

A General Inequality Related to Variational Inequalities and its Consequences

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ABSTRACT

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty subset of X . Let $g: K \rightarrow \mathbb{R}$ be a real valued map. In this paper we study the following problem: "Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ ". Some results on this problem have been studied by Takahasi, Park and Kim and Yen. Behera and Panda proved several results on this problem in the setting of a Banach space. This problem includes as special cases, many problems on variational inequalities and generalized variational inequalities studied by many authors. As a consequence of the main results, we also consider the problem, "Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ " where $S: K \rightarrow 2^K$ is any point-to-set map which, includes as a special case, the classical quasi-variational inequality problem. A generalization of Minty's Lemma is also studied.

Keywords: Partition of unity, upper and lower **semicontinuous** point-to setmap, monotone operator

1. INTRODUCTION

Let X be a Hausdorff topological vector space with dual X^* and K a nonempty **closed and convex** subset of X . Let the value of $u \in X^*$ at $x \in X$ be denoted by (u, x) . Let $g: K \rightarrow \mathbb{R}$ be a map (possibly nonlinear). The classical minimization problem for the pair (g, K) is to find $x_0 \in K$ such that

$$g(x_0) = \min_{y \in K} g(y).$$

If we define a function $f: K \times K \rightarrow \mathbb{R}$ as $f(x, y) = g(y) - g(x)$ for all $x, y \in K$, then the above problem reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

If $T: K \rightarrow X^*$, then the nonlinear variational inequality problem is to find $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$. If we define $f: K \times K \rightarrow \mathbb{R}$ as $f(x, y) = (Tx, y - x)$, then it reduces to the problem of finding $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Motivated by the above facts, Behera and Panda [3] considered the following problem:

P_1 : Given $f: K \times K \rightarrow \mathbb{R}$, find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Certain variation of this problem is known as equilibrium problem. The readers are advised to refer [5], [6] and [13] for a detailed discussion on equilibrium problem and its generalization. Some similar results are also available in [20].

Special cases

1. If $z \in K$ is fixed and $f(x, y) = (T((z+x)/2), y-x)$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(T((z+x_0)/2), y-x_0) \geq 0$ for all $y \in K$, which is the *variational-type inequality problem* originally introduced by Behera and Panda[2].

2. If $f(x, y) = (Tx - Ax, y - x)$ for some maps $T, A : K \rightarrow X^*$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, y - x_0) \geq (Ax_0, y - x_0)$ for all $y \in K$ which is the *strongly nonlinear variational inequality problem* studied by Nanda[10] and Noor[11].

3. If $f(x, y) = (Tx - Ax, g(y) - g(x))$ for some maps $T, A : K \rightarrow X^*$, $g : K \rightarrow K$, then P_1 reduces to the problem of finding $x_0 \in K$ such that $(Tx_0, g(y) - g(x_0)) \geq (Ax_0, g(y) - g(x_0))$ for all $y \in K$, which is the *strongly nonlinear implicit variational inequality problem* introduced and studied by Noor[12] in connection with the solution of the differential equations of odd order.

Some results on Problem P_1 are available in Takahasi[18] and Park and Kim[14]. Behera and Panda [3] proved several results on the existence of solution of this problem in the setting of Banach spaces. Many authors used results on Problem P_1 to prove the existence of solution of variational and generalized variational inequality problems.

In this paper, we prove some results on the existence of solution of Problem P_1 under different assumptions in the setting of Hausdorff topological vector spaces. We also use a result on the existence of solution of Problem P_1 for the study the following problem, which is a generalization of the quasi-variational inequality problem introduced by Benoussan and Lions [4] in connection with impulse control and subsequently studied by Baicchi and Capello [1] and Mosco [9].

P_2 : Find $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in S(x_0)$ where $S : K \rightarrow 2^K$ is any point-to-set map.

We also prove a theorem on the uniqueness of solution of Problem P_1 which would serve as a generalization of the well-known Minty's Lemma (see [7], p.6).

2. PRILIMINARIES

In this section we recall some definitions and known results which will be needed in the sequel. Throughout this section, X is a real Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X .

2.1 Definition. A mapping $T : K \rightarrow X^*$ is said to be monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in K$. T is said to be strictly monotone if strict inequality holds whenever $x \neq y$. T is said to be pseudo-monotone if $x, y \in K$ and $(Tx, x - y) \geq 0$ then $(Ty, x - y) \geq 0$.

