightarrow \mathbb{R}$ be a function such that

- (a) for each $y \in K$, the map $x \mapsto F(x, y)$ is upper semicontinuous,
- (b) for each $x \in K$, the map $y \mapsto F(x, y)$ is convex,
- (c) $F(x, x) \geq \alpha$ for all $x \in K$ with some real number α .

Then there exists $x_0 \in K$ such that $F(x_0, y) \geq \alpha$ for all $y \in K$.

2.8 Theorem. (Rudin [19]) Let K be a compact subset of a topological space and let $\{V_1, V_2, \dots, V_n\}$ be a finite open covering of K . Then there exists a family $\{\beta_1, \beta_2, \dots, \beta_n\}$ of continuous real valued function on K such that $\beta_i(x) = 0$ outside V_i , $0 \leq \beta_i(x) \leq 1$ for all $i \in \{1, 2, \dots, n\}$ and for all $x \in K$, and $\sum_{i=1}^n \beta_i(x) = 1$ for all $x \in K$.

2.9 Theorem. (Tarafdar [25]) Let K be a nonempty subset of a Hausdorff topological vector space X and let $F: K \rightarrow 2^K$ be a point-to-set map such that

- (a) for each $x \in K$, $F(x)$ is nonempty,
- (b) for each $y \in K$, $F^{-1}(y) = \{x \in K: y \in F(x)\}$ contains a relatively open subset A_y of K (A_y may be empty for some y),
- (c) $\bigcup_{x \in K} A_x = K$,

132 (d) K contains a nonempty set K_0 contained in a compact convex subset K_1 of K
 133 such that $D = \bigcap_{x \in K_0} A_x^C$ is compact (D may be empty).

134 Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.

135
 136 **2.10 Theorem.** ([23], p. 34). Let X be a Hausdorff topological vector space with dual X^* and
 137 K a nonempty convex subset of X . Let $S: K \rightarrow 2^K$ be a point-to-set upper semi-continuous
 138 map such that for each $x \in K$, $S(x)$ is nonempty and bounded. Then for each $p \in X^*$ the map
 139 $f_p: K \rightarrow \mathbb{R}$ defined by $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semi-continuous.

140
 141 **2.11 Theorem.** ([20], p.201). Let X be a locally convex Hausdorff topological vector space
 142 with dual X^* and K a nonempty closed and convex subset of X . Let x^* be a point of X not in K .
 143 Then there exists $p \in X^*$ such that $(p, x^*) > \sup_{x \in K} (p, x)$.

144
 145

146 3. EXISTENCE OF SOLUTION

147

148 The following theorems on the existence of solution of problem P_1 are the main results of
 149 this paper.

150

151 **3.1 Theorem.** Let K be a nonempty compact convex subset of a Hausdorff topological vector
 152 space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 153 (a) $f(x, x) \geq 0$ for each $x \in K$,
 154 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
 155 (c) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K .

156 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

157

158 *Proof.* Same as the proof of Theorem 2.7. Indeed Takahasi [24] assumed that for each $y \in K$,
 159 the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is upper semi-continuous and for each $x \in K$, the map $y \mapsto$
 160 $f(x, y)$ of K into $\mathbb{R}$ is convex which are stronger than hypotheses (b) and (c) of this theorem.

161

162 The following theorem generalizes Theorem 3.1 to an arbitrary nonempty convex subset K of
 163 X .

164

165 **3.2 Theorem.** Let K be a nonempty convex subset of a Hausdorff topological vector space X
 166 and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 167 (a) $f(x, x) \geq 0$ for each $x \in K$,
 168 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
 169 (c) for each $y \in K$, the set $\{x \in C : f(x, y) < 0\}$ is open in K for every compact
 170 subset C of K ,
 171 (d) there exists a nonempty compact subset L of K such that for each $x \in K -$
 172 L , there exists $u \in L$ for which $f(x, u) < 0$.

173 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

174

175 *Proof.* For each $y \in K$, let

176
$$A(y) = \{x \in L : f(x, y) \geq 0\}.$$

177 From hypothesis (c) it follows that $A(y)$ is a nonempty compact subset of K . The conclusion
 178 of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. To ascertain this, it is sufficient to show that the
 179 family $\{A(y) : y \in K\}$ has the finite intersection property.

