

A Nonlocal Boundary Value mixed Problem for second-order hyperbolic-parabolic equations in nonclassical function spaces

DJIBIBE Moussa Zakari^{*1} and TCHARIE Kokou²

¹ University of Lomé, Department of Mathematics

PO Box 1515 Lomé, Togo

² University of Lomé, Department of Mathematics

PO Box 1515 Lomé, Togo

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Abstract

Aims/ We use an energy method to solve a following boundary-value problem for hypobolic-parabolic equation with an integral condition :

$$\left\{ \begin{array}{l} \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v}{\partial x} \right) + b(x, t)v = F(x, t) \\ v(x, 0) = \psi_1(x), \quad 0 \leq x \leq \ell, \\ \frac{\partial v}{\partial t}(x, 0) = \psi_2(x), \quad 0 \leq x \leq \ell, \\ u(\ell, t) = E(t), \quad 0 \leq t \leq T, \\ \int_0^\ell xv(x, t) dx = H(t), \quad 0 \leq t \leq T. \end{array} \right.$$

The proof is based on an energy inequality and on the fact that the range of the operator generated is dense.

Keywords: Hyperbolic-Parabolic; A priori Estimate; Nonlocal conditions; Mixed Problem.

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^{*}DJIBIBE Moussa Zakari : zakari.djibibe@gmail.com

1 Introduction

In the present paper, we consider the problem of determining a function $v = v(x, t)$ satisfying, in a weak sense, the linear hyperbolic-parabolic equation

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v}{\partial x} \right) + b(x, t)v = F(x, t), \quad (1.1)$$

supplemented by the initial conditions

$$v(x, 0) = \psi_1(x), \quad 0 \leq x \leq \ell, \quad (1.2)$$

$$\frac{\partial v}{\partial t}(x, 0) = \psi_2(x), \quad 0 \leq x \leq \ell, \quad (1.3)$$

the Dirichlet condition

$$v(\ell, t) = E(t), \quad 0 \leq t \leq T, \quad (1.4)$$

and the integral condition

$$\int_0^\ell xv(x, t) dx = H(t), \quad 0 \leq t \leq T, \quad (1.5)$$

where, ψ_1, ψ_2, E, G and H are sufficiently regular given function, and T and ℓ are the positive constants.

Condition 1.1. For all (x, t) in the closed of $(0, \ell) \times [0, T]$, we conditione that

$$a_0 \leq a(x, t) \leq a_1,$$

$$a_2 \leq \frac{\partial a}{\partial x}(x, t) \leq a_3,$$

$$a_4 \leq \frac{\partial a}{\partial t}(x, t) \leq a_5,$$

$$a_6 \leq \frac{\partial^2 a}{\partial x \partial t}(x, t) \leq a_7,$$

where $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 arer positive constants.

Condition 1.2. For all (x, t) in the closed of $(0, \ell) \times [0, T]$, we conditione that

$$b_0 \leq b(x, t) \leq b_1,$$

$$b_2 \leq \frac{\partial b}{\partial x}(x, t) \leq b_3,$$

$$b_4 \leq \frac{\partial b}{\partial t}(x, t) \leq b_5,$$

$$b_6 \leq \frac{\partial^2 b}{\partial x \partial t}(x, t) \leq b_7,$$

where $b_0, b_1, b_2, b_3, b_4, b_5, b_6$ and b_7 arer positive constants.

The data satisfies the following compatibility conditions : for consistency, we have

$$\psi_1(0) = E(0), \quad \text{and} \quad \int_0^\ell x\psi_1(x) dx = H(0).$$

The importance of problems with integral conditions has been pointed out by Samarskii[17]. Mathematical modelling by evolution problems with a nonlocal constraint of the form

$\int_0^\ell \gamma(x)u(x, t) dx = \Gamma(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

See for instance Benouar-Yurchuk [2], Benouar-Bouziani [3]-[4], Bouziani [5]-[8], Cannon et al [10]-[11], Ionkin [12]-[13], Kamynin [14] and Yurchuk [15]-[16]. Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [8]-[17], Byszewski et al [18]-[20], Gasymov [22], Ionkin [12]-[13], Lazhar [24], Mouravey-Philipovski [25], Said-Nadia [26] and Yurtchuk [27].

The results and the method used here are a further elaboration of those in [2].

This basic tool is the energy inequality method which, of course, requires appropriate multipliers and functional spaces.

In this paper, we extend this method to the study of a mixed-type hyperbolic-parabolic equations with Dirichlet and integral conditions.

The remainder of the paper is organized as follows. After this introduction, in section 2, we present some preliminaries and basic definitions. Then in Section 3, we establish a priori estimate and give its several applications. Finally, in section 4, we prove existence of generalized solution.