2.2 Definition. A function $h : K \rightarrow \mathbb{R}$ is said to be upper semi-continuous, if for each real number λ , the set $\{x \in K : h(x) < \lambda\}$ is open; f is said to be lower semi-continuous if $-f$ is upper semi-continuous.

2.3 Definition. A function $h: K \rightarrow \mathbb{R}$ is said to be quasi-convex if for each real number λ , the set $\{x \in K: h(x) < \lambda\}$ is convex; function f is said to be quasi-concave if $-f$ is quasi-convex.

2.4 Definition. Let M and N be two topological spaces, and $h: M \rightarrow 2^N$, a point-to-set map. Then h is said to be upper semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \subset U$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \subset U$; h is said to be lower semi-continuous at $x_0 \in M$ if for each open set U of N with $h(x_0) \cap U \neq \emptyset$, there exists a neighborhood $n(x_0)$ of x_0 in M such that if $x \in n(x_0)$ then $h(x) \cap U \neq \emptyset$; h is said to be upper semi-continuous (lower semi-continuous) on M if h is upper semi-continuous (lower semi-continuous) at each point of M .

2.5 Theorem. (Isac [8]) Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let f and g be two real valued functions on $K \times K$ having the following properties :

- (a) for all $x, y \in K$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq 0$,
- (b) for each fixed $y \in K$, $f(x, y)$ is a lower semicontinuous function of x on K ,
- (c) for each fixed $x \in K$, $g(x, y)$ is a quasi-concave function of x on K .

Then there exists a point $\hat{x} \in K$ such that $f(\hat{x}, y) \leq 0$ for all $y \in K$.

2.6 Theorem. (Shih and Tan [17]) Let X be a Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X . Let $S: K \rightarrow 2^K$ be an upper semicontinuous point-to-set map such that for each $x \in K$, $S(x)$ is nonempty and bounded. Then for each $p \in X^*$, the map $f_p: K \rightarrow \mathbb{R}$ defined by $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semicontinuous.

2.7 Theorem. (Takahasi [18]) Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X and let $F: K \times K \rightarrow \mathbb{R}$ be a function such that

- (a) for each $y \in K$, the map $x \mapsto F(x, y)$ is upper semicontinuous,
- (b) for each $x \in K$, the map $y \mapsto F(x, y)$ is convex,
- (c) $F(x, x) \geq \alpha$ for all $x \in K$ with some real number α .

Then there exists $x_0 \in K$ such that $F(x_0, y) \geq \alpha$ for all $y \in K$.

2.8 Theorem. (Rudin [15]) Let K be a compact subset of a topological space and let $\{V_1, V_2, \dots, V_n\}$ be a finite open covering of K . Then there exists a family $\{\beta_1, \beta_2, \dots, \beta_n\}$ of continuous real valued function on K such that $\beta_i(x) = 0$ outside V_i , $0 \leq \beta_i(x) \leq 1$ for all $i \in \{1, 2, \dots, n\}$ and for all $x \in K$, and $\sum_{i=1}^n \beta_i(x) = 1$ for all $x \in K$.

2.9 Theorem. (Tarafdar [19]) Let K be a nonempty subset of a Hausdorff topological vector space X and let $F: K \rightarrow 2^K$ be a point-to-set map such that

- (a) for each $x \in K$, $F(x)$ is nonempty,
- (b) for each $y \in K$, $F^{-1}(y) = \{x \in K: y \in F(x)\}$ contains a relatively open subset A_y of K (A_y may be empty for some y),
- (c) $\bigcup_{x \in K} A_x = K$,
- (d) K contains a nonempty set K_0 contained in a compact convex subset K_1 of K such that $D = \bigcap_{x \in K_0} A_x^c$ is compact (D may be empty).

Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.

2.10 Theorem. ([17], p. 34). Let X be a Hausdorff topological vector space with dual X^* and K a nonempty convex subset of X . Let $S: K \rightarrow 2^K$ be a point-to-set upper semi-continuous map such that for each $x \in K$, $S(x)$ is nonempty and bounded. Then for each $p \in X^*$ the map $f_p: K \rightarrow \mathbb{R}$ defined by $f_p(x) = \sup_{y \in S(x)} (p, y)$ is upper semi-continuous.