180 Let $y_1, y_2, \dots, y_n$ be arbitrary elements of K and K_0 , the convex hull of $L \cup \{y_1, y_2, \dots, y_n\}$. Then
 181 K_0 is a nonempty compact convex subset of K . By Theorem 3.1, there exists $\tilde{x}_0 \in K_0$ such
 182 that

$$183 \quad f(\tilde{x}_0, y) \geq 0 \quad \dots(1)$$

184 for all $y \in K_0$. We claim that $\tilde{x}_0 \in L$; for if $\tilde{x}_0 \notin L$, then $\tilde{x}_0 \in K_0 - L \subset K - L$ and by hypothesis
 185 (d), there exists $u \in L$ such that $f(\tilde{x}_0, u) < 0$ which contradicts (1) when $y = u$. Thus $\tilde{x}_0 \in L$,
 186 and in particular, $\tilde{x}_0 \in A(y_i)$ for $i = 1, 2, \dots, n$; that is $\tilde{x}_0 \in \bigcap_{i=1}^n A(y_i)$. Hence, $\bigcap_{i=1}^n A(y_i) \neq \emptyset$
 187 which proves that the family $\{A(y) : y \in K\}$ has the finite intersection property. This
 188 completes the proof.

189

190

191 If X is a normed linear space and K is locally compact, we have the following result.

192

193 **3.3 Theorem.** Let K be a nonempty locally compact convex subset of a real normed linear
 194 space X with $0 \in K$, and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 195 (a) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
 196 (b) for all $y \in K$ and $r > 0$, the set $\{x \in K_r : f(x, y) < 0\}$ is open in K , where
 197 $K_r = \{x \in K : \|x\| \leq r\}$,
 198 (c) $f(x, x) = 0$ for each $x \in K$ and for each $r > 0$,
 199 (d) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

200 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

201 *Proof.* For each $r > 0$, the set K_r is a nonempty compact and convex subset of K . Hence by
 202 Theorem 3.1 there exists $x_r \in K_r$ such that

$$203 \quad f(x_r, y) \geq 0 \quad \dots(2)$$

204 for all $y \in K_r$. We claim that $\|x_r\| < r$; for if $\|x_r\| = r$, then by hypothesis (d), $f(x_r, 0) < 0$
 205 which contradicts (2) when $y = 0$. Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently
 206 small so that

$$207 \quad y_r = \lambda x + (1-\lambda)x_r \in K_r.$$

208 Now by hypotheses (a) and (b)

$$209 \quad \begin{aligned} 0 \leq f(x_r, y_r) &= f(x_r, \lambda x + (1-\lambda)x_r) \\ &\leq \lambda f(x_r, x) + (1-\lambda)f(x_r, x_r) = \lambda f(x_r, x). \end{aligned}$$

210 Since $\lambda > 0$, it follows that $f(x_r, x) \geq 0$. Since x is arbitrary, the proof is complete.

211

212 If X is a reflexive Banach space then local compactness of K can be relaxed from the above
 213 theorem.

214

215 **3.4 Theorem.** Let K be a nonempty closed and convex subset of a reflexive real Banach
 216 space X with $0 \in K$ and $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 217 (a) $f(x, x) = 0$ for each $x \in K$,
 218 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 219 (c) the set $\{x \in C : f(x, y) < 0\}$ is weakly open in K for each $y \in K$, and for each
 220 weakly compact subset C of K .

221 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ under each of the following
 222 conditions:

- 223 (i) K is bounded,
 224 (ii) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

225

226 *Proof.* Equip K with the weak topology. If K is bounded, then it becomes weakly compact and
 227 in this case, the proof follows from Theorem 3.1.

228

229 If K is not bounded let

$$K_r = \{x \in K : \|x\| \leq r\}.$$

230 Then K_r is a nonempty closed, convex and bounded subset of K . By the first part of the
 231 theorem, there exists $x_r \in K_r$ such that $f(x_r, y) \geq 0$ for all $y \in K_r$. The remaining part of the
 232 proof follows from the proof of Theorem 3.3 and hence it is omitted.