2 Preliminaries

In point of view of the used method, it is preferable to transform inhomogeneous boundary conditions to homogeneous ones by introducing a new unknown function z defined as follows :

$$z(x, t) = v(x, t) - u(x, t)$$

where

$$u(x, t) = \frac{(-6x^2 + 12\ell x - 5\ell^2)}{\ell^2} E(t) + \frac{(2x + \ell)(x - \ell)}{\ell} G(t) + \frac{(x - \ell)^2}{\ell^2} H(t).$$

Then the problem becomes

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial z}{\partial t} - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) + b(x, t)z = f(x, t) \quad (2.1)$$

$$z(x, 0) = \varphi_2(x), \quad 0 \leq x \leq \ell, \quad (2.2)$$

$$\frac{\partial z}{\partial t}(x, 0) = \varphi_2(x), \quad 0 \leq x \leq \ell, \quad (2.3)$$

$$z(\ell, t) = 0, \quad 0 \leq t \leq T, \quad (2.4)$$

$$\int_0^\ell xz(x, t) dx = 0, \quad 0 \leq t \leq T, \quad (2.5)$$

where

$$\begin{aligned} f(x, t) = & F(x, t) + \frac{(6x^2 - 12\ell x + 5\ell^2)}{\ell^2} (E_1'(t) + E_1'') - \frac{(2x + \ell)(x - \ell)}{\ell} (E_2'(t) + E_2'')(t) \\ & - \frac{(x - \ell)^2}{\ell^2} (G'(t) + G''(t)) + \frac{(6x^2 - 12\ell x + 5\ell^2)b(x, t)}{\ell^2} E(t) + \frac{\partial a}{\partial x} \frac{(x - \ell)^2}{\ell^2} H(t) \\ & - \frac{(2x + \ell)(x - \ell)b(x, t)}{\ell} G(t) - \frac{(x - \ell)^2 b(x, t)}{\ell^2} H(t) + \frac{\partial a}{\partial x} \frac{(2x + \ell)(x - \ell)}{\ell} G(t) \\ & + \frac{\partial a}{\partial x} \frac{(-6x^2 + 12\ell x - 5\ell^2)}{\ell^2} E(t) + \frac{(4x - \ell)a(x, t)}{\ell} G(t) - \frac{12(x - \ell)a(x, t)}{\ell^2} E(t) \\ & + \frac{2(x - \ell)a(x, t)}{\ell^2} H(t), \end{aligned}$$

$$\begin{aligned} \varphi_1(x) = & \psi_1(x) + \frac{(6x^2 - 12\ell x + 5\ell^2)}{\ell^2} E(0) - \frac{(2x + \ell)(x - \ell)}{\ell} G(0) - \frac{(x - \ell)^2}{\ell^2} H(0), \\ \varphi_2(x) = & \psi_2(x) + \frac{(6x^2 - 12\ell x + 5\ell^2)}{\ell^2} E'(0) - \frac{(2x + \ell)(x - \ell)}{\ell} G'(0) - \frac{(x - \ell)^2}{\ell^2} H'(0). \end{aligned}$$

Here we conditione that the function φ_1 and φ_2 satisfy conditions of (2.4) and (2.5), that is

$$\varphi_1(0) = 0 \quad \text{and} \quad \int_0^\ell x \varphi_1(x) dx = 0.$$

Instead of searching for the function $v(x, t)$, we search for the function z . So the solution of problem (1.1), (1.2), (1.3), (1.4) and (1.5) will be given by $v(x, t) = z(x, t) + u(x, t)$.

The problem (2.1), (2.2), (2.3), (2.4) and (2.5) can be considered as solving the operator equation

$$Az = (f, \varphi_1, \varphi_2) = \mathcal{F}, \quad (2.6)$$

where A is an operator defined on E into F . E is the Banach space of functions $L^2(\Omega)$ satisfying the conditions (2.4) and (2.5) with the norm

$$\|z\|_E^2 = \sup_{0 \leq t \leq T} \left\{ \int_0^\ell x z^2 dx + \int_0^\ell x \left(\frac{\partial z}{\partial x} \right)^2 dx + \int_0^\ell x \left(\frac{\partial z}{\partial t} \right)^2 dx \right\},$$

and F is the Hilbert space $L^2(\Omega) \times L^2(0, \ell) \times L^2(0, \ell)$ which consists of elements $\mathcal{F} = (f, \varphi_1, \varphi_2)$ with the norm

$$\|\mathcal{F}\|_F^2 = \int_0^\ell x \varphi_1^2(x) dx + \int_0^\ell x \left(\frac{d\varphi_1}{dx} \right)^2 dx + \int_0^\ell x \varphi_2^2(x) dx + \int_\Omega x f^2(x, t) dx dt.$$

Let $D(A)$ be the set of all functions $u \in L^2(\Omega)$, for $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t} \in L^2(\Omega)$ and satisfying conditions (2.4), (??) and (2.5).

3 Energy inequality and its consequences

Theorem 3.1. *Let condition 1.1 be fulfilled. Then the a priori estimate*

$$\|z\|_E \leq \lambda \|Az\|_F, \quad (3.1)$$

holds for any function $z \in D(A)$, where λ is a positive constant independent of z .

