

ON THE SUPERSTABILITY OF A GENERALIZATION OF THE COSINE EQUATION

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ABSTRACT

The aim of this paper is to investigate the stability problem for the functional equation:

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \quad (E_{g,f})$$

and the superstability of the d'Alembert's equation:

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \quad (A)$$

under the conditions from which the differences of each equation are bounded by $\varphi(x)$, $\psi(x)$ and $\min(\varphi(x), \psi(y))$ where G is an arbitrary group, not necessarily abelian, f, g are complex valued functions, φ, ψ are real valued functions and σ is an involution of G .

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1. INTRODUCTION

There is a strong stability phenomenon which is known as a superstability. An equation of homomorphism is called superstable if each approximate homomorphism is actually a true homomorphism. This property was first observed by J. Baker et al. [4] in the following Theorem: Let V be a vector space. If a function $f : V \rightarrow \mathbb{R}$ satisfies the inequality

$$|f(x+y) - f(x)f(y)| \leq \varepsilon,$$

for some $\varepsilon > 0$ and for all $x, y \in V$. Then either f is a bounded function or

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in V.$$

In light of this result, the stability of a class of functional equations has been investigated by Badora, Baker, Dragomir, Gavruta, Ger, Kabbaj, Kim, Rassias, Roukbi, Tyrala,

34 Székelyhidi, Zeglami etc. The interested reader should refer to [1-6, 8-12, 14-21, 24-28] for a
35 thorough account on the subject of stability of functional equations.

36 In [2], R. Badora and R. Ger have improved the superstability problem of the classical
37 d'Alembert's functional equation

$$38 \quad f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G, \quad (C)$$

39 under the condition

$$40 \quad |f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \text{ or } \varphi(y).$$

41 Namely, the following theorem holds true.

42 **Theorem 1.** (R. Badora, R. Ger [2]) *Let $(G, +)$ be an Abelian group, $f : G \rightarrow C$ and let*
43 *$\varphi : G \rightarrow R$ satisfy the inequality*

$$44 \quad |f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \text{ or } \psi(y) \quad \text{for all } x, y \in G.$$

45 *Then either f is bounded or f satisfies the classical d'Alembert's equation (C).*

46
47 In [21] A. Roukbi, D. Zeglami and S. Kabbaj proved the superstability of the equation
48

$$49 \quad f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \quad (E_{f,g})$$

50 without imposing any conditions on the group G . Equation $(E_{f,g})$ is called the Wilson

51 functional equation (see [22]) and sometimes, the first generalization of the d'Alembert's
52 functional equation.

53 In the present paper, we consider, in both abelian and non abelian groups and without any
54 conditions on f , the stability problem of the functional equation

$$55 \quad f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \quad (E_{g,f})$$

56 under the condition

57 $|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x), \psi(y) \text{ or } \min(\varphi(x), \psi(y))$ where G is any
58 one group and σ is an involution of G , i. e. $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all
59 $x, y \in G$. The equation $(E_{g,f})$ is called, sometimes, second generalization of the cosine
60 equation. As a consequence, we obtain the superstability of the d'Alembert's functional
61 equation

$$62 \quad f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \quad (A)$$

63 which proved by Roukbi, Zeglami and Kabbaj [21] on any group, by Redouani, Elqorachi
64 and Rassias [18] on step 2 nilpotent groups and by Baker, Badora and Ger, Gavruta, Kim,
65 etc ([3], [2], [9], [16], ...) in the case where G is an abelian group.

66 In this paper, let G be any one group, e denote its neutral element, C the field of
67 complex numbers and R the field of real numbers. We may assume that f and g are
68 complex valued functions on G , $\varphi, \psi : G \rightarrow R$ are mappings, λ, δ are nonnegative real

69 constants, and σ is an involution of G i. e. $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all
 70 $x, y \in G$. In the case that $f(e) \neq 0$ we put $\tilde{f} = \frac{1}{f(e)} f$.