233

234 The following theorem generalizes Theorem 3.4 to a Hausdorff topological vector space.

235

236 **3.5 Theorem.** Let K be a nonempty convex subset of a Hausdorff topological vector space
 237 X and L , a nonempty compact and convex subset of K . Let $f: K \times K \rightarrow \mathbb{R}$ be a map such
 238 that

- 239 (a) $f(x, x) = 0$ for each $x \in L$,
- 240 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
- 241 (c) for each $y \in K$, the set $\{x \in L : f(x, y) < 0\}$ is open in L ,
- 242 (d) for every x in the boundary of L , there exists $u \in L$ such that $f(x, u) < 0$.

243 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

244 *Proof.* By theorem 3.1 there exists $x_0 \in L$ such that

$$245 \quad f(x_0, y) \geq 0 \quad \dots (3)$$

246 for all $y \in L$. We claim that $x_0 \in \text{int } L$ (where $\text{int } L$ stands for the interior of L), for if $x_0 \notin$
 247 $\text{int } L$, then by hypothesis (d) there exists $u \in L$ such that $f(x_0, u) < 0$ which contradicts (3)
 248 when $y = u$.

249 Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently small so that

$$y_\lambda = \lambda x + (1-\lambda)x_0 \in L.$$

250 By hypotheses (a) and (b)

$$0 \leq f(x_0, y_\lambda) \leq \lambda f(x_0, x) + (1-\lambda)f(x_0, x_0) = \lambda f(x_0, x).$$

251 Since $\lambda > 0$, it follows that $f(x_0, x) \geq 0$. This completes the proof.

252

253 The following theorem is a variant of Theorem 3.1 in which, the hypotheses (a) and (b) of
 254 Theorem 3.1 have been replaced by a single generalized condition.

255

256 **3.6 Theorem.** Let K be a nonempty compact convex subset of a Hausdorff topological
 257 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 258 (a) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K ,
- 259 (b) for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K and
 260 $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$, $\max_{1 \leq i \leq n} f(u, y_i) \geq 0$.

261 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

262

263 *Proof.* Assume that the assertion is false. Then for each $x \in K$, there exists $y \in K$ such that
 264 $f(x, y) < 0$. For each $y \in K$, let

$$265 \quad A(y) = \{x \in K : f(x, y) < 0\}.$$

266 Then by hypothesis (a), $A(y)$ is open for every $y \in K$. Further $K = \bigcup_{y \in K} A(y)$. Since K is
 267 compact, there exists $\{y_1, y_2, \dots, y_n\} \subset K$ such that $K = \bigcup_{i=1}^n A(y_i)$. Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be a
 268 partition of unity subordinate to the covering $\{A(y_i) : i = 1, 2, \dots, n\}$ and let

269 $S = \text{conv}(\{y_1, y_2, \dots, y_n\})$. Then S is a compact convex subset of K . Define a continuous map
 270 $p : K \rightarrow K$ as $p(x) = \sum_{i=1}^n \beta_i(x) y_i$. Since $p(x)$ is a convex linear combination of points of the
 271 set S , $p(x) \in S$ for each $x \in K$. In particular, p maps S into S . By Browuer's fixed point
 272 theorem p has a fixed point; that is, there exists $\hat{x} \in K$ such that $\hat{x} = p(\hat{x})$.

273

274 Since $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a partition of unity, $\beta_i(\hat{x}) > 0$ for at least one i . But $\beta_i(\hat{x}) > 0$ implies
 275 that $\hat{x} \in A(y_i)$ so that

$$276 \quad f(\hat{x}, y_i) < 0. \quad \dots(4)$$

277 If

$$J = \{i: 1 \leq i \leq n \text{ and } \beta_i(\hat{x}) > 0\}$$

278 then

$$\hat{x} = p(\hat{x}) = \sum_{i=1}^n \beta_i(\hat{x}) y_i = \sum_{i \in J} \beta_i(\hat{x}) y_i$$

279 so that $\hat{x} \in \text{conv}\{y_i: i \in J\}$. Now by hypothesis (b), there exists $i_0 \in J$ such that $f(\hat{x}, y_{i_0}) \geq 0$
 280 which contradicts (4). This contradiction proves the theorem.