Proof

Multiplying the equation (2.1) with $Lz = (t - \tau) \int_0^t \int_0^x \frac{\partial z}{\partial t}(\xi, s) d\xi ds$ and integrating the results obtained over $\Omega_\tau = (0, \ell) \times (0, T)$. Observe that

$$\begin{aligned} & \int_{\Omega_\tau} (t - \tau) \frac{\partial^2 z}{\partial t^2} \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt + \int_{\Omega_\tau} (t - \tau) \frac{\partial z}{\partial t} \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt \\ & - \int_{\Omega_\tau} (t - \tau) \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt \\ & + \int_{\Omega_\tau} b(x, t) (t - \tau) z \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt \\ & = \int_{\Omega_\tau} (t - \tau) f(x, t) \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt. \end{aligned} \quad (3.2)$$

Successive integration by parts of integrals on the left-hand of (3.2) are straight-foward but somewhat tedious. We give only their results

$$\begin{aligned} & \int_{\Omega_\tau} (t - \tau) \frac{\partial^2 z}{\partial t^2} \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt = - \int_0^\ell z(x, \tau) \left(\int_0^x z(\xi, \tau) d\xi \right) dxdt \\ & + \int_0^\ell z(x, \tau) \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt + \int_{\Omega_\tau} z \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt \\ & - \int_{\Omega_\tau} (t - \tau) \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \int_{\Omega_\tau} (t - \tau) \frac{\partial z}{\partial t} \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt = - \int_{\Omega_\tau} (t - \tau) z \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt \\ & - \int_{\Omega_\tau} z \left(\int_0^x z(\xi, t) d\xi \right) dxdt + \int_{\Omega_\tau} z \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt, \end{aligned} \quad (3.4)$$

$$\begin{aligned} & - \int_{\Omega_\tau} (t - \tau) \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt \\ & = \int_{\Omega_\tau} (t - \tau) a(x, t) z \frac{\partial z}{\partial x} dxdt - \int_{\Omega_\tau} (t - \tau) a(x, t) \varphi_1(x) \frac{\partial z}{\partial x} dxdt, \end{aligned} \quad (3.5)$$

$$\begin{aligned} & \int_{\Omega_\tau} (t - \tau) f(x, t) \left(\int_0^t \int_0^x \frac{\partial z}{\partial s}(\xi, s) d\xi ds \right) dxdt = - \int_{\Omega_\tau} (t - \tau) f(x, t) \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt \\ & + \int_{\Omega_\tau} (t - \tau) f(x, t) \left(\int_0^x z(\xi, t) d\xi \right) dxdt. \end{aligned} \quad (3.6)$$

By substituting (3.3), (3.4), (3.5) and (3.6) in (3.2), we obtain

$$\begin{aligned}
 & \int_{\Omega_\tau} (t - \tau) a(x, t) z \frac{\partial z}{\partial x} dx dt - \int_{\Omega_\tau} (t - \tau) a(x, t) \varphi_1(x) \frac{\partial z}{\partial x} dx dt \\
 & - \int_{\Omega_\tau} (t - \tau - 1) z \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dx dt - \int_{\Omega_\tau} (t - \tau) \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial z}{\partial t} \right) dx dt \\
 & - \int_0^\ell z(x, \tau) \left(\int_0^x z(\xi, \tau) d\xi \right) dx dt + \int_0^\ell z(x, \tau) \left(\int_0^x \varphi_1(\xi) d\xi \right) dx dt \\
 & - \int_{\Omega_\tau} z \left(\int_0^x z(\xi, t) d\xi \right) dx dt + \int_{\Omega_\tau} z \left(\int_0^x \varphi_1(\xi) d\xi \right) dx dt \\
 & = - \int_{\Omega_\tau} (t - \tau) f(x, t) \left(\int_0^x \varphi_1(\xi) d\xi \right) dx dt + \int_{\Omega_\tau} (t - \tau) f(x, t) \left(\int_0^x z(\xi, t) d\xi \right) dx dt. \quad (3.7)
 \end{aligned}$$

Multiplying the equation (2.1) with $\frac{\partial z}{\partial t}$ and integrating the results obtained over $\Omega_\tau = (0, \ell) \times (0, \tau)$. Observe that

$$\begin{aligned}
 & \int_{\Omega_\tau} \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial t} dx dt + \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dx dt - \int_{\Omega_\tau} \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial t} dx dt \\
 & + \int_{\Omega_\tau} b(x, t) z \frac{\partial z}{\partial t} dx dt = \int_{\Omega_\tau} f(x, t) \frac{\partial z}{\partial t} dx dt. \quad (3.8)
 \end{aligned}$$

The standard integration by parts of the terms on the left-hand side of (3.8) leads to

$$\int_{\Omega_\tau} \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial t} dx dt = \frac{1}{2} \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx - \frac{1}{2} \int_0^\ell \varphi_2^2(x) dx, \quad (3.9)$$

$$\begin{aligned}
 & - \int_{\Omega_\tau} \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial t} dx dt = \int_0^\tau a(0, t) \frac{\partial z}{\partial t}(0, t) \frac{\partial z(0, t)}{\partial x} dt + \frac{1}{2} \int_0^\ell a(x, \tau) \left(\frac{\partial z}{\partial x} \right)^2 dx \\
 & - \frac{1}{2} \int_{\Omega_\tau} \frac{\partial a(x, t)}{\partial t} \left(\frac{\partial z}{\partial x} \right)^2 dx dt - \frac{1}{2} \int_0^\ell a_0(x) \left(\frac{d\varphi_1(x)}{dx} \right)^2 dx - \int_0^\tau a(\ell, t) \frac{\partial z}{\partial t}(\ell, t) \frac{\partial z(\ell, t)}{\partial x} dt, \quad (3.10)
 \end{aligned}$$

$$\int_{\Omega_\tau} b(x, t) z \frac{\partial z}{\partial t} dx dt = \frac{1}{2} \int_0^\ell b(x, \tau) z^2(x, \tau) dx - \frac{1}{2} \int_0^\ell b_0(x) \varphi_1^2(x) dx - \frac{1}{2} \int_{\Omega_\tau} \frac{\partial b(x, t)}{\partial t} z^2 dx dt. \quad (3.11)$$