2.11 Theorem. ([16], p.201). Let X be a locally convex Hausdorff topological vector space with dual X^* and K a nonempty closed and convex subset of X . Let x^* be a point of X not in K . Then there exists $p \in X^*$ such that $(p, x^*) > \sup_{x \in K} (p, x)$.

3. EXISTENCE OF SOLUTION

The following theorems on the existence of solution of problem P_1 are the main results of this paper.

3.1 Theorem. Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- (a) $f(x, x) \geq 0$ for each $x \in K$,
- (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
- (c) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K .

Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Proof. Same as the proof of Theorem 2.7. Indeed Takahasi [18] assumed that for each $y \in K$, the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is upper semi-continuous and for each $x \in K$, the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex which are stronger than hypotheses (b) and (c) of this theorem.

The following theorem generalizes Theorem 3.1 to an arbitrary nonempty convex subset K of X .

3.2 Theorem. Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- (a) $f(x, x) \geq 0$ for each $x \in K$,
- (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
- (c) for each $y \in K$, the set $\{x \in C : f(x, y) < 0\}$ is open in K for every compact subset C of K ,
- (d) there exists a nonempty compact subset L of K such that for each $x \in K - L$, there exists $u \in L$ for which $f(x, u) < 0$.

Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

Proof. For each $y \in K$, let

$$A(y) = \{x \in L : f(x, y) \geq 0\}.$$

From hypothesis (c) it follows that $A(y)$ is a nonempty compact subset of K . The conclusion of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. To ascertain this, it is sufficient to show that the family $\{A(y) : y \in K\}$ has the finite intersection property.

Let $y_1, y_2, \dots, y_n$ be arbitrary elements of K and K_0 , the convex hull of $L \cup \{y_1, y_2, \dots, y_n\}$. Then K_0 is a nonempty compact convex subset of K . By Theorem 3.1, there exists $\tilde{x}_0 \in K_0$ such that

$$f(\tilde{x}_0, y) \geq 0 \quad \dots (1)$$

for all $y \in K_0$. We claim that $\tilde{x}_0 \in L$; for if $\tilde{x}_0 \notin L$, then $\tilde{x}_0 \in K_0 - L \subset K - L$ and by hypothesis (d), there exists $u \in L$ such that $f(\tilde{x}_0, u) < 0$ which contradicts (1) when $y = u$. Thus $\tilde{x}_0 \in L$, and in particular, $\tilde{x}_0 \in A(y_i)$ for $i = 1, 2, \dots, n$; that is $\tilde{x}_0 \in \bigcap_{i=1}^n A(y_i)$. Hence, $\bigcap_{i=1}^n A(y_i) \neq \emptyset$.

182 $\square$ which proves that the family $\{A(y) : y \in K\}$ has the finite intersection property. This
 183 completes the proof.
 184
 185

186 If X is a normed linear space and K is locally compact, we have the following result.
 187

188 **3.3 Theorem.** Let K be a nonempty locally compact convex subset of a real normed linear
 189 space X with $0 \in K$, and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 190 (a) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi-convex for each $x \in K$,
 191 (b) for all $y \in K$ and $r > 0$, the set $\{x \in K_r : f(x, y) < 0\}$ is open in K , where
 192 $K_r = \{x \in K : \|x\| \leq r\}$,
 193 (c) $f(x, x) = 0$ for each $x \in K$ and for each $r > 0$,
 194 (d) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.

195 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

196 *Proof.* For each $r > 0$, the set K_r is a nonempty compact and convex subset of K . Hence by
 197 Theorem 3.1 there exists $x_r \in K_r$ such that

$$198 \quad f(x_r, y) \geq 0 \quad \dots (2)$$

199 for all $y \in K_r$. We claim that $\|x_r\| < r$; for if $\|x_r\| = r$, then by hypothesis (d), $f(x_r, 0) < 0$
 200 which contradicts (2) when $y = 0$. Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently
 201 small so that

$$202 \quad y_r = \lambda x + (1-\lambda)x_r \in K_r.$$

203 Now by hypotheses (a) and (b)

$$204 \quad \begin{aligned} 0 \leq f(x_r, y_r) &= f(x_r, \lambda x + (1-\lambda)x_r) \\ &\leq \lambda f(x_r, x) + (1-\lambda) f(x_r, x_r) = \lambda f(x_r, x). \end{aligned}$$

