BERNOULLI POLYNOMIALS AND CONVOLUTION SUMS

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Abstract In this paper, we study the summations $\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b)+f(b+c)+f(c+a))$ and $\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4\\ a+b+c+d=n}} (f(a+b+c)+f(b+c+d)+f(c+d+a))$. Especially, for the first summation we obtain some relations related to Bernoulli and Euler polynomials.

1. Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan. For $n \in \mathbb{N}$ with $l \in \mathbb{N} \cup \{0\}$, we define some notations which also appear in many areas of number theory:

$$\sigma_l(n) := \sum_{d|n} d^l, \quad S_l(n) = \sum_{i=1}^{n-1} i^l \quad (n \neq 1), \quad S_l(1) = 0,$$

and $[x]$ called the *floor* of x, denotes the greatest integer that does not exceed x.

The Bernoulli polynomials $B_m(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!},$$

play an important role in various areas of mathematics, including number theory and the theory of finite differences (see [2]). The Bernoulli polynomials satisfy the following well-known identities :

$$B_m(x+1) - B_m(x) = mx^{m-1} \quad (m \in \mathbb{N})$$

and

$$\sum_{j=0}^{n} j^m = \frac{B_{m+1}(n+1) - B_{m+1}(0)}{m+1} \quad (m \in \mathbb{N}), \tag{1}$$

where $\mathbb{N}$ denotes the set of positive integers. We set $B_m := B_m(0)$. It is obvious , from the way the polynomials $B_m(x)$ are constructed , that all the B_m are rational numbers. It can be shown that $B_{2m+1} = 0$ for $m \geq 1$, and is alternatively positive and negative for even m.

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Let n be a positive integer and let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. In 1858 Liouville stated the surprising result that the sum called the third identity of Liouvile

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (f(a-b) - f(a+b))$$
$$= f(0)\left(\sigma_1(n) - \sigma_0(n)\right) + \sum_{\substack{d\in\mathbb{N}\\ d|n}} \left(1 + \frac{2n}{d} - d\right) f(d) - 2\sum_{\substack{d\in\mathbb{N}\\ d|n}} \left(\sum_{v=1}^{d} f(v)\right)$$

can be evaluated in terms of sums over the positive integers d dividing n (see [5, p. 247]). In 2000 Huard, Ou, Spearman and Williams (see [7]) proved a far reaching generalization of Liouville's Identity as Proposition 1.1. We introduce Proposition 1.1 which is the motivation of our results Lemma 2.1 and Lemma 2.3.

Proposition 1.1. [7, Theorem 13.1] *Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$f(a,b,x,y) - f(x,y,a,b) = f(-a,-b,x,y) - f(x,y,-a,-b) \tag{2}$$

for all integers a, b, x and y. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (f(a,b,x,-y) - f(a,-b,x,y) + f(a,a-b,x+y,y) \\
&\qquad - f(a,a+b,y-x,y) + f(b-a,b,x,x+y) - f(a+b,b,x,x-y)) \\
&= \sum_{\substack{d\in\mathbb{N}\\ d|n}} \sum_{\substack{x\in\mathbb{N}\\ x<d}} (f(0,\frac{n}{d},x,d) + f(\frac{n}{d},0,d,x) + f(\frac{n}{d},\frac{n}{d},d-x,-x) \\
&\qquad - f(x,x-d,\frac{n}{d},\frac{n}{d}) - f(x,d,0,\frac{n}{d}) - f(d,x,\frac{n}{d},0)),
\end{aligned} \tag{3}$$

where the sum on the left hand side of (3) is over all positive integers a, b, x, y satisfying $ax + by = n$, the inner sum on the right hand side is over all positive integers x satisfying $x < d$, and the outer sum on the right hand side is over all positive integers d dividing n. (See also [3, Theorem 1]*)*

In this paper we will find a special formula of the summation

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$$

and so , by using this, we will present some relations related to Bernoulli polynomial and the generalized Riemann zeta function :

Theorem 1.2. *Let $n \in \mathbb{N}$. If $l \in \mathbb{N}$, then we have*

$$A(l,n) := \sum_{i=0}^{l}\binom{l}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{l+1-i+j}\sum_{m=0}^{l-i+j}(-1)^{m+j}$$
$$\times\binom{l+1-i+j}{m}B_m\sum_{k=1}^{n-2}(n-k)^{i-j}(n-k-1)^{l+1-i+j-m}$$
$$=\frac{(l+1)\left(B_{l+2}(n)-B_{l+2}\right)-(l+2)\left(B_{l+1}(n)-B_{l+1}\right)}{(l+2)(l+1)}.$$