Substituting (3.9), (3.10) and (3.11) into (3.8), we obtain

$$\begin{aligned}
 & \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \frac{1}{2} \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^\ell a(x, \tau) \left(\frac{\partial z}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^\ell b(x, \tau) z^2(x, \tau) dx \\
 & + \int_0^\tau a(0, t) \frac{\partial z(0, t)}{\partial t} \frac{\partial z(\ell, t)}{\partial x} dt - \int_0^\tau a(\ell, t) \frac{\partial z(\ell, t)}{\partial t} \frac{\partial z(\ell, t)}{\partial x} dt = \frac{1}{2} \int_{\Omega_\tau} \frac{\partial b(x, t)}{\partial t} z^2 dx dt \\
 & + \frac{1}{2} \int_{\Omega_\tau} \frac{\partial a(x, t)}{\partial t} \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \frac{1}{2} \int_0^\ell a_0(x) \left(\frac{d\varphi_1(x)}{dx} \right)^2 dx + \frac{1}{2} \int_0^\ell \varphi_2^2(x) dx \\
 & + \frac{1}{2} \int_0^\ell b_0(x) \varphi_1^2(x) dx + \int_{\Omega_\tau} f(x, t) \frac{\partial z}{\partial t} dx dt. \quad (3.12)
 \end{aligned}$$

Combining the equalities (3.7) with (3.12), we get

$$\begin{aligned}
 & \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dxdt + \frac{1}{2} \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^\ell a(x, \tau) \left(\frac{\partial z}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^\ell b(x, \tau) z^2(x, \tau) dx \\
 &= \int_{\Omega_\tau} (\tau - t) a(x, t) z \frac{\partial z}{\partial x} dxdt + \int_{\Omega_\tau} (\tau - t) a(x, t) \varphi_1(x) \frac{\partial z}{\partial x} dxdt \\
 &+ \int_{\Omega_\tau} (\tau + 1 - t) z \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt + \int_{\Omega_\tau} (\tau - t) \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt \\
 &+ \int_0^\ell z(x, \tau) \left(\int_0^x z(x, \tau) d\xi \right) dxdt - \int_0^\tau a(0, t) \frac{\partial z(0, t)}{\partial t} \frac{\partial z(0, t)}{\partial x} dt \\
 &- \int_0^\ell z(x, \tau) \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt + \int_{\Omega_\tau} z \left(\int_0^x z(\xi, t) d\xi \right) dxdt - \int_{\Omega_\tau} z \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt \\
 &+ \int_{\Omega_\tau} (\tau - t) f(x, \tau) \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt - \int_{\Omega_\tau} (\tau - t) f(x, t) \left(\int_0^x z(\xi, t) d\xi \right) dxdt \\
 &+ \frac{1}{2} \int_{\Omega_\tau} \frac{\partial b(x, t)}{\partial t} z^2 dxdt + \frac{1}{2} \int_{\Omega_\tau} \frac{\partial a(x, t)}{\partial t} \left(\frac{\partial z}{\partial x} \right)^2 dxdt + \frac{1}{2} \int_0^\ell a(x, \tau) \left(\frac{d\varphi_1(x)}{dx} \right)^2 dx \\
 &+ \frac{1}{2} \int_0^\ell b_0(x) \varphi_1^2(x) + \frac{1}{2} \int_0^\ell \varphi_2^2(x) dx + \int_{\Omega_\tau} f(x, t) \frac{\partial z}{\partial t} dxdt + \int_0^\tau a(\ell, t) \frac{\partial z}{\partial t}(\ell, t) \frac{\partial z(\ell, t)}{\partial x} dt, \quad (3.13)
 \end{aligned}$$

In light the Young's inequality and the inequality of Poincare type

$$\int_0^\ell \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right)^2 dx \leq \frac{\ell}{2} \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx, \quad (3.14)$$

and the condition 1.1 and 1.2, certains terms in the right-hand of (3.13) are then majorized as follows :

$$\int_{\Omega_\tau} (\tau - t) a(x, t) z \frac{\partial z}{\partial x} dxdt \leq \frac{a_1 T}{2} \int_{\Omega_\tau} z^2 dxdt + \frac{a_1 T}{2} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial x} \right)^2 dxdt, \quad (3.15)$$

$$\int_{\Omega_\tau} (\tau - t) a(x, t) \varphi_1(x) \frac{\partial z}{\partial x} dxdt \leq \frac{a_1 T^2}{2} \int_0^\ell \varphi_1^2(x) dx + \frac{a_1 T}{2} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial x} \right)^2 dxdt, \quad (3.16)$$

$$\int_{\Omega_\tau} (\tau + 1 - t) z \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt \leq \frac{T+1}{2} \int_{\Omega_\tau} z^2 dxdt + \frac{(T+1)\ell}{4} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dxdt, \quad (3.17)$$

$$\int_{\Omega_\tau} (\tau - t) \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial z}{\partial t} d\xi \right) dxdt \leq \frac{T(2+\ell)}{4} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dxdt, \quad (3.18)$$