205 Since $\lambda > 0$, it follows that $f(x_r, x) \geq 0$. Since x is arbitrary, the proof is complete.
 206

207 If X is a reflexive Banach space then local compactness of K can be relaxed from the above
 208 theorem.
 209

210 **3.4 Theorem.** Let K be a nonempty closed and convex subset of a reflexive real Banach
 211 space X with $0 \in K$ and $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 212 (a) $f(x, x) = 0$ for each $x \in K$,
 213 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 214 (c) the set $\{x \in C : f(x, y) < 0\}$ is weakly open in K for each $y \in K$, and for each
 215 weakly compact subset C of K .

216 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ under each of the following
 217 conditions:

- 218 (i) K is bounded,
 219 (ii) there exists $r > 0$ such that $f(x, 0) < 0$ whenever $\|x\| = r$.
 220

221 *Proof.* Equip K with the weak topology. If K is bounded, then it becomes weakly compact and
 222 in this case, the proof follows from Theorem 3.1.
 223

224 If K is not bounded let

$$K_r = \{x \in K : \|x\| \leq r\}.$$

225 Then K_r is a nonempty closed, convex and bounded subset of K . By the first part of the
 226 theorem, there exists $x_r \in K_r$ such that $f(x_r, y) \geq 0$ for all $y \in K_r$. The remaining part of the
 227 proof follows from the proof of Theorem 3.3 and hence it is omitted.

228
 229 The following theorem generalizes Theorem 3.4 to a Hausdorff topological vector space.

230
 231 **3.5 Theorem.** Let K be a nonempty convex subset of a Hausdorff topological vector space
 232 X and L , a nonempty compact and convex subset of K . Let $f: K \times K \rightarrow \mathbb{R}$ be a map such
 233 that

- 234 (a) $f(x, x) = 0$ for each $x \in L$,
 235 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 236 (c) for each $y \in K$, the set $\{x \in L : f(x, y) < 0\}$ is open in L ,
 237 (d) for every x in the boundary of L , there exists $u \in L$ such that $f(x, u) < 0$.

238 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

239 *Proof.* By theorem 3.1 there exists $x_0 \in L$ such that

240
$$f(x_0, y) \geq 0 \quad \dots (3)$$

 241 for all $y \in L$. We claim that $x_0 \in \text{int } L$ (where $\text{int } L$ stands for the interior of L), for if $x_0 \notin$
 242 $\text{int } L$, then by hypothesis (d) there exists $u \in L$ such that $f(x_0, u) < 0$ which contradicts (3)
 243 when $y = u$.

244 Now let $x \in K$ be arbitrary and choose $\lambda > 0$ sufficiently small so that

$$y_\lambda = \lambda x + (1-\lambda)x_0 \in L.$$

245 By hypotheses (a) and (b)

$$0 \leq f(x_0, y_\lambda) \leq \lambda f(x_0, x) + (1-\lambda)f(x_0, x_0) = \lambda f(x_0, x).$$

246 Since $\lambda > 0$, it follows that $f(x_0, x) \geq 0$. This completes the proof.

247
 248 The following theorem is a variant of Theorem 3.1 in which, the hypotheses (a) and (b) of
 249 Theorem 3.1 have been replaced by a single generalized condition.

250
 251 **3.6 Theorem.** Let K be a nonempty compact convex subset of a Hausdorff topological
 252 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 253 (a) for each $y \in K$, the set $\{x \in K : f(x, y) < 0\}$ is open in K ,
 254 (b) for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K and
 255 $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$, $\max_{1 \leq i \leq n} f(u, y_i) \geq 0$.