71 A typical example of the involution σ is the group involution $\sigma(x) = x^{-1}, x \in G$. Another is
 72 the adjoint $A \rightarrow A^*$ in the matrix group $GL(n, C)$ of $n \times n$ invertible matrices, A third one
 73 is

$$74 \quad \sigma \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

$$75 \quad \text{on the Heisenberg group } H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} / x, y, z \in R \right\}.$$

76 77 2. SOLUTIONS OF THE EQUATION $(E_{g,f})$

78 We start with solutions of the d'Alembert's functional equation: In 2008, Th. Davison [7]
 79 proved the following result:

80 **Lemma 1.** Let G be a topological group and $f : G \rightarrow C$ a continuous function with
 81 $f(e) = 1$ satisfying

$$82 \quad f(xy) + f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G.$$

83 Then there is a continuous (group) homomorphism $h : G \rightarrow SL(2, C)$ such that

$$84 \quad f(x) = \frac{1}{2} \text{tr}(h(x)) \text{ for all } x, y \in G.$$

85 Giving solutions of equation (A) the theory of representations is introduced by H. Stetkær in
 86 [23]. Precisely, he proved that:

87 **Lemma 2.** Let S be a semigroup. The non-zero continuous solutions f of (A) on S are
 88 the functions of the form

$$89 \quad f(x) = \frac{1}{2} \text{tr}(\pi(x)), \quad x \in G \quad (2.1)$$

90 where π ranges over the 2-dimensional continuous representations of S for which

$$91 \quad \pi(\sigma(x)) = \text{adj}(\pi(x)) \quad (2.2)$$

$$92 \quad \text{for all } x \in S \text{ and } \text{adj} : \text{Mat}_2(C) \rightarrow \text{Mat}_2(C), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

93 Note that the equation (A) is raised by Kannappan in the case where G is abelian [13].

94 Using Lemma 2 we directly prove the following fact concerning solutions of equation $(E_{g,f})$.

95 **Theorem 2.** Let G be any group. Then $f, g : G \rightarrow C$ satisfy the equation $(E_{g,f})$ if and
 96 only if

97 i) $f = 0$ and g is arbitrary, or

98 ii) $f \neq 0$ and $f(x) = \alpha g(x)$ for all $x \in G$, where $\alpha \in C - \{0\}$ and g is a solution of (A).

99 Furthermore, the non-zero continuous solutions f, g of $(E_{g,f})$ on G are functions of the
 100 form : $g = \frac{1}{2} \chi_\pi$; $f = \frac{\alpha}{2} \chi_\pi$ where $\alpha \in C - \{0\}$ and π ranges over the 2-dimensional
 101 continuous representations of G satisfying (2.2).

102 **Proof.** Assume that $f \neq 0$. Setting $y = e$ in $(E_{g,f})$ we have $f(x) = g(x)f(0)$
 103 for all $x \in G$. From which we conclude that $f(0) \neq 0$. Putting $\alpha := f(0)$ we get
 104 $f(x) = \alpha g(x)$ for all $x \in G$. So, from $(E_{g,f})$ we obtain

$$105 \quad \alpha g(xy) + \alpha g(x\sigma(y)) = 2g(x)\alpha g(y), \quad x, y \in G,$$

106 for all $x, y \in G$. Then g is a solution of (A) and $f = \alpha g$. The rest of the proof follows
 107 from Lemma 2.
 108

109 3. STABILITY OF THE EQUATION $(E_{g,f})$

110
 111 **Lemma 3.** Assume that functions $f, g : G \rightarrow C$ and $\psi : G \rightarrow R$ satisfy the inequality

$$112 \quad |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y), \quad \text{for all } x, y \in G \quad (3.1)$$

113 such that $f \neq 0$. Then f is unbounded if and only if g is unbounded too.

114 **Proof.** If $f(e) = 0$. Putting $y = e$ in (3.1) we get $|f(x)| \leq \frac{\psi(e)}{2}$, for all $x \in G$ i.e. f is
 115 bounded. Let $M = \sup|f|$ and choose $a \in G$ such that $f(a) \neq 0$ then we get from the
 116 inequality (3.1) that $|g(x)| \leq \frac{1}{2f(a)}(2M + \psi(a))$ for all $x \in G$, i.e. g is bounded too.