281

282 The following theorem generalizes Theorem 3.6 to an arbitrary nonempty convex subset K of
 283 X .

284

285 **3.7 Theorem.** Let X be a locally convex Hausdorff topological vector space, K a nonempty
 286 convex subset of X , K_0 a nonempty compact convex subset of K , and L a nonempty
 287 compact subset of K . Let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 288 (a) for each $y \in K$, the set $\{x \in L: f(x, y) \geq 0\}$ is closed in K ,
 289 (b) for any finitesubset $\{y_1, y_2, \dots, y_n\}$ of K , and
 290 $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$ $\max_{1 \leq i \leq n} f(u, y_i) \geq 0$.
 291 (c) for each $x \in K - L$, there exists $u \in \text{conv}(K_0 \cup \{x\})$ such that $f(x, u) < 0$.

292 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

293

294 *Proof.* For each $y \in K$, let

$$295 \quad A(y) = \{x \in L: f(x, y) \geq 0\}.$$

296 From (b) it follows that $A(y)$ is closed and consequently compact for each $y \in K$.
 297 The assertion of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. For this, it is sufficient to prove that the
 298 family $\{A(y): y \in K\}$ has the finite intersection property.

299

300 Let $\{y_1, y_2, \dots, y_n\} \subset K$ and $S = \text{conv}(K_0 \cup \{y_1, y_2, \dots, y_n\})$. Clearly S is a non-empty compact
 301 convex subset of K . By Theorem 3.6 there exists $\hat{x} \in S$ such that

$$302 \quad f(\hat{x}, y) \geq 0 \quad \dots(5)$$

303 for all $y \in S$. We claim that $\hat{x} \in L$; for if $\hat{x} \notin L$, that is, $\hat{x} \in S - L \subset K - L$, then there exists
 304 $u \in \text{conv}(K_0 \cup \{\hat{x}\})$ such that $f(\hat{x}, y) < 0$ which contradicts (5) when $y = u$.
 305 Thus $\hat{x} \in L$, and in particular, $\hat{x} \in A(y_i)$ for each $i \in \{1, 2, \dots, n\}$, that is, $\hat{x}_0 \in \bigcap_{i=1}^n A(y_i)$.
 306 Hence, $\bigcap_{i=1}^n A(y_i) \neq \emptyset$ showing that the family $\{A(y): y \in K\}$ has the finite intersection
 307 property. This completes the proof.

308

309 The following theorem also deals with the existence of solution of problem P_1 under pseudo-
 310 monotonicity type assumptions.

311

312 **3.8 Theorem.** Let K be a nonempty closed and convex subset of a Hausdorff topological
 313 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 314 (a) $f(x, x) \geq 0$ for each $x \in K$,

- 315 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex and lower semicontinuous for each
 316 $x \in K$,
 317 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line
 318 segments of K to the usual topology of $\mathbb{R}$,
 319 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$,
 320 (e) there exists a nonempty set K_0 contained in a compact convex subset K_1 of
 321 K such that $D = \bigcap_{y \in K_0} \{y \in K : f(x, y) \leq 0\}$ is either compact or empty.

322 Then there exists $x_0 \in K$ such that

$$323 \quad f(x_0, y) \geq 0 \quad \dots(6)$$

324 for all $y \in K$.

325
 326 Note: If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then condition (d) of the
 327 above theorem is equivalent to the fact that T is pseudo-monotone.

328 To prove the above theorem, we need the following lemma.

329

330 **3.9 Lemma.** Let K be a nonempty closed and convex subset of a Hausdorff topological
 331 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 332 (a) $f(x, x) \geq 0$ for each $x \in K$,
 333 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 334 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line segments
 335 of K to the usual topology of $\mathbb{R}$,
 336 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$.