256 Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

257
 258 *Proof.* Assume that the assertion is false. Then for each $x \in K$, there exists $y \in K$ such that
 259 $f(x, y) < 0$. For each $y \in K$, let

$$A(y) = \{x \in K : f(x, y) < 0\}.$$

261 Then by hypothesis (a), $A(y)$ is open for every $y \in K$. Further $K = \bigcup_{y \in K} A(y)$. Since K is
 262 compact, there exists $\{y_1, y_2, \dots, y_n\} \subset K$ such that $K = \bigcup_{i=1}^n A(y_i)$. Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be a
 263 partition of unity subordinate to the covering $\{A(y_i) : i = 1, 2, \dots, n\}$ and let
 264 $S = \text{conv}(\{y_1, y_2, \dots, y_n\})$. Then S is a compact convex subset of K . Define a continuous map
 265 $p: K \rightarrow K$ as $p(x) = \sum_{i=1}^n \beta_i(x) y_i$. Since $p(x)$ is a convex linear combination of points of the
 266 set S , $p(x) \in S$ for each $x \in K$. In particular, p maps S into S . By Browder's fixed point
 267 theorem p has a fixed point; that is, there exists $\hat{x} \in K$ such that $\hat{x} = p(\hat{x})$.

268

269 Since $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a partition of unity, $\beta_i(\hat{x}) > 0$ for at least one i . But $\beta_i(\hat{x}) > 0$ implies
 270 that $\hat{x} \in A(y_i)$ so that
 271 $f(\hat{x}, y_i) < 0$ (4)

272 If

$$J = \{i: 1 \leq i \leq n \text{ and } \beta_i(\hat{x}) > 0\}$$

273 then

$$\hat{x} = p(\hat{x}) = \sum_{i=1}^n \beta_i(\hat{x}) y_i = \sum_{i \in J} \beta_i(\hat{x}) y_i$$

274 so that $\hat{x} \in \text{conv}\{y_i: i \in J\}$. Now by hypothesis (b), there exists $i_0 \in J$ such that $f(\hat{x}, y_{i_0}) \geq 0$
 275 which contradicts (4). This contradiction proves the theorem.

276

277 The following theorem generalizes Theorem 3.6 to an arbitrary nonempty convex subset K of
 278 X .

279

280 **3.7 Theorem.** Let X be a locally convex Hausdorff topological vector space, K a nonempty
 281 convex subset of X , K_0 a nonempty compact convex subset of K , and L a nonempty
 282 compact subset of K . Let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 283 (a) for each $y \in K$, the set $\{x \in L: f(x, y) \geq 0\}$ is closed in K ,
 284 (b) for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K , and
 285 $u \in \text{conv}(\{y_1, y_2, \dots, y_n\})$ $\max_{1 \leq i \leq n} f(u, y_i) \geq 0$.
 286 (c) for each $x \in K - L$, there exists $u \in \text{conv}(K_0 \cup \{x\})$ such that $f(x, u) < 0$.

287 Then there exists $x_0 \in L$ such that $f(x_0, y) \geq 0$ for all $y \in K$.

288

289 *Proof.* For each $y \in K$, let

$$290 A(y) = \{x \in L: f(x, y) \geq 0\}.$$

291 From (b) it follows that $A(y)$ is closed and consequently compact for each $y \in K$.
 292 The assertion of the theorem follows if $\bigcap_{y \in K} A(y) \neq \emptyset$. For this, it is sufficient to prove that the
 293 family $\{A(y): y \in K\}$ has the finite intersection property.

294

295 Let $\{y_1, y_2, \dots, y_n\} \subset K$ and $S = \text{conv}(K_0 \cup \{y_1, y_2, \dots, y_n\})$. Clearly S is a non-empty compact
 296 convex subset of K . By Theorem 3.6 there exists $\hat{x} \in S$ such that

$$297 f(\hat{x}, y) \geq 0 \quad \dots (5)$$

298 for all $y \in S$. We claim that $\hat{x} \in L$; for if $\hat{x} \notin L$, that is, $\hat{x} \in S - L \subset K - L$, then there exists
 299 $u \in \text{conv}(K_0 \cup \{\hat{x}\})$ such that $f(\hat{x}, y) < 0$ which contradicts (5) when $y = u$. Thus $\hat{x} \in L$, and in
 300 particular, $\hat{x} \in A(y_i)$ for each $i \in \{1, 2, \dots, n\}$, that is, $\hat{x}_0 \in \bigcap_{i=1}^n A(y_i)$. Hence, $\bigcap_{i=1}^n A(y_i) \neq \emptyset$
 301 showing that the family $\{A(y): y \in K\}$ has the finite intersection property. This completes the
 302 proof.

303

304 The following theorem also deals with the existence of solution of problem P_1 under pseudo-
 305 monotonicity type assumptions.