117 If $f(e)$ is a non zero complex number, substituting y by e in (3.1) we obtain

$$118 \quad |f(x) - f(e)g(x)| \leq \frac{\psi(e)}{2},$$

119 for all $x \in G$, which shows that f is unbounded is equivalent to g is unbounded too.

120 **Lemma 4.** Assume that functions $f, g : G \rightarrow C$ and $\psi : G \rightarrow R$ satisfy the inequality

$$121 \quad |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y), \quad \text{for all } x, y \in G$$

122 Such that $f(e) = 1$. Then

$$123 \quad i) |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y) + \psi(e), \quad \text{for all } x, y \in G. \quad (3.2)$$

124 ii) f is unbounded if and only if g is also unbounded.

125 **Proof.** i) Assume that $f(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show that

$$126 \quad |f(x) - g(x)| \leq \frac{\psi(e)}{2} \quad (3.3)$$

127 for all $x \in G$. Let $F(x) := f(x) - g(x)$. By virtue of inequality (3.3), we have

$$128 \quad g(x) = f(x) - F(x) \text{ and } |F(x)| \leq \frac{\psi(e)}{2} \quad (3.4)$$

129 for all $x \in G$. By the definition of F and the use of (3.1) we have

$$\begin{aligned} 130 \quad |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| &= |f(xy) - F(xy) + f(x\sigma(y)) - F(x\sigma(y)) - 2g(x)f(y)| \\ 131 \quad &\leq |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| + |F(xy)| + |F(x\sigma(y))| \\ 132 \quad &\leq \psi(y) + \psi(e). \end{aligned}$$

133 ii) Follows from (3.3) and it is also a particular case of Lemma 3.

134 **Lemma 5.** Assume that functions $f, g : G \rightarrow C$ and $\varphi : G \rightarrow R$ satisfy the inequality

$$135 \quad |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x), \quad (3.5)$$

136 for all $x, y \in G$ such that $f(e) = 1$. Then

$$137 \quad |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq 2\varphi(x), \text{ for all } x, y \in G. \quad (3.6)$$

138 **Proof.** i) Assume that $f(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show
139 that

$$140 \quad |f(x) - g(x)| \leq \frac{\varphi(x)}{2} \quad (3.7)$$

141 for all $x \in G$. Let $F(x) := f(x) - g(x)$. By virtue of inequality (3.7), we have

$$142 \quad g(x) = f(x) - F(x) \text{ and } |F(x)| \leq \frac{\varphi(x)}{2} \quad (3.8)$$

143 for all $x \in G$. Using (3.5) and (3.8) we get

$$\begin{aligned}
& |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| = |f(xy) - F(xy) + f(x\sigma(y)) - F(x\sigma(y)) - 2g(x)f(y)| \\
& \leq |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| + |F(xy)| + |F(x\sigma(y))| \\
& \leq 2\varphi(x)
\end{aligned}$$

Theorem 3. Assume that functions $f, g : G \rightarrow C$ and $\psi : G \rightarrow R$ satisfy the inequality

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y),$$

for all $x, y \in G$ such that $f \neq 0$. Then either g (or f) is bounded or

$$\tilde{f}(xy) + \tilde{f}(x\sigma(y)) + \tilde{f}(yx) + \tilde{f}(\sigma(y)) = 4\tilde{f}(x)\tilde{f}(y), \quad (3.9)$$

for all $x, y \in G$, where $\tilde{f} = \frac{1}{f(e)} f$.

Proof. i) Assume that f, g satisfy the inequality (3.1) such that g is unbounded (which is equivalent - by lemma 3- to f is also unbounded).

First case: We start with the following particular case $f(e) = 1$. For all $x, y, z \in G$ we have