337 Then the following are equivalent:

- 338 (A) $x_0 \in K$ and $f(x_0, y) \geq 0$ for all $y \in K$.
 339 (B) $x_0 \in K$ and $f(y, x_0) \leq 0$ for all $y \in K$.

340

341 **Note:** If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$, then Lemma 3.9 reduces
 342 to Minty's Lemma ([10], p. 6).

343

344 *Proof.* Assume that (A) holds. Then for some $x_0 \in K$ and for all $y \in K$, $f(x_0, y) \geq 0$. By
 345 hypothesis (d), $f(y, x_0) \leq 0$ for all $y \in K$ from which (B) follows.

346

347 Now assume that (B) holds. Then for some $x_0 \in K$, $f(y, x_0) \leq 0$ for all $y \in K$. Fix $y \in K$,
 348 $t \in (0, 1)$ and set $y_t = (1 - t)x_0 + ty$. Then $y_t \in K$ and by (B), $f(y_t, x_0) \leq 0$ for each $t \in (0, 1)$.
 349 Now it follows from hypotheses (a) and (b) that

$$0 \leq f(y_t, y_t) \leq (1 - t)f(y_t, x_0) + tf(y_t, y).$$

350 Thus

$$f(y_t, y) \geq -\frac{1-t}{t}f(y_t, x_0) \geq 0$$

351 for each $t \in (0, 1)$. Since the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous
 352 from the line segments of K to the usual topology of $\mathbb{R}$, taking limit $t \rightarrow 0^+$ in the above
 353 inequality, we get $f(x_0, y) \geq 0$. Since $y \in K$ is arbitrary (B) follows. This completes the proof
 354 of the lemma.

355

356 **3.10 Proof of Theorem 3.8.** Assume that inequality (6) has no solution. Then, for each
 357 $x \in K$, there exists $u \in K$ such that

$$358 \quad f(x, u) < 0 \quad \dots(7)$$

359 and by Lemma 3.9, there exists $v \in K$ such that

360 $f(v, x) > 0$ (8)
 361 Thus for each $x \in K$, it follows from (7) that the set

$$F(x) = \{y \in K: f(x, y) < 0\}$$

 362 is nonempty, which is also convex by hypothesis (b) of the theorem. Again for each $y \in K$,
 363 since

$$F^{-1}(y) = \{x \in K: f(x, y) < 0\},$$

 364 it follows that

$$\begin{aligned} [F^{-1}(y)]^c &= \{x \in K: f(x, y) \geq 0\} \\ &\subset \{x \in K: f(y, x) \leq 0\} \quad (\text{by (d)}) \\ &= G(y) \quad (\text{say}). \end{aligned}$$

 365 Thus

$$F^{-1}(y) \supset [G(y)]^c = A(y) \quad (\text{say}).$$

 366 Since for each $x \in K$, the map $y \mapsto f(x, y)$ is lower semicontinuous, it follows that $G(y)$ is
 367 closed in K . Thus for each $y \in K$, $A(y)$ is open in K .
 368
 369 Now, let $x \in K$ be arbitrary. By (8) there exists $v \in K$ such that $f(v, x) > 0$, so that $x \in$
 370 $A(v)$. Thus $\bigcup_{y \in K} A(y) = K$. Finally, $D = \bigcap_{y \in K_0} G(y) = \bigcap_{y \in K_0} [A(y)]^c$ is either empty or
 371 compact by hypothesis (e), so that all the conditions of Theorem 2.9 are fulfilled by F . Thus
 372 there exists $x_0 \in K$ such that $x_0 \in F(x_0)$, which is tantamount to $f(x_0, x_0) < 0$, a contradiction
 373 to hypothesis (a) of the theorem. Hence the conclusion of the theorem follows.
 374
 375
 376
 377
 378

379 4. UNIQUENESS OF SOLUTIONS

380
 381 The following theorem characterizes the uniqueness of solution of the Problem P_1 under
 382 certain conditions.

383
 384 **4.1 Theorem.** Let K be a nonempty subset of a topological vector space X and
 385 let $f: K \times K \rightarrow \mathbb{R}$ a map such that

- 386 (a) for all $x, y \in K$, $f(x, y) + f(y, x) \leq 0$,
 387 (b) equality does not hold in (a) unless $x = y$.