$$\begin{aligned}
 & \int_0^\ell z(x, \tau) \left(\int_0^x z(x, \tau) d\xi \right) dxdt + \int_{\Omega_\tau} z \left(\int_0^x z(\xi, t) d\xi \right) dxdt \\
 & \leq \frac{(2+\ell)}{4} \int_0^\ell z^2(x, \tau) dx + \frac{2+\ell}{8} \int_{\Omega_\tau} z^2 dxdt, \quad (3.19)
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^\ell z(x, \tau) \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt - \int_{\Omega_\tau} z \left(\int_0^x \varphi_1(\xi) d\xi \right) dxdt \\
 & \leq \frac{\alpha}{2} \int_0^\ell z^2(x, \tau) dx + \frac{\ell(1+\alpha T)}{4\alpha} \int_0^\ell \varphi_1^2(x) dx + \frac{1}{2} \int_{\Omega_\tau} z^2(x, t) dxdt, \quad (3.20)
 \end{aligned}$$

$$\begin{aligned}
 & \int_{\Omega_\tau} (\tau - t) f(x, t) \left(\int_0^x \varphi_1(\xi) d\xi \right) dx dt - \int_{\Omega_\tau} (\tau - t) f(x, t) \left(\int_0^x z(\xi, t) d\xi \right) dx dt \\
 & + \int_{\Omega_\tau} f(x, t) \frac{\partial z}{\partial t} dx dt \leq \frac{1+2T}{2} \int_{\Omega_\tau} f^2(x, t) dx dt + \frac{\ell T}{4} \int_0^\ell \varphi_1^2(x) dx \\
 & + \frac{1}{2} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \frac{\ell T}{4} \int_{\Omega_\tau} z^2(x, t) dx dt. \quad (3.21)
 \end{aligned}$$

Combining the inequalities (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21), choosing $\alpha + 3 + \ell = b_0$, and using condition 1.1 and 1.2, we obtain

$$\begin{aligned}
 & \frac{1}{2} \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx + \frac{a_0}{2} \int_0^\ell \left(\frac{\partial z}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^\ell z^2(x, \tau) dx \leq \frac{1}{2} \int_0^\ell \varphi_2^2(x) dx + \frac{a_1}{2} \int_0^\ell \left(\frac{d\varphi_1(x)}{dx} \right)^2 dx \\
 & + \left[\frac{\ell(1+\alpha T)}{4\alpha} + \frac{\ell T}{4} + \frac{a_1 T^2}{2} + \frac{b_1}{2} \right] \int_0^\ell \varphi_1^2(x) dx + \left[\frac{(1+a_1)T}{2} + \frac{2+\ell}{8} + \frac{1}{2} + \frac{\ell T}{4} + \frac{b_5}{2} \right] \int_{\Omega_\tau} z^2 dx dt \\
 & + \left[a_1 T + \frac{a_5}{2} \right] \int_{\Omega_\tau} \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \frac{2T\ell + \ell + 2T}{4} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \frac{1+4T}{8} \int_{\Omega_\tau} f^2(x, t) dx dt. \quad (3.22)
 \end{aligned}$$

We also have, by straight forward calculations ,

$$\begin{aligned}
 & \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx + \int_0^\ell \left(\frac{\partial z}{\partial x} \right)^2 dx + \int_0^\ell z^2(x, \tau) dx \\
 & \leq M_0 \left(\int_0^\ell \varphi_2^2(x) dx + \int_0^\ell \left(\frac{d\varphi_1(x)}{dx} \right)^2 dx + \int_0^\ell \varphi_1^2(x) dx + \int_{\Omega_\tau} f^2(x, t) dx dt \right) \\
 & + M_1 \left\{ \int_{\Omega_\tau} z^2 dx dt + \int_{\Omega_\tau} \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dx dt \right\}, \quad (3.23)
 \end{aligned}$$

where

$$\begin{aligned}
 M_0 &= \frac{\max \left(\frac{1}{2}, \frac{\ell(1+\alpha T)}{4\alpha} + \frac{\ell T}{4} + \frac{a_1 T^2}{2} + \frac{b_1}{2}, \frac{1+4T}{8} \right)}{\min \left(\frac{1}{2}, \frac{a_0}{2} \right)} \\
 M_1 &= \frac{\max \left(\frac{(1+a_1)T}{2} + \frac{2+\ell}{8} + \frac{1}{2} + \frac{\ell T}{4} + \frac{b_5}{2}, \frac{2a_1 T + a_5}{2}, \frac{2T\ell + \ell + 2T}{4} \right)}{\min \left(\frac{1}{2}, \frac{a_0}{2} \right)}
 \end{aligned}$$

We eliminate the last term on the right-hand side of (3.23). To do that we use GRONWALL'S Lemma to we obtain

$$\begin{aligned}
 & \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx + \int_0^\ell \left(\frac{\partial z}{\partial x} \right)^2 dx + \int_0^\ell z^2(x, \tau) dx \leq \\
 & M_0 e^{M_1 T} \left(\int_0^\ell \varphi_2^2(x) dx + \int_0^\ell \left(\frac{d\varphi_1(x)}{dx} \right)^2 dx + \int_0^\ell \varphi_1^2(x) dx + \int_{\Omega_\tau} f^2(x, t) dx dt \right). \quad (3.24)
 \end{aligned}$$

The right-hand side of (3.24) is independent of τ . By taking the least upper bound of the left side with respect to τ from 0 to T , we get the estimate (3.1) with $\lambda = \sqrt{M_0} e^{\frac{M_1 T}{2}}$. This complete the proof of Theorem 3.1. $\square$

Definition 3.1. A solution of the equation

$$\bar{A}z = (f, \varphi_1, \varphi_2) \quad (3.25)$$

is called a strong solution of problem (2.1), (2.2), (2.3), (2.4), (??) and (2.5).