Furthermore, if $l\in\mathbb{N}\cup\{0\}$, then we obtain

$$A(l,n)=S_{l+1}(n)-S_l(n).$$

Corollary 1.3. *Let $t,l\in\mathbb{N}\cup\{0\}$. Then we have*

(a)
$$\sum_{l=0}^{t}A(l,n)=S_{t+1}(n)-S_0(n).$$

(b) *Let $\overline{A(l,n)}=S_{l+1}(n)+S_l(n)$. So*

$$\sum_{l=0}^{t}A(l,n)\overline{A(l,n)}=S_{t+1}^2(n)-S_0^2(n).$$

Corollary 1.4. *Let $l,n\in\mathbb{N}$. Then we have*

$$2^{l+1}A\left(l,\left[\frac{n+1}{2}\right]\right)-A(l,n)+2^{l+1}S_{l+1}\left(\left[\frac{n+1}{2}\right]\right)$$
$$=\frac{E_{l+1}(0)-E_l(0)-(-1)^n\left(E_{l+1}(n)-E_l(n)\right)}{2},$$

where Euler polynomials $E_l(x)$ are defined by $\sum_{l=0}^{\infty}E_l(x)\frac{t^l}{l!}=\frac{2e^{xt}}{e^t+1}$.

2. Proofs of Lemma 2.1, Theorem 1.2 and Corollary 1.3

Using the four basic theta functions, one is led to the formula (see [7, p. 14]) :

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2n\\ a,b,x,y\ \text{odd}}}\left((a+b)^{2k}-(a-b)^{2k}\right)=\sum_{\substack{m\in\mathbb{N}\\ m|n\\ \frac{n}{m}\ \text{odd}}}2^{2k}m^{2k+1},\quad (k,n\in\mathbb{N})\,. \tag{4}$$

There are several arithmetical formulae due to Liouville. A classical one of Liouville very often used in elementary number theory (completely avoiding the hard analysis) is that (see [4, p. 144]) : For $n\in\mathbb{N}$ and for $f:\mathbb{Z}\to\mathbb{C}$ an even function, the identity

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2n\\ a,b,x,y\ \text{odd}}}\left(f(a+b)-f(a-b)\right)=\sum_{\substack{m\in\mathbb{N}\\ m|n\\ \frac{n}{m}\ \text{odd}}}\left(f(2m)-f(0)\right) \tag{5}$$

holds.

Lemma 2.1. *Let $n \geq 3$ be a positive integer. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) = 3\sum_{k=1}^{n-2} (n-k-1)\, f(n-k).$$

Proof. We note that

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\
&= 3\sum_{a+b+c=n} f(a+b) = 3\sum_{k\geq 2} f(k) \sum_{\substack{a+b+c=n \\ a+b=k}} 1 \\
&= 3\left\{(n-2)f(n-1) + (n-3)f(n-2) + \cdots + (n-(n-1))\, f(2)\right\} \\
&= 3\sum_{k=1}^{n-2} (n-k-1)\, f(n-k).
\end{aligned}$$

□

We can see the arithmetic function $F_l(n)$ given by

$$F_l(n) := \begin{cases} 1, & l \mid n, \\ 0, & l \nmid n \end{cases}$$

for $l, n \in \mathbb{N}$ in [7, p. 32]. Using this function we introduce Example 2.2 and Example 2.4.

Example 2.2. *Let us consider the function $f(x) = F_2(x)$ in Lemma 2.1. If n is even, then the left hand side of Lemma 2.1 is*