$$\begin{aligned}
& 2|g(z)||f(xy) + f(x\sigma(y)) + f(yx) + f(\sigma(y)x) - 4f(x)f(y)| \\
& = |2g(z)f(xy) + 2g(z)f(x\sigma(y)) + 2g(z)f(yx) + 2g(z)f(\sigma(y)x) - 8g(z)f(x)f(y)| \\
& \leq |f(zxy) + f(z\sigma(y)\sigma(x)) - 2g(z)f(xy)| \\
& \quad + |f(zx\sigma(y)) + f(zy\sigma(x)) - 2g(z)f(x\sigma(y))| \\
& \quad + |f(zyx) + f(z\sigma(x)\sigma(y)) - 2g(z)f(yx)| \\
& \quad + |f(z\sigma(y)x) + f(z\sigma(x)y) - 2g(z)f(\sigma(y)x)| \\
& \quad + |f(zxy) + f(zx\sigma(y)) - 2g(zx)f(y)| \\
& \quad + |f(zyx) + f(zy\sigma(x)) - 2g(zy)f(x)| \\
& \quad + |f(z\sigma(y)x) + f(z\sigma(y)\sigma(x)) - 2g(z\sigma(y))f(x)|
\end{aligned}$$

$$\begin{aligned}
& + |f(z\sigma(x)y) + f(z\sigma(x)\sigma(y)) - 2g(z\sigma(x))f(y)| \\
& + 2|f(y)| |g(zx) + g(z\sigma(x)) - 2g(z)f(x)| \\
& + 2|f(x)| |g(zy) + g(z\sigma(y)) - 2g(z)f(y)|.
\end{aligned}$$

By virtue of inequalities (3.1) and (3.2), we have

$$\begin{aligned}
& 2|g(z)| |f(xy) + f(x\sigma(y)) + f(yx) + f(\sigma(y)x) - 4f(x)f(y)| \\
& \leq \psi(xy) + \psi(x\sigma(y)) + \psi(yx) + \psi(\sigma(y)x) + 2\psi(x) \\
& + 2\psi(y) + 2|f(y)|(\psi(x) + \psi(e)) + 2|f(x)|(\psi(y) + \psi(e)) \quad (3.10)
\end{aligned}$$

If we fix x, y , the right hand side of the above inequality is bounded function of z . Since g is unbounded, from (3.10), we conclude that $f (= \tilde{f})$ is a solution of the equation (3.9), which ends the proof in this case.

General case: If f is a non-zero function such that $f(e) = 0$ then g and f are bounded (Proof of Lemma 3). For the case that $f(e)$ is any one non-zero complex number, dividing the two sides of the inequality (3.1) by $\alpha = f(e)$ we find that

$$|\tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y)| \leq \frac{\psi(y)}{|\alpha|}, \quad \text{for all } x, y \in G,$$

where $\tilde{f} = \frac{1}{\alpha} f$. We see that the pair $(\tilde{f}, g)$ satisfies the inequality (3.1) with $\tilde{f}(e) = 1$ which shows, using the first case result, that either f (or g) is bounded or $\tilde{f}$ satisfies the equation (3.9) which finished this proof.

As a consequence of Theorem 3, we have the following result on the superstability of the equation (A).

Corollary 1. Assume that functions $f : G \rightarrow C$ and $\psi : G \rightarrow R$ satisfy the inequality

$$|f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \psi(y), \quad (3.11)$$

for all $x, y \in G$. Then either f is bounded or f satisfies the d'Alembert's long functional equation (3.9). Further, in the latter case, if G is abelian then f satisfies the equation (A).

189 **Proof.** Assume that f is unbounded function satisfying (3.11). Putting $f = g$ in Theorem
 190 3 we get that $\tilde{f}$ is a solution of the equation (3.9). Substituting y by e in (3.11) we obtain
 191 $|f(x)(f(e)-1)| \leq \frac{\psi(e)}{2}$ for all $x \in G$. This inequality shows that $f(e) = 1$ because f is
 192 unbounded. So $f = \tilde{f}$ is a solution of (3.9) and if G is abelian then f satisfies the
 193 equation (A).

194 In the following theorem the stability of the equation $(E_{g,f})$ will be investigated on any
 195 group. For $f = 0$ the pair (f, g) is a trivial solution of the equation $(E_{g,f})$.