Corollary 3.2. By passing to the limit, the estimate (3.1) can be extended to strong solutions, that is we have the inequality

$$\|z\|_E \leq \lambda \|\bar{A}z\|_F, \quad \forall v \in D(\bar{A}). \quad (3.26)$$

From the inequality, we deduce the following statements.

Corollary 3.3. The range $R(\bar{A})$ of the operator $\bar{A}$ is closed and $\overline{R(\bar{A})} = R(\bar{A})$.

Corollary 3.4. A strong solution of the problem (2.1), (2.2), (2.3), (2.4), (??) and (2.5) is unique and depends continuously on $\mathcal{F} = (f, \varphi_1, \varphi_2) \in F$.

Remark 3.1. Hence, to prove that a strong solution of the problem (2.1), (2.2), (2.3), (2.4), (??) and (2.5) exists for any element $(f, \varphi_1, \varphi_2) \in F$, it remains to prove that $\overline{R(A)} = F$, where A is an operator define on F by

$$A(z) = \left(\frac{\partial^2 z}{\partial t^2} + \frac{\partial z}{\partial t} - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right), \varphi_1, \varphi_2 \right).$$

4 Solvability of the problem

Theorem 4.1. If for some $w \in \left\{ z : \int_{\Omega} z^2(x, t) < +\infty \right\}$ and for all element

$$z \in D_0(A) = \left\{ z \in D(A) : z(x, 0) = \frac{\partial z}{\partial t}(x, 0) = 0 \right\},$$

we have

$$\int_0^T \int_0^\ell \frac{\partial^2 z}{\partial t^2} w(x, t) dx dt + \int_0^T \int_0^\ell \frac{\partial z}{\partial t} w(x, t) dx dt - \int_0^T \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) w(x, t) dx dt = 0, \quad (4.1)$$

then w vanishes almost everywhere in Ω .

Proof

Replacing in (4.1), w by $w = -\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi + \frac{\partial z}{\partial t}$, it follows that

$$\begin{aligned} & \int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2} \left(-\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi + \frac{\partial z}{\partial t} \right) dx dt \\ & + \int_0^\tau \int_0^\ell \frac{\partial z}{\partial t} \left(-\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi + \frac{\partial z}{\partial t} \right) dx dt \\ & - \int_0^\tau \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \left(-\frac{\partial^3 z}{\partial t^3} - \int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi + \frac{\partial z}{\partial t} \right) dx dt = 0. \end{aligned} \quad (4.2)$$

Integrating by parts and taking into account (2.5) each term of (4.2), we obtain

$$- \int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2} \frac{\partial^3 z}{\partial t^3} dx dt = -\frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x, \tau)}{\partial t^2} \right)^2 dx + \frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x, 0)}{\partial t^2} \right)^2 dx, \quad (4.3)$$

$$\int_0^\tau \int_0^\ell \frac{\partial z}{\partial t} \frac{\partial z}{\partial t} dx dt = \int_0^\tau \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx dt, \quad (4.4)$$

$$\int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial t} dx dt = \frac{1}{2} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial t} \right)^2 dx, \quad (4.5)$$

$$- \int_0^\tau \int_0^\ell \frac{\partial z}{\partial t} \frac{\partial^3 z}{\partial t^3} dx dt = \int_0^\tau \int_0^\ell \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt - \int_0^\ell \frac{\partial z(x, \tau)}{\partial t} \frac{\partial^2 z(x, \tau)}{\partial t^2} dx, \quad (4.6)$$

$$\begin{aligned} - \int_0^\tau \int_0^\ell \frac{\partial^2 z}{\partial t^2} \left(\int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi \right) dx dt &= - \int_0^\ell \frac{\partial z(x, \tau)}{\partial t} \left(\int_0^x \frac{\partial z(\xi, \tau)}{\partial t} d\xi \right) dx \\ &\quad + \int_{\Omega_\tau} \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial^2 z}{\partial t^2} d\xi \right) dx dt, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \int_0^\tau \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \frac{\partial^3 z}{\partial t^3} dx dt &= \int_0^\ell a(x, \tau) \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial t^2} dx + \int_0^\ell \frac{\partial a(x, \tau)}{\partial x} \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial t^2} dx \\ &\quad - \int_0^\ell \int_0^\tau a(x, t) \frac{\partial^2 z}{\partial t^2} \frac{\partial^3 z}{\partial t \partial x^2} dx dt - \int_0^\ell \int_0^\tau \frac{\partial^2 a}{\partial x \partial t} \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial x} dx dt, \end{aligned} \quad (4.8)$$

$$- \int_0^\tau \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial t} dx dt = - \int_0^\tau \int_0^\ell a(x, t) \frac{\partial^2 z}{\partial x^2} \frac{\partial z}{\partial t} dx dt - \int_0^\tau \int_0^\ell \frac{\partial a(x, t)}{\partial x} \frac{\partial z}{\partial x} \frac{\partial z}{\partial t} dx dt, \quad (4.9)$$

$$\int_0^\tau \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \left(\int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi \right) dx dt = - \int_0^\ell \int_0^\tau a(x, t) \frac{\partial z}{\partial t} \frac{\partial z}{\partial x} dx dt. \quad (4.10)$$