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ n\in\mathbb{N} \\ a+b+c=2n}} (f(a+b) + f(b+c) + f(c+a)) \\
&= 3\sum_{a+b+c=2n} F_2(a+b) = 3\sum_{\substack{a+b+c=2n \\ 2\mid(a+b)}} 1
\end{aligned}$$

and the right hand side of Lemma 2.1 is

$$
\begin{aligned}
&3\sum_{k=1}^{2n-2}(2n-k-1)\,f(2n-k) = 3\sum_{k=1}^{2n-2}(2n-k-1)\,F_2(2n-k)\\
&= 3\sum_{\substack{k=1\\2|(2n-k)}}^{2n-2}(2n-k-1) = 3\sum_{\substack{k=1\\2|k}}^{2n-2}(2n-k-1)\\
&= 3\sum_{k=1}^{n-1}(2n-2k-1) = 3(n-1)^2,
\end{aligned}
$$

where we use $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. *These facts show that*

$$
\sum_{\substack{a+b+c=2n\\2|(a+b)}} 1 = (n-1)^2 \tag{6}
$$

Also, if n *is odd, then we obtain*

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\n\in\mathbb{N}\\a+b+c=2n-1}}(f(a+b)+f(b+c)+f(c+a))\\
&= 3\sum_{a+b+c=2n-1}F_2(a+b) = 3\sum_{\substack{a+b+c=2n-1\\2|(a+b)}}1\\
&= 3\sum_{k=1}^{2n-3}(2n-2-k)\,F_2(2n-1-k) = 3\sum_{\substack{k=1\\2|(2n-1-k)}}^{2n-3}(2n-2-k)\\
&= 3\sum_{\substack{k=1\\2\nmid k}}^{2n-3}(2n-2-k) = 3\left\{\sum_{k=1}^{2n-3}(2n-2-k) - \sum_{\substack{k=1\\2|k}}^{2n-3}(2n-2-k)\right\}\\
&= 3\left\{\sum_{k=1}^{2n-3}(2n-2-k) - \sum_{k=1}^{n-2}(2n-2-2k)\right\} = 3(n-1)^2.
\end{aligned} \tag{7}
$$

Therefore, by (6) *and* (7), *we conclude that*

$$
\sum_{\substack{n\in\mathbb{N}\\a+b+c=n\\2|(a+b)}} 1 = \left(\left[\frac{n+1}{2}\right]-1\right)^2.
$$

We can prove Lemma 2.3 in a similar manner as in the proof of Lemma 2.1 as follows :

Lemma 2.3. *Let $n \geq 4$ be a positive integer. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4\\ a+b+c+d=n}} (f(a+b+c)+f(b+c+d)+f(c+d+a)) = \frac{3}{2}\sum_{k=1}^{n-1}(k-1)(k-2)f(k).$$

Proof. We observe that

$$\begin{aligned}
&\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4\\ a+b+c+d=n}} (f(a+b+c)+f(b+c+d)+f(c+d+a))\\
&= 3\sum_{a+b+c+d=n} f(a+b+c) = 3\sum_{k\geq 3} f(k) \sum_{\substack{a+b+c+d=n\\ a+b+c=k}} 1\\
&= 3\sum_{k=3}^{n-1} f(k)\left\{1+2+...+(k-2)\right\} = 3\sum_{k=3}^{n-1} f(k)\cdot\frac{(k-1)(k-2)}{2}\\
&= \frac{3}{2}\sum_{k=1}^{n-1}(k-1)(k-2)f(k).
\end{aligned}$$

□

Example 2.4. *We apply the function $f(x) = F_2(x)$ in Lemma 2.3. If n is even, then the left hand side of Lemma 2.3 is*

$$\begin{aligned}
&\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4\\ n\in\mathbb{N}\\ a+b+c+d=2n}} (f(a+b+c)+f(b+c+d)+f(c+d+a))\\
&= 3\sum_{a+b+c+d=2n} F_2(a+b+c) = 3\sum_{\substack{a+b+c+d=2n\\ 2|(a+b+c)}} 1
\end{aligned}$$

and the right hand side of Lemma 2.3 is

$$\begin{aligned}
&\frac{3}{2}\sum_{k=1}^{2n-1}(k-1)(k-2)f(k) = \frac{3}{2}\sum_{k=1}^{2n-1}(k-1)(k-2)F_2(k)\\
&= \frac{3}{2}\sum_{\substack{k=1\\ 2|k}}^{2n-1}(k-1)(k-2) = \frac{3}{2}\sum_{k=1}^{n-1}(2k-1)(2k-2)\\
&= \frac{3}{2}\sum_{k=1}^{n-1}\left\{4k^2-6k+2\right\} = \frac{1}{2}\left\{4n^3-15n^2+17n-6\right\},
\end{aligned}$$

where we use $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. These facts lead us to

$$\sum_{\substack{a+b+c+d=2n\\ 2|(a+b+c)}} 1 = \frac{1}{6}\left\{4n^3-15n^2+17n-6\right\}. \tag{8}$$