388 Then, if problem P_1 is solvable, it has a unique solution.

389
 390 *Proof.* If possible let x_1, x_2 be two solutions of P_1 . Then

$$f(x_1, y) \geq 0$$

391 and

$$f(x_2, y) \geq 0$$

392 for all $y \in K$. Putting $y = x_1$ in the former inequality and $y = x_2$ in the latter, we get

$$f(x_1, x_2) \geq 0$$

393 and

$$f(x_2, x_1) \geq 0.$$

394 Adding the last two inequalities we get,

$$f(x_1, x_2) + f(x_2, x_1) \geq 0.$$

395 This inequality combined with hypothesis (a) gives

$$f(x_1, x_2) + f(x_2, x_1) = 0.$$

396 Now an application of hypothesis (b) yields $x_1 = x_2$. This completes the proof.

397 *Note:* If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then condition (a) of the
 398 above theorem is equivalent to the fact that T is monotone.

399

400 The following examples illustrate Theorem 4.1. Example 1 shows that the fulfillment of
 401 conditions (a) and (b) does not guarantee the existence of a solution to problem P_1 .

Examples (2) and (3) show that if conditions (a) and (b) are fulfilled and problem P_1 is solvable, then it has a unique solution.

Example 1. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define let $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = -|x - y|.$$

Then

$$f(x, y) + f(y, x) = -2|x - y|$$

and

$$f(x, y) + f(y, x) = 0$$

if $|x - y| = 0$, that is $x = y$. Thus hypotheses (a) and (b) of Theorem 4.1 are fulfilled by f . But $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $|x_0 - y| \leq 0$ for all $y \geq 0$. Thus in this case no solution exists.

Example 2. Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = x(y - x).$$

Then

$$f(x, y) + f(y, x) = -(x - y)^2 \leq 0$$

and

$$f(x, y) + f(y, x) = 0$$

if and only if $(x - y)^2 = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $x_0(y - x_0) \geq 0$ for all $y \geq 0$, so that $x_0 = 0$ is the only solution in this case.

Example 3. Let $X = K = \mathbb{R}$, and define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = -x^2|x - y|.$$

Then

$$f(x, y) + f(y, x) = -(x^2 + y^2)|x - y| \leq 0$$

and

$$f(x, y) + f(y, x) = 0$$

if and only if $|x - y| = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $x_0^2|x_0 - y| = 0$ for all $y \in \mathbb{R}$ so that $x_0 = 0$ is the only solution in this case also.

425

426

5. SOME CONSEQUENCES

428

In this section, we discuss some consequences of the theorems proved in Section 3.

430

The following theorem deals with the existence of solution of problem P_2 .

432

5.1 Theorem. Let X be a locally convex Hausdorff topological vector space with dual X^* , K a nonempty compact convex subset of X , $f: K \times K \rightarrow \mathbb{R}$ a point-to-point map, and $S: K \rightarrow 2^K$, a point-to-set upper semicontinuous map such that

- (a) $f(x, x) \geq 0$ for each $x \in K$,
- (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi convex for each $x \in K$,
- (c) the set $\{x \in K: \inf_{y \in S(x)} f(x, y) < 0\}$ is empty or open in K .

Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $f(x_0, y) \geq 0$ for all $y \in S(x_0)$.

440

Proof. Assume that the assertion is false. Then for each $x \in K$ either $x \notin S(x)$, or there exists $y \in S(x)$ such that $f(x, y) < 0$. If $x \notin S(x)$, then by Theorem 2.11, there exists $p \in X^*$ such that