306

307 **3.8 Theorem.** Let K be a nonempty closed and convex subset of a Hausdorff topological
 308 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 309 (a) $f(x, x) \geq 0$ for each $x \in K$,
 310 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex and lower semicontinuous for each
 311 $x \in K$,
 312 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line
 313 segments of K to the usual topology of $\mathbb{R}$,
 314 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$,

315 (e) there exists a nonempty set K_0 contained in a compact convex subset K_1 of
 316 K such that $D = \bigcap_{y \in K_0} \{y \in K: f(x, y) \leq 0\}$ is either compact or empty.

317 Then there exists $x_0 \in K$ such that

$$318 f(x_0, y) \geq 0 \quad \dots(6)$$

319 for all $y \in K$.

320
 321 Note: If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then condition (d) of the
 322 above theorem is equivalent to the fact that T is pseudo-monotone.

323 To prove the above theorem, we need the following lemma.

324

325 **3.9 Lemma.** Let K be a nonempty closed and convex subset of a Hausdorff topological
 326 vector space X and let $f: K \times K \rightarrow \mathbb{R}$ be a map such that

- 327 (a) $f(x, x) \geq 0$ for each $x \in K$,
 328 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is convex for each $x \in K$,
 329 (c) the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous from the line segments
 330 of K to the usual topology of $\mathbb{R}$,
 331 (d) $x \in K, y \in K$ and $f(x, y) \geq 0$ implies $f(y, x) \leq 0$.

332 Then the following are equivalent:

- 333 (A) $x_0 \in K$ and $f(x_0, y) \geq 0$ for all $y \in K$.
 334 (B) $x_0 \in K$ and $f(y, x_0) \leq 0$ for all $y \in K$.

335

336 **Note:** If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$, then Lemma 3.9 reduces
 337 to Minty's Lemma ([7], p. 6).

338

339 *Proof.* Assume that (A) holds. Then for some $x_0 \in K$ and for all $y \in K, f(x_0, y) \geq 0$. By
 340 hypothesis (d), $f(y, x_0) \leq 0$ for all $y \in K$ from which (B) follows.

341

342 Now assume that (B) holds. Then for some $x_0 \in K, f(y, x_0) \leq 0$ for all $y \in K$. Fix $y \in K$,
 343 $t \in (0, 1)$ and set $y_t = (1 - t)x_0 + ty$. Then $y_t \in K$ and by (B), $f(y_t, x_0) \leq 0$ for each $t \in (0, 1)$.
 344 Now it follows from hypotheses (a) and (b) that

$$0 \leq f(y_t, y_t) \leq (1 - t)f(y_t, x_0) + tf(y_t, y).$$

345 Thus

$$f(y_t, y) \geq -\frac{1-t}{t}f(y_t, x_0) \geq 0$$

346 for each $t \in (0, 1)$. Since the map $x \mapsto f(x, y)$ of K into $\mathbb{R}$ is lower semicontinuous
 347 from the line segments of K to the usual topology of $\mathbb{R}$, taking limit $t \rightarrow 0^+$ in the above
 348 inequality, we get $f(x_0, y) \geq 0$. Since $y \in K$ is arbitrary (B) follows. This completes the proof
 349 of the lemma.

350

351 **3.10 Proof of Theorem 3.8.** Assume that inequality (6) has no solution. Then, for each
 352 $x \in K$, there exists $u \in K$ such that

$$353 f(x, u) < 0 \quad \dots(7)$$

354 and by Lemma 3.9, there exists $v \in K$ such that

$$355 f(v, x) > 0. \quad \dots(8)$$

356 Thus for each $x \in K$, it follows from (7) that the set

$$F(x) = \{y \in K: f(x, y) < 0\}$$

357 is nonempty, which is also convex by hypothesis (b) of the theorem. Again for each $y \in K$,
 358 since

$$F^{-1}(y) = \{x \in K: f(x, y) < 0\},$$

359 it follows that

$$\begin{aligned} [F^{-1}(y)]^c &= \{x \in K: f(x, y) \geq 0\} \\ &\subset \{x \in K: f(y, x) \leq 0\} \quad (\text{by (d)}) \\ &= G(y) \quad (\text{say}). \end{aligned}$$

362 Thus

$$363 \quad F^{-1}(y) \supset [G(y)]^c = A(y) \quad (\text{say}).$$

364 Since for each $x \in K$, the map $y \mapsto f(x, y)$ is lower semicontinuous, it follows that $G(y)$ is
365 closed in K . Thus for each $y \in K$, $A(y)$ is open in K .