196 **Theorem 4.** Assume that functions $f, g : G \rightarrow C$ and $\varphi, \psi : G \rightarrow R$ satisfy the inequality

$$197 \quad |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x) \text{ and } \psi(y), \quad (3.12)$$

198 for all $x, y \in G$ such that $f \neq 0$. Then either f (or g) is bounded or the pair (f, g)
 199 satisfies the equation

$$200 \quad g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y), \quad x, y \in G. \quad (3.13)$$

201 Furthermore in the latter case the function $\tilde{f}$ satisfies the equation (3.9).

202 **Proof.** Assume that f, g satisfy inequality (3.12) such that $f \neq 0$. If $f(e) = 0$, we have
 203 seen in Proof of Lemma 3 that f and g are bounded. Suppose that f (or g) is
 204 unbounded then we necessarily have $f(e) \neq 0$. That $\tilde{f}$ satisfies the equation (3.9) is
 205 proven in Theorem 3.

206 **First case:** We start with the case $f(e) = 1$. For all $x, y, z \in G$ we have

$$\begin{aligned} 207 \quad & 2|f(z)| |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \\ 208 \quad & = |2f(z)g(xy) + 2f(z)g(x\sigma(y)) - 4f(z)g(x)f(y)| \\ 209 \quad & \leq |f(xyz) + f(xy\sigma(z)) - 2g(xy)f(z)| \\ 210 \quad & + |f(x\sigma(y)z) + f(x\sigma(y)\sigma(z)) - 2g(x\sigma(y))f(z)| \\ 211 \quad & + |f(xyz) + f(x\sigma(z)\sigma(y)) - 2g(x)f(yz)| \\ 212 \quad & + |f(xy\sigma(z)) + f(xz\sigma(y)) - 2g(x)f(y\sigma(z))| \end{aligned}$$

$$\begin{aligned}
& + |f(x\sigma(y)z) + f(x\sigma(z)y) - 2g(x)f(\sigma(z)y)| \\
& + |f(x\sigma(y)\sigma(z)) + f(xzy) - 2g(x)f(zy)| \\
& + |f(x\sigma(z)y) + f(x\sigma(z)\sigma(y)) - 2g(x\sigma(z))f(y)| \\
& + |f(xzy) + f(xz\sigma(y)) - 2g(xz)f(y)| \\
& + |2g(x)\{f(yz) + f(y\sigma(z)) + f(zy) + f(\sigma(z)y) - 4f(y)f(z)\}| \\
& + 2|f(y)| |g(xz) + g(x\sigma(z)) - 2g(x)f(z)|
\end{aligned}$$

In virtue of inequalities (3.12) and (3.6), we obtain

$$\begin{aligned}
& 2|f(z)| |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \\
& \leq \varphi(xy) + \varphi(x\sigma(y)) + 4\varphi(x) + 2\psi(y) + 4\varphi(x)|f(y)| \\
& + 2|g(x)| |f(yz) + f(y\sigma(z)) + f(zy) + f(\sigma(z)y) - 4f(y)f(z)|.
\end{aligned}$$

Since g is unbounded (which is equivalent to f is unbounded) then, according to Theorem 3, f is a solution of the equation (3.9). So we conclude that

$$2|f(z)| |g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(xy) + \varphi(x\sigma(y)) + 4\varphi(x) + 2\psi(y) + 4\varphi(x)|f(y)|. \quad (3.14)$$

Again the right hand side of (3.14) as a function of z is bounded for all fixed x, y . Since f is unbounded, from (3.14), we see that the pair (g, f) satisfies the equation:

$$g(xy) + g(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, .$$

General case: Now we suppose that $f(e)$ is a nonzero complex number. Dividing the two sides of the inequality (3.12) by $\alpha = f(e)$ we find that

$$|\tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y)| \leq \frac{\varphi(x)}{|\alpha|} \quad \text{and} \quad \frac{\psi(y)}{|\alpha|} \quad \text{for all } x, y \in G,$$

where $\tilde{f} = \frac{1}{\alpha} f$. We see that the pair $(\tilde{f}, g)$ satisfies the inequality (3.12) with $\tilde{f}(e) = 1$ which shows, using the first case result, that either $\tilde{f}$ (or g) is bounded or the pair $(\tilde{f}, g)$ satisfies the equation (3.13) which finished this proof.

233 As another consequence of Theorem 4, we have the following result on the superstability of
 234 the equation (A) on any group which generalizes the Baker's result on the classical
 235 d'Alembert functional equation on an abelian group [3, Theorem 5].