Similarly, if n is odd, then we have

$$
\begin{aligned}
&\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4\\ n\in\mathbb{N}\\ a+b+c+d=2n-1}} (f(a+b+c)+f(b+c+d)+f(c+d+a)) \\
&= 3\sum_{a+b+c+d=2n-1} F_2(a+b+c) = 3\sum_{\substack{a+b+c+d=2n-1\\ 2|(a+b+c)}} 1 \\
&= \frac{3}{2}\sum_{k=1}^{2n-2}(k-1)(k-2)F_2(k) = \frac{3}{2}\sum_{\substack{k=1\\ 2|k}}^{2n-2}(k-1)(k-2) \\
&= \frac{3}{2}\sum_{k=1}^{n-1}(2k-1)(2k-2) = \frac{1}{2}\left\{4n^3-15n^2+17n-6\right\}.
\end{aligned} \tag{9}
$$

Therefore, by (8) *and* (9)*, we conclude that*

$$
\sum_{\substack{n\in\mathbb{N}\\ a+b+c+d=n\\ 2|(a+b+c)}} 1 = \frac{1}{6}\left\{4\left[\frac{n+1}{2}\right]^3 - 15\left[\frac{n+1}{2}\right]^2 + 17\left[\frac{n+1}{2}\right] - 6\right\}.
$$

***Proof of* Theorem 1.2.** Let $l \in \mathbb{N}\cup\{0\}$. We apply $f(x)=x^l$ in Lemma 2.1. Then the left hand side is

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b)+f(b+c)+f(c+a)) \\
&= 3\sum_{a+b+c=n} f(a+b) = 3\sum_{a+b+c=n}(a+b)^l \\
&= 3\sum_{a+b+c=n}\sum_{i=0}^{l}\binom{l}{i}a^{l-i}b^i \\
&= 3\sum_{i=0}^{l}\binom{l}{i}\sum_{a+b+c=n} a^{l-i}b^ic^0.
\end{aligned} \tag{10}
$$

Here, we can observe that

$$
\begin{aligned}
&\sum_{a+b+c=n} a^{l-i}b^ic^0 \\
&= \sum_{a+b=n-1} a^{l-i}b^i1^0 + \sum_{a+b=n-2} a^{l-i}b^i2^0 + \cdots + \sum_{a+b=2} a^{l-i}b^i(n-2)^0 \\
&= \sum_{a+b=n-1} a^{l-i}b^i + \sum_{a+b=n-2} a^{l-i}b^i + \cdots + \sum_{a+b=2} a^{l-i}b^i.
\end{aligned} \tag{11}
$$

Then we can calculate the first term of (11) as follows:

$$
\begin{aligned}
&\sum_{a+b=n-1} a^{l-i}b^i = \sum_{a=1}^{n-2} a^{l-i}\left\{(n-1)-a\right\}^i \\
&= \sum_{a=1}^{n-2} a^{l-i} \sum_{j=0}^{i} \binom{i}{j} (n-1)^{i-j}(-1)^j a^j \\
&= \sum_{j=0}^{i} \binom{i}{j} (n-1)^{i-j}(-1)^j \sum_{a=1}^{n-2} a^{l-i+j} \\
&= \sum_{j=0}^{i} \binom{i}{j} (n-1)^{i-j}(-1)^j \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^m \binom{l+1-i+j}{m} \\
&\qquad \times B_m (n-2)^{l+1-i+j-m} \\
&= \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
&\qquad \times (n-1)^{i-j}(n-2)^{l+1-i+j-m},
\end{aligned}
$$

where we use the Faulhaber's sum (see [1]) as

$$
\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j n^{p+1-j}
$$

with the Bernoulli numbers B_j. Also we can apply the same method into the other terms of (11). So (11) becomes

$$
\begin{aligned}
&\sum_{a+b+c=n} a^{l-i}b^i c^0 \\
&= \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
&\qquad \times \Big\{ (n-1)^{i-j}(n-2)^{l+1-i+j-m} + (n-2)^{i-j}(n-3)^{l+1-i+j-m} \\
&\qquad + \cdots + (n-(n-2))^{i-j}\,(n-(n-1))^{l+1-i+j-m} \Big\} \\
&= \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
&\qquad \times \sum_{k=1}^{n-2} (n-k)^{i-j}(n-k-1)^{l+1-i+j-m}.
\end{aligned}
$$