443

$$(p, x) - \sup_{y \in S(x)} (p, y) > 0.$$

444 Let

$$V_0 = \{x \in K: \inf_{y \in S(x)} f(x, y) < 0\},$$

445 and for each $p \in X^*$ let

$$V(p) = \{x \in K: (p, x) - \sup_{y \in S(x)} (p, y) > 0\}.$$

446 Then by hypothesis (c), V_0 is open in K . By Theorem 2.6, since the map $x \mapsto \sup_{y \in S(x)} (p, y)$
 447 is upper semicontinuous for each $p \in X^*$, it follows that the map $x \mapsto (p, x) - \sup_{y \in S(x)} (p, y)$ is
 448 lower semicontinuous. Hence $V(p)$ is open in K for each $p \in X^*$. Further, $K = V_0 \cup$
 449 $\bigcup_{p \in X^*} V(p)$. Since K is compact, there exists $\{p_1, p_2, \dots, p_n\} \subset X^*$ such that $K = V_0 \cup$
 450 $[\bigcup_{i=1}^n V(p_i)]$. Let $\{\beta_0, \beta_1, \dots, \beta_n\}$ be a partition of unity subordinate to the covering
 451 $\{V_0, V(p_1), \dots, V(p_n)\}$, and define $g: K \times K \rightarrow \mathbb{R}$ as

$$g(x, y) = \beta_0(x)f(x, y) + \sum_{i=1}^n \beta_i(x)(p_i, y - x).$$

452 Then it is clear that for each $x \in K$, $g(x, x) \geq 0$, the map $y \mapsto g(x, y)$ is quasi-convex, for
 453 each $y \in K$, the map $x \mapsto g(x, y)$ is upper semicontinuous so that the set $\{x \in K: g(x, y) <$
 454 $0\}$ is open in K . Thus all the conditions of Theorem 3.1 are fulfilled. Hence there exists $\hat{x} \in K$
 455 such that

$$456 \quad g(\hat{x}, y) \geq 0 \quad \dots(9)$$

457 for all $y \in K$, which is equivalent to

$$\beta_0(\hat{x})f(\hat{x}, y) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, y - \hat{x}) \geq 0.$$

458 Since $\{\beta_0, \beta_1, \dots, \beta_n\}$ is a partition of unity $\beta_i(\hat{x}) > 0$ for at least one i . If $\beta_0(\hat{x}) > 0$ then $\hat{x} \in V_0$
 459 so that $\inf_{y \in S(\hat{x})} f(\hat{x}, y) < 0$. Let $\hat{y} \in S(\hat{x})$ be such that $f(\hat{x}, \hat{y}) < 0$. If $\beta_i(\hat{x}) > 0$ then $\hat{x} \in V(p_i)$
 460 so that

$$(p_i, \hat{x}) - \sup_{y \in S(\hat{x})} (p_i, y) > 0$$

461 from which it follows that

$$(p_i, \hat{x}) > \sup_{y \in S(\hat{x})} (p_i, y) > (p_i, \hat{y})$$

462 which implies that $(p_i, \hat{y} - \hat{x}) < 0$. Thus,

$$g(\hat{x}, \hat{y}) = \beta_0(\hat{x})f(\hat{x}, \hat{y}) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, \hat{y} - \hat{x}) < 0$$

463 which contradicts (9) when $y = \hat{y}$. This contradiction proves the theorem.

464

465 The following result on the existence of solution of the quasi-variational inequality problem is
 466 a direct consequence of the above theorem.

467

468 **5.2 Corollary.** Let X be a locally convex Hausdorff topological vector space with dual X^* , K a
 469 nonempty compact convex subset of X , $T: K \rightarrow X^*$ an operator and $S: K \rightarrow 2^K$ a point-to-set
 470 upper semi-continuous map such that the set $\{x \in K: \inf_{y \in S(x)} (Tx, y - x) < 0\}$ is empty or
 471 open in K . Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $(Tx_0, y - x_0) \geq 0$ for all $y \in S(x_0)$.

472

473 *Proof.* Follows directly from Theorem 5.1 with $f(x, y) = (Tx, y - x)$.