366

367 Now, let $x \in K$ be arbitrary. By (8) there exists $v \in K$ such that $f(v, x) > 0$, so that $x \in$
368 $A(v)$. Thus $\bigcup_{y \in K} A(y) = K$. Finally, $D = \bigcap_{y \in K_0} G(y) = \bigcap_{y \in K_0} [A(y)]^c$ is either empty or
369 compact by hypothesis (e), so that all the conditions of Theorem 2.9 are fulfilled by F . Thus
370 there exists $x_0 \in K$ such that $x_0 \in F(x_0)$, which is tantamount to $f(x_0, x_0) < 0$, a contradiction
371 to hypothesis (a) of the theorem. Hence the conclusion of the theorem follows.

372

373

374 4. UNIQUENESS OF SOLUTIONS

375

376 The following theorem characterizes the uniqueness of solution of the Problem P_1 under
377 certain conditions.

378

379 **4.1 Theorem.** Let K be a nonempty subset of a topological vector space X and
380 let $f: K \times K \rightarrow \mathbb{R}$ a map such that

- 381 (a) for all $x, y \in K$, $f(x, y) + f(y, x) \leq 0$,
382 (b) equality does not hold in (a) unless $x = y$.

383 Then, if problem P_1 is solvable, it has a unique solution.

384

385 *Proof.* If possible let x_1, x_2 be two solutions of P_1 . Then

$$f(x_1, y) \geq 0$$

386 and

$$f(x_2, y) \geq 0$$

387 for all $y \in K$. Putting $y = x_1$ in the former inequality and $y = x_2$ in the latter, we get

$$f(x_1, x_2) \geq 0$$

388 and

$$f(x_2, x_1) \geq 0.$$

389 Adding the last two inequalities we get,

$$f(x_1, x_2) + f(x_2, x_1) \geq 0.$$

390 This inequality combined with hypothesis (a) gives

$$f(x_1, x_2) + f(x_2, x_1) = 0.$$

391 Now an application of hypothesis (b) yields $x_1 = x_2$. This completes the proof.

392 *Note:* If we set $f(x, y) = (Tx, y - x)$ for some operator $T: K \rightarrow X^*$ then condition (a) of the
393 above theorem is equivalent to the fact that T is monotone.

394

395 The following examples illustrate Theorem 4.1. Example 1 shows that the fulfillment of
396 conditions (a) and (b) does not guarantee the existence of a solution to problem P_1 .
397 Examples (2) and (3) show that if conditions (a) and (b) are fulfilled and problem P_1 is
398 solvable, then it has a unique solution.

399

400 **Example 1.** Let $X = \mathbb{R}$, $K = [0, \infty)$, and define let $f: K \times K \rightarrow \mathbb{R}$ as

$$401 \quad f(x, y) = -|x - y|.$$

402 Then

403 and
$$f(x, y) + f(y, x) = -2|x - y|$$

$$f(x, y) + f(y, x) = 0$$

 404 if $|x - y| = 0$, that is $x = y$. Thus hypotheses (a) and (b) of Theorem 4.1 are fulfilled by f .
 405 But $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to $|x_0 - y| \leq 0$ for all $y \in K$. Thus in this case no
 406 solution exists.

407
 408 **Example 2.** Let $X = \mathbb{R}$, $K = [0, \infty)$, and define $f: K \times K \rightarrow \mathbb{R}$ as

409
$$f(x, y) = x(y - x).$$

410 Then

411
$$f(x, y) + f(y, x) = -(x - y)^2 \leq 0$$

and

$$f(x, y) + f(y, x) = 0$$

412 if and only if $(x - y)^2 = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to
 413 $x_0(y - x_0) \geq 0$ for all $y \geq 0$, so that $x_0 = 0$ is the only solution in this case.