So (10) can be written as

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b)+f(b+c)+f(c+a)) \\
&= 3\sum_{i=0}^{l}\binom{l}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{l+1-i+j}\sum_{m=0}^{l-i+j}(-1)^{m+j} \\
&\quad\times\binom{l+1-i+j}{m}B_m\sum_{k=1}^{n-2}(n-k)^{i-j}(n-k-1)^{l+1-i+j-m}.
\end{aligned}
\tag{12}
$$

On the other hand, the right hand side of Lemma 2.1 is

$$
\begin{aligned}
&3\sum_{k=1}^{n-2}(n-k-1)(n-k)^l \\
&= 3\sum_{k=1}^{n-2}\left\{(n-k)^{l+1}-(n-k)^l\right\} \\
&= 3\left[\left\{(n-1)^{l+1}-(n-1)^l\right\}+\left\{(n-2)^{l+1}-(n-2)^l\right\}\right. \\
&\qquad \left.+\cdots+\left\{3^{l+1}-3^l\right\}+\left\{2^{l+1}-2^l\right\}\right] \\
&= 3\left\{\sum_{k=1}^{n-1}k^{l+1}-\sum_{k=1}^{n-1}k^l\right\}.
\end{aligned}
\tag{13}
$$

Equating (12) and (13), we obtain

$$
\begin{aligned}
A(l,n) &:= \sum_{i=0}^{l}\binom{l}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{l+1-i+j}\sum_{m=0}^{l-i+j}(-1)^{m+j} \\
&\quad\times\binom{l+1-i+j}{m}B_m\sum_{k=1}^{n-2}(n-k)^{i-j}(n-k-1)^{l+1-i+j-m} \\
&= \sum_{k=1}^{n-1}k^{l+1}-\sum_{k=1}^{n-1}k^l.
\end{aligned}
\tag{14}
$$

For $l\in\mathbb{N}\cup\{0\}$, we can write $A(l,n)=S_{l+1}(n)-S_l(n)$.
Now, let $l\in\mathbb{N}$. Then, by (1) and $B_l=B_l(0)$, the right hand side of (14) is

$$
\begin{aligned}
&\sum_{k=1}^{n-1}k^{l+1}-\sum_{k=1}^{n-1}k^l \\
&= \frac{B_{l+2}(n)-B_{l+2}(0)}{l+2}-\frac{B_{l+1}(n)-B_{l+1}(0)}{l+1} \\
&= \frac{(l+1)\left(B_{l+2}(n)-B_{l+2}\right)-(l+2)\left(B_{l+1}(n)-B_{l+1}\right)}{(l+2)(l+1)},
\end{aligned}
\tag{15}
$$

so that we have

$$A(l,n) = \frac{(l+1)\left(B_{l+2}(n) - B_{l+2}\right) - (l+2)\left(B_{l+1}(n) - B_{l+1}\right)}{(l+2)(l+1)}.$$

□

Remark 2.5. We recall the Riemann zeta function and the generalized Riemann (or Hurwitz) zeta function are given by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta(s,q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}$, respectively. Equation (13) can be written as

$$\begin{aligned} &3\sum_{k=1}^{n-2}(n-k-1)(n-k)^l \\ &= 3\sum_{k=1}^{n-2}\left\{(n-k)^{l+1} - (n-k)^l\right\} \\ &= 3\left\{\zeta(-l,n) - \zeta(-l-1,n) - \zeta(-l) + \zeta(-l-1)\right\}. \end{aligned} \tag{16}$$

Indeed, observe that

$$\begin{aligned} &\sum_{k=1}^{n-2}\left\{(n-k)^{l+1} - (n-k)^l\right\} \\ &= \left\{1^{l+1} + 2^{l+1} + 3^{l+1} + \cdots + (n-2)^{l+1} + (n-1)^{l+1}\right\} \\ &\quad - \left\{1^l + 2^l + 3^l + \cdots + (n-2)^l + (n-1)^l\right\} \end{aligned}$$