474

475

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480

481 REFERENCES

482

- 483 1. Baiocchi C, Capelo A. Variational and quasi variational inequalities, John Wiley and
484 Sons: New York; 1984.
- 485 2. Behera A, Panda GK. A generalization of Browder's theorem. Bull Inst
486 Math Acad Sinica. 1993; 21(2): 183–186.
- 487 3. Behera A, Panda GK. A variational-type Inequality and Minty's like lemma. Opsearch.
488 1996; 33(4): 213–220.
- 489 4. Behera A, Panda GK. Variational inequality problem in Hausdorff topological vector
490 space. Indian J Pure Appl Math, 1997; 28(2): 181–187.
- 491 5. Behera A, Panda GK. An inequality associated with variational inequalities in Banach
492 spaces and its consequences. Bull Inst Math. Acad Sinica. 1999; 27(3): 227–245.
- 493 6. Ben-Israel A, Mond B. What is invexity?. J Austral Math Soc: Ser B. 1986; 28:
494 1–9.
- 495 7. Bensoussan A, Lions JL. Nouvelle formulation des problemes de controle impulsionnel et
496 applications. C R Acad Sci: Ser A. 1973; 29: 1189–1192. French
- 497 8. Blum E, Oettli W. From optimization and variational inequalities to equilibrium problems,
498 Math Student. 1994; 63 : 123–145.
- 499 9. Chadli O, Chbani Z, Riahi H. Equilibrium problems with generalized monotone
500 bifunctions and applications to Variational Inequalities. JOTA. 2000; 105(2): 299–323.
- 501 10. Chipot M. Variational Inequalities and Flow in Porous Media. Appl Math Sci. 52,
502 Springer-Verlag: New York/Berlin; 1984.
- 503 11. Dien N.H. Some remarks on variational-like and quasi-variational-like inequalities. Bull
504 Austral Math Soc. 1992; 46: 335–342.
- 505 12. Isac G. Complementarity problems: Lecture notes in Math. 1528, Springer-Verlag: New
506 York; 1992.
- 507 13. Mosco U. Implicit variational problems and quasi-variational inequalities. Nonlinear
508 Operators and the Calculus of Variations. Lecture notes in Mathematics, No. 453.
509 Springer-Verlag: New York/Berlin. 1976; 83–156.
- 510 14. Nanda S. Strongly nonlinear variational inequality problem and its alternative form and
511 complementarity problem. Indian J Pure Appl Math. 1991; 22(5): 355 – 359.
- 512 15. Noor M A. Strongly nonlinear variational inequalities, C R Math Rep Acad Sci Canada
513 4: 1982; 213–218.
- 514 16. Noor M A. An iterative algorithm for variational inequalities. J Math Anal Appl. 1991;
515 158(2): 448–455.
- 516 17. Noor MA, Oettli W. On general nonlinear complementarity problems and quasi-equilibria,
517 Le Matematiche (Catania), 49(1994): 313–331.
- 518 18. Park S, Kim I. Nonlinear variational inequalities and fixed-point theorems. Bull Koer Math
519 Soc. 1989; 26(2): 139–149.
- 520 19. Rudin W. Real and Complex Analysis, 2nd ed., Tata McGraw-Hill: New Delhi; 1981.
- 521 20. Royden HL. Real Analysis, Macmillan Publishing Company: New York; 1988.
- 522 21. Siddiqi AH, Ansari QH, Khan MF, Variational-like inequalities for multivalued maps,
523 Indian J Pure Appl Math. 1999; 30(2): 161–166.

- 524 22. Siddiqi AH, KhaliqA, Ansari QH. On variational-like inequalities, Ann Sci Math Quebec.
525 1994; 18(1): 39–48.
- 526 23. Shih MH, Tan KK, Generalized quasi-variational inequalities in locally convex topological
527 vector spaces.J Math Anal Appl. 1985;108(2): 333–343.
- 528 24. Takahashi W.Nonlinear variational inequalities and fixed-point theorems, J Math Soc,
529 Japan. 1976; 28(1): 168–181.
- 530 25. Tarafdar E. A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-
531 Mazurkiewicz theorem. J Math Anal Appl. 1987; 128(2): 475–479.
- 532 26. Yang XQ, Chen GY.A class of non-convex functions and pre-variational inequalities.J
533 Math Anal Appl. 1992; 169(2): 359–373.
- 534 27. Yen CL. A minimax inequality and its application to variational inequalities. Pacific J
535 Math. 1981; (2):477–481.