236 **Corollary 2.** [21] Let $\delta > 0$ be given. Assume that the function $f : G \rightarrow C$ satisfies the
 237 inequality

$$238 \quad |f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \delta,$$

239 for all $x, y \in G$. Then either f is bounded or f is a solution of the equation (A).
 240 Further, in the latter case if f is continuous on G then it has the form (2.1).

241 **Proof.** Using similar techniques as in Proof of Corollary 1 we see that if f is unbounded
 242 then we have $f(e) = 1$ implying that $\tilde{f} = f$. The rest of the proof follows on putting $f = g$
 243 in Theorem 4 (iii).

244 From above Theorems we get also the superstability of the equation $(E_{g,f})$ on two
 245 particular cases:

246 **Corollary 3.** Let G be an Abelian group (or at least f is central), and let $f, g : G \rightarrow C$
 247 and $\varphi, \psi : G \rightarrow R$ satisfy the inequality

$$248 \quad |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x) \text{ and } \psi(y), \quad (3.15)$$

249 for all $x, y \in G$. Then there are the following possibilities:

250 i) If $f = 0$, then g is arbitrary.

251 ii) If $g = 0$, then f is bounded.

252 iii) If $f \neq 0 \neq g$ and f is bounded, then g is bounded, too.

253 iv) If $g \neq 0$ and f is unbounded, then g is unbounded, too. Moreover g is a
 254 solutions of (A) and the pair (f, g) satisfies equations $(E_{g,f})$ and $(E_{g,f})$.

255 **Proof.** (ii) If $g = 0$ then the inequality (3.15) has a form $|f(xy) + f(x\sigma(y))| \leq \varphi(x)$ and
 256 $\psi(y)$ for all $x, y \in G$. Put $y = e$, we get $|f(x)| \leq \frac{\psi(e)}{2}$ for all $x \in G$ i.e. f is
 257 bounded.

258 (iii) If $f \neq 0 \neq g$ and f is bounded, let $M = \sup|f|$ and choose $a \in G$ such that
 259 $f(a) \neq 0$ then we get from the inequality (3.15) that $|g(x)| \leq \frac{1}{2|f(a)|}(2M + \psi(a))$ for all
 260 $x \in G$, i.e. g is bounded too.

261 To get (iv) we use Theorem 4 in which we have seen that if f is unbounded then
 262 $f(e) \neq 0$,

$$263 \quad g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y), \quad x, y \in G$$

264 and

$$265 \quad \tilde{f}(xy) + \tilde{f}(x\sigma(y)) + \tilde{f}(yx) + \tilde{f}(\sigma(y)x) = 4\tilde{f}(x)\tilde{f}(y), \quad \text{for all } x, y \in G.$$

266 If G is abelian or at least f is central (i.e. $f(xy) = f(yx)$ for all $x, y \in G$) then we get

$$267 \quad \tilde{f}(xy) + \tilde{f}(x\sigma(y)) = 2\tilde{f}(x)\tilde{f}(y), \quad (3.16)$$

268 for all $x, y \in G$. Dividing the two sides of the inequality (3.15) by $\alpha = f(e)$ we find that

$$269 \quad \left| \tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y) \right| \leq \frac{\varphi(x)}{|\alpha|} \quad \text{and} \quad \frac{\psi(y)}{|\alpha|}, \quad (3.17)$$

270 for all $x, y \in G$. When we substitute (3.16) into (3.17) we get that

$$271 \quad \left| 2\tilde{f}(y)(\tilde{f}(x) - g(x)) \right| \leq \frac{\varphi(x)}{|\alpha|} \quad \text{and} \quad \frac{\psi(y)}{|\alpha|}, \quad (3.18)$$

272 for all $x, y \in G$. Since f is unbounded then so is $\tilde{f}$. Consequently (3.18) implies $\tilde{f} = g$.

273 Thus g is a solution of (A). Substituting $\tilde{f}$ by g on the second (resp. the last) Factor of
 274 the right hand side of (3.16) the expression reduces to $(E_{g,f})$ and $(E_{f,g})$.