and

$$\begin{aligned} &\zeta(-l-1) - \zeta(-l-1,n) = \sum_{n=1}^{\infty} n^{l+1} - \sum_{k=0}^{\infty}(k+n)^{l+1} \\ &= \left\{1^{l+1} + 2^{l+1} + \cdots + (n-1)^{l+1} + n^{l+1} + (n+1)^{l+1} + \cdots\right\} \\ &\quad - \left\{n^{l+1} + (n+1)^{l+1} + (n+2)^{l+1} + (n+3)^{l+1} + \cdots\right\} \\ &= 1^{l+1} + 2^{l+1} + 3^{l+1} + \cdots + (n-2)^{l+1} + (n-1)^{l+1}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \zeta(-l) - \zeta(-l,n) &= \sum_{n=1}^{\infty} n^l - \sum_{k=0}^{\infty}(k+n)^l \\ &= 1^l + 2^l + 3^l + \cdots + (n-2)^l + (n-1)^l. \end{aligned}$$

Thus we can rewrite (14) as

$$A(l,n) = \zeta(-l,n) - \zeta(-l-1,n) - \zeta(-l) + \zeta(-l-1).$$

Example 2.6. *If $l = 3$ in* Theorem 1.2, *then we have*

$$A(3,n) = \frac{1}{60}n\left(12n^4 - 45n^3 + 50n^2 - 15n - 2\right).$$

***Proof of* Corollary 1.3.** (a) From Theorem 1.2, we obtain

$$\sum_{l=0}^{t} A(l,n) = \sum_{l=0}^{t}\left\{\sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^{l}\right\} = \sum_{k=1}^{n-1}\sum_{l=0}^{t}\left\{k^{l+1}-k^{l}\right\}$$
$$= \sum_{k=1}^{n-1}\left\{(k-1)+(k^2-k)+(k^3-k^2)+\cdots+(k^t-k^{t-1})+(k^{t+1}-k^t)\right\}$$
$$= \sum_{k=1}^{n-1}\left\{k^{t+1}-1\right\} = S_{t+1}(n) - S_0(n).$$

(b) By the definition of $\overline{A(l,n)} = S_{l+1}(n) + S_l(n)$, we have

$$\sum_{l=0}^{t} A(l,n)\overline{A(l,n)} = \sum_{l=0}^{t}\left\{S_{l+1}^2(n) - S_l^2(n)\right\}$$
$$= (S_1^2(n) - S_0^2(n)) + (S_2^2(n) - S_1^2(n)) + (S_3^2(n) - S_2^2(n)) + \cdots$$
$$+ (S_t^2(n) - S_{t-1}^2(n)) + (S_{t+1}^2(n) - S_t^2(n))$$
$$= S_{t+1}^2(n) - S_0^2(n).$$

□

***Proof of* Corollary 1.4**. In [6, (2.1), Theorem 2.1] we can see that

$$T_l(n) := \sum_{i=0}^{n-1}(-1)^i i^l,$$
$$T_l(n) = \frac{E_l(0) - (-1)^n E_l(n)}{2} = 2^{l+1} S_l\left(\left[\frac{n+1}{2}\right]\right) - S_l(n),$$

for $l, n \in \mathbb{N}$. These facts lead us to

$$\begin{aligned} &T_{l+1}(n) - T_l(n) \\ &= \frac{E_{l+1}(0) - (-1)^n E_{l+1}(n)}{2} - \frac{E_l(0) - (-1)^n E_l(n)}{2} \\ &= \frac{E_{l+1}(0) - E_l(0) - (-1)^n\left(E_{l+1}(n) - E_l(n)\right)}{2}. \end{aligned} \tag{17}$$

On the other hand, $T_{l+1}(n) - T_l(n)$ can be rewritten as

$$\begin{aligned} &T_{l+1}(n) - T_l(n) \\ &= \left\{2^{l+2} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) - S_{l+1}(n)\right\} - \left\{2^{l+1} S_l\left(\left[\frac{n+1}{2}\right]\right) - S_l(n)\right\} \\ &= 2^{l+1}\left\{S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) - S_l\left(\left[\frac{n+1}{2}\right]\right)\right\} + 2^{l+1} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) \\ &\quad - \left\{S_{l+1}(n) - S_l(n)\right\} \\ &= 2^{l+1} A\left(l, \left[\frac{n+1}{2}\right]\right) + 2^{l+1} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) - A(l,n), \end{aligned} \tag{18}$$

where we use Theorem 1.2. Equating (17) and (18), we obtain the proof.

□

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