Substituting the equalities (4.3) (4.4), (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) into (4.2), we obtain

$$\begin{aligned} & - \frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x, \tau)}{\partial t^2} \right)^2 dx + \frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x, 0)}{\partial t^2} \right)^2 dx + \int_0^\tau \int_0^\ell \left(\frac{\partial z}{\partial t} \right)^2 dx dt \\ & + \frac{1}{2} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial t} \right)^2 dx + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt - \int_0^\ell \frac{\partial z(x, \tau)}{\partial t} \frac{\partial^2 z(x, \tau)}{\partial t^2} dx \\ & - \int_0^\ell \frac{\partial z(x, \tau)}{\partial t} \left(\int_0^x \frac{\partial z(\xi, \tau)}{\partial t} d\xi \right) dx + \int_0^\ell \int_0^\tau \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial^2 z}{\partial t^2} d\xi \right) dx dt \\ & - \int_0^\tau \int_0^\ell \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi \right) dx dt - \int_0^\ell \int_0^\tau a(x, t) \frac{\partial z}{\partial t} \frac{\partial z}{\partial x} dx dt \\ & - \int_0^\ell \int_0^\tau \frac{\partial^2 a}{\partial x \partial t} \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial x} dx dt - \int_0^\tau \int_0^\ell a(x, t) \frac{\partial^2 z}{\partial x^2} \frac{\partial z}{\partial t} dx dt - \int_0^\tau \int_0^\ell \frac{\partial a(x, t)}{\partial x} \frac{\partial z}{\partial x} \frac{\partial z}{\partial t} dx dt \\ & + \int_0^\ell a(x, \tau) \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial t^2} dx + \int_0^\ell \frac{\partial a(x, \tau)}{\partial x} \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial t^2} dx - \int_0^\ell \int_0^\tau a(x, t) \frac{\partial^2 z}{\partial t^2} \frac{\partial^3 z}{\partial t \partial x^2} dx dt = 0. \end{aligned} \quad (4.11)$$

The application of Young's inequality and (3.14) to the terms of the above equality gives

$$- \int_0^\ell \frac{\partial z(x, \tau)}{\partial t} \frac{\partial^2 z(x, \tau)}{\partial t^2} dx \leq \frac{1}{2} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^\ell \left(\frac{\partial^2 z(x, \tau)}{\partial t^2} \right)^2 dx, \quad (4.12)$$

$$-\int_0^\ell \frac{\partial z(x, \tau)}{\partial t} \left(\int_0^x \frac{\partial z(\xi, \tau)}{\partial t} d\xi \right) dx \leq \frac{8+\ell}{8} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial t} \right)^2 dx, \quad (4.13)$$

$$\int_{\Omega_\tau} \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial^2 z}{\partial t^2} \right) dx dt \leq \frac{1}{2} \int_{\Omega_\tau} \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \frac{\ell}{4} \int_{\Omega_\tau} \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt, \quad (4.14)$$

$$-\int_0^\ell \int_0^\tau \frac{\partial z}{\partial t} \left(\int_0^x \frac{\partial z}{\partial t}(\xi, t) d\xi \right) dx dt \leq \frac{2+\ell}{4} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt, \quad (4.15)$$

$$\begin{aligned} & -\int_0^\ell \int_0^\tau a(x, t) \frac{\partial z}{\partial t} \frac{\partial z}{\partial x} dx dt - \int_0^\ell \int_0^\tau \frac{\partial a(x, t)}{\partial x} \frac{\partial z}{\partial x} \frac{\partial z}{\partial t} dx dt \\ & \leq \frac{a_1 + a_3 \varepsilon}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \frac{a_1 \varepsilon + a_3}{2\varepsilon} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \end{aligned} \quad (4.16)$$

$$-\int_0^\ell \int_0^\tau \frac{\partial^2 a}{\partial x \partial t} \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial x} dx dt \leq \frac{a_7}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \frac{a_7}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt, \quad (4.17)$$

$$\int_0^\ell \frac{\partial a(x, \tau)}{\partial x} \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial t^2} dx \leq \frac{a_3}{2} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial x} \right)^2 dx + \frac{a_3}{2} \int_0^\ell \left(\frac{\partial^2 z(x, \tau)}{\partial t^2} \right)^2 dx, \quad (4.18)$$

$$\int_0^\ell a(x, \tau) \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial t^2} dx \leq \frac{a_1}{2} \int_0^\ell \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx + \frac{a_1}{2} \int_0^\ell \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx, \quad (4.19)$$

$$-\int_0^\ell \int_0^\tau a(x, t) \frac{\partial^2 z}{\partial t^2} \frac{\partial^3 z}{\partial t \partial x^2} dx dt \leq \frac{a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt + \frac{a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx dt. \quad (4.20)$$

$$-\int_0^\ell \int_0^\tau a(x, t) \frac{\partial^2 z}{\partial x^2} \frac{\partial z}{\partial t} dx dt \leq \frac{a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx dt + \frac{a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt, \quad (4.21)$$