275 **Corollary 4.** Let G be any group, and let $f, g : G \rightarrow C$ satisfy the inequality (3.15) such
 276 that $g(\sigma(x)) = g(x)$ for all $x \in G$. Then if f is unbounded, then g is unbounded, too.
 277 Moreover g is a solutions of (A) and (f, g) satisfies equations $(E_{g,f})$ and $(E_{f,g})$.

278 **Proof.** Suppose that f, g satisfy (3.15) with $g(\sigma(x)) = g(x)$ for all $x \in G$. If f is
 279 unbounded, using Theorem 4, we obtain the equality (3.13). By putting $y = e$ in (3.13) it is

280 easy to see that $\tilde{f}(y) = \frac{1}{g(e)} g(y)$ (the case $g(e) = 0$ does not occur here due to our
 281 assumption that f is unbounded). Using this equality and (3.13) we get

$$\begin{aligned} \tilde{f}(xy) + \tilde{f}(x\sigma(y)) &= \frac{1}{g(e)} (g(xy) + g(x\sigma(y))) \\ 282 \quad &= \frac{1}{g(e)} (2g(x)\tilde{f}(y)) \\ &= 2\tilde{f}(x)\tilde{f}(y), \end{aligned}$$

283 and the rest of the proof runs along the same lines as in proof of Corollary 3 (iv).

284 **Remarks.**

285 i) In the case where G is an abelian group and f, g satisfy the inequality (3.15) we
 286 know –according to Corollary 3– that if f is unbounded then g is a solutions of
 287 (A) but does not always f as shown by the example: Let $f, g : R \rightarrow R$ be
 288 functions with $g(x) = ch(x) := \frac{e^{ix} + e^{-ix}}{2}$ and $f(x) = 3ch(x)$ and let
 289 $\sigma(x) = -x$ for all $x \in R$. Then

$$290 \quad |f(x+y) + f(x+\sigma(y)) - 2g(x)f(y)| = 0,$$

291 but f is unbounded and f does not satisfy the equation (A).

292 ii) Let $f, g : R \rightarrow R$ be functions with $f(x) = x^2 + 1$ and $g(x) = 1$ for all $x \in IR$,
 293 and let $\sigma(x) = -x$. Then

$$294 \quad |f(x+y) + f(x+\sigma(y)) - 2g(x)f(y)| = 2x^2 = \varphi(x),$$

295 and $f(0) = 1$ but f is unbounded and f, g do not satisfy the equation

$$296 \quad g(x+y) + g(x+\sigma(y)) = 2g(x)\tilde{f}(y). \quad (3.19)$$

297 This shows that the condition

$$298 \quad |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \psi(y), \quad x, y \in G \quad (3.20)$$

299 is essential in the case (iii) of Theorem 4. This example shows also that the
 300 condition (3.20) is essential in Theorem 3.

301 iii) Let $f, g : R \rightarrow R$ be functions with $f(x) = ch(x)$ and $g(x) = 1 + ch(x)$ for all
 302 $x \in R$, and let $\sigma(x) = -x$. Then

303 $|f(x+y) + f(x+\sigma(y)) - 2g(x)f(y)| = 2ch(y) = \psi(y),$

304 and $f(0) = 1$ but f is unbounded and f, g do not satisfy the equation (3.19).
 305 This shows that the condition

306 $|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x), \quad x, y \in G,$

307 is essential in the case (iii) of Theorem 4.

308 iv) The obtained results in this paper can be extended to the equation

309 $f(xy) + f(x\sigma(y)) = \lambda g(x)f(y), \quad x, y \in G, \text{ and } \lambda \text{ is a complex constant.}$

310 It can be also extended to the commutative semi simple Banach algebra on any
 311 group as in [14-16] in the case where G is an abelian group.

312 v) If we apply the combinaison of cases

313 (a) $g = f$ or $g \neq f$.

314 (b) $\sigma(x)$, $\sigma(x) = x$, or $\sigma(x) = -x$.

315 (c) $\varphi(x) = \psi(x) = \delta$ or $\varphi(x) = \psi(x) \neq \delta$.

316 (d) The group G is abelian or non abelian.

317 to Theorem 3 and Theorem 4, we obtain some results of the papers [1-6, 8-12,
 318 14-21, 24-28].

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320

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322

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