If we substitute (4.12), (4.13), (4.14), (4.14), (4.15), (4.16), (4.17), (4.18), (4.19), (4.20) and (4.21) into (4.11), we obtain

$$\begin{aligned} & -\frac{a_3}{2} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial x} \right)^2 dx - \frac{a_1 + 1}{2} \int_0^\ell \left(\frac{\partial^2 z(x, \tau)}{\partial x^2} \right)^2 dx - T \int_0^\ell \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx \\ & -\frac{\ell}{8} \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial t} \right)^2 dx - \frac{a_3 + a_1 \varepsilon - 1}{2} \int_0^\ell \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx \\ & \leq \frac{(2 + a_7)\varepsilon + a_1 \varepsilon + a_3}{2\varepsilon} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \frac{2 + a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx dt \\ & + \frac{3 + \ell 2a_1 + 2a_3 \varepsilon + 2a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt \\ & + \frac{\ell + 2a_7}{4} \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt + \frac{2 + a_1}{2} \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx dt. \end{aligned} \quad (4.22)$$

Hence, if $\varepsilon > 0$, and $a_3 + \varepsilon a_1 - 1 = 1$, then (4.26) implies that

$$\begin{aligned} & - \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial x} \right)^2 dx - \int_0^\ell \left(\frac{\partial^2 z(x, \tau)}{\partial x^2} \right)^2 dx - \int_0^\ell \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx - \int_0^\ell \left(\frac{\partial z(x, \tau)}{\partial t} \right)^2 dx \\ & - \int_0^\ell \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx \leq \beta \left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt \right. \\ & \left. + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx dt \right]. \end{aligned} \quad (4.23)$$

Where

$$\beta = \frac{\beta_2}{\beta_1},$$

$$\beta_1 = \min \left(\frac{a_3}{2}, \frac{1+a_1}{2}, T, \frac{\ell}{4}, \frac{1}{2} \right),$$

$$\beta_2 = \max \left(\frac{(2+a_7)(2+a_3)+2a_1}{2+a_3}, \frac{\ell+2a_7}{4}, \frac{2+a_1}{2} \right).$$

It follows from (4.27),

$$\begin{aligned} & - \frac{\partial}{\partial \tau} \left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt \right. \\ & \left. + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx dt \right] \leq \beta \left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx dt \right. \\ & \left. + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx dt \right]. \end{aligned} \quad (4.24)$$

An integration of (4.28) over $[0, \tau]$ gives

$$\begin{aligned} & \left[\int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial z}{\partial t} \right)^2 dx dt \right. \\ & \left. + \int_0^\ell \int_0^\tau \left(\frac{\partial^2 z}{\partial t^2} \right)^2 dx dt + \int_0^\ell \int_0^\tau \left(\frac{\partial^3 z}{\partial t \partial x^2} \right)^2 dx dt \right] e^{\beta \tau} \leq 0 \quad \forall \tau \in [0, T]. \end{aligned} \quad (4.25)$$

It follows the above inequality (4.21) that $w = 0$ almost everywhere on the Domaine $(0, \ell) \times [0, \tau]$. The length τ does not depend on the origin, so we can proceed in the same way a finite number of times to show that $w = 0$ in $(0, \ell) \times [0, T]$. Thus, the proof of Theorem (4.1) is completed. $\square$

Theorem 4.2. Under the conditions of the Theorem 4.1, the range $R(A)$ of the operator is dense in F .

Proof

Since F is a Hilbert space, $R(A)$ is dense in F is equivalent to the orthogonality of the vector $\eta = (\eta_0, \eta_1, \eta_2) \in F$ to the set $R(A)$, that is, if and only if the relation

$$\begin{aligned} & \int_0^T \int_0^\ell \frac{\partial^2 z}{\partial t^2} \eta_0(x, t) dx dt + \int_0^T \int_0^\ell \frac{\partial z}{\partial t} \eta_0(x, t) dx dt - \int_0^T \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial z}{\partial x} \right) \eta_0(x, t) dx dt \\ & + \int_0^\ell \varphi_1(x) \eta_1(x) dx + \int_0^\ell \varphi_2(x) \eta_2(x) dx = 0, \end{aligned} \quad (4.26)$$

where z runs over E and $\eta = (\eta_0, \eta_1, \eta_2) \in F$, implies $\eta = 0$.

Putting $z \in D_0(A)$ in (4.26), we obtain

$$\int_0^T \int_0^\ell \frac{\partial^2 z}{\partial t^2} \eta_0(x, t) dx dt + \int_0^T \int_0^\ell \frac{\partial z}{\partial t} \eta_1(x, t) dx dt - \int_0^T \int_0^\ell \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial}{\partial x} \right) \eta_2(x, t) dx dt. \quad (4.27)$$

Therefore, the Theorem(4.1) implies that $\eta_0 = 0$. Thus (4.26) becomes

$$\int_0^\ell \varphi_1(x) \eta_1(x) dx + \int_0^\ell \varphi_2(x) \eta_2(x) dx = 0. \quad (4.28)$$

Since the operators $z(x, t) \mapsto \varphi_1(x)$ and $z(x, t) \mapsto \frac{\partial z}{\partial t}(x, 0) = \varphi_2(x)$ are dense in $L^2(0, \ell)$, the last relation (4.28) gives $\eta_1 = \eta_2 = 0$. Hence $\eta \equiv 0$, and thus the closed $\overline{R(A)}$ of $R(A)$ is equal to F . This completed the proof of Theorem 4.2. $\square$

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