414
 415 **Example 3.** Let $X = K = \mathbb{R}$, and define $f: K \times K \rightarrow \mathbb{R}$ as

$$f(x, y) = -x^2|x - y|.$$

416 Then

$$f(x, y) + f(y, x) = -(x^2 + y^2)|x - y| \leq 0$$

417 and

$$f(x, y) + f(y, x) = 0$$

418 if and only if $|x - y| = 0$ i.e. $x = y$. Furthermore, $f(x_0, y) \geq 0$ for all $y \in K$ is equivalent to
 419 $x_0^2|x_0 - y| = 0$ for all $y \in \mathbb{R}$ so that $x_0 = 0$ is the only solution in this case also.

420

421

422 5. SOME CONSEQUENCES

423

424 In this section, we discuss some consequences of the theorems proved in Section 3.

425

426 The following theorem deals with the existence of solution of problem P_2 .

427

428 **5.1 Theorem.** Let X be a locally convex Hausdorff topological vector space with dual X^* , K a
 429 nonempty compact convex subset of X , $f: K \times K \rightarrow \mathbb{R}$ a point-to-point map, and $S: K \rightarrow 2^K$, a
 430 point-to-set upper semicontinuous map such that

431 (a) $f(x, x) \geq 0$ for each $x \in K$,

432 (b) the map $y \mapsto f(x, y)$ of K into $\mathbb{R}$ is quasi convex for each $x \in K$,

433 (c) the set $\{x \in K: \inf_{y \in S(x)} f(x, y) < 0\}$ is empty or open in K .

434 Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $f(x_0, y) \geq 0$ for all $y \in S(x_0)$.

435

436 *Proof.* Assume that the assertion is false. Then for each $x \in K$ either $x \notin S(x)$, or there exists
 437 $y \in S(x)$ such that $f(x, y) < 0$. If $x \notin S(x)$, then by Theorem 2.11, there exists $p \in X^*$ such
 438 that

$$(p, x) - \sup_{y \in S(x)} (p, y) > 0.$$

439 Let

$$V_0 = \left\{ x \in K: \inf_{y \in S(x)} f(x, y) < 0 \right\},$$

440 and for each $p \in X^*$ let

$$V(p) = \left\{ x \in K : (p, x) - \sup_{y \in S(x)} (p, y) > 0 \right\}.$$

441 Then by hypothesis (c), V_0 is open in K . By Theorem 2.6, since the map $x \mapsto \sup_{y \in S(x)} (p, y)$
 442 is upper semicontinuous for each $p \in X^*$, it follows that the map $x \mapsto (p, x) - \sup_{y \in S(x)} (p, y)$ is
 443 lower semicontinuous. Hence $V(p)$ is open in K for each $p \in X^*$. Further, $K = V_0 \cup$
 444 $\bigcup_{p \in X^*} V(p)$. Since K is compact, there exists $\{p_1, p_2, \dots, p_n\} \subset X^*$ such that $K = V_0 \cup$
 445 $[\bigcup_{i=1}^n V(p_i)]$. Let $\{\beta_0, \beta_1, \dots, \beta_n\}$ be a partition of unity subordinate to the covering
 446 $\{V_0, V(p_1), \dots, V(p_n)\}$, and define $g: K \times K \rightarrow \mathbb{R}$ as

$$g(x, y) = \beta_0(x)f(x, y) + \sum_{i=1}^n \beta_i(x)(p_i, y - x).$$

447 Then it is clear that for each $x \in K$, $g(x, x) \geq 0$, the map $y \mapsto g(x, y)$ is quasi-convex, for
 448 each $y \in K$, the map $x \mapsto g(x, y)$ is upper semicontinuous so that the set $\{x \in K : g(x, y) <$
 449 $0\}$ is open in K . Thus all the conditions of Theorem 3.1 are fulfilled. Hence there exists $\hat{x} \in K$
 450 such that

$$451 \quad g(\hat{x}, y) \geq 0 \quad \dots(9)$$

452 for all $y \in K$, which is equivalent to

$$\beta_0(\hat{x})f(\hat{x}, y) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, y - \hat{x}) \geq 0.$$

453 Since $\{\beta_0, \beta_1, \dots, \beta_n\}$ is a partition of unity $\beta_i(\hat{x}) > 0$ for at least one i . If $\beta_0(\hat{x}) > 0$ then $\hat{x} \in V_0$
 454 so that $\inf_{y \in S(\hat{x})} f(\hat{x}, y) < 0$. Let $\hat{y} \in S(\hat{x})$ be such that $f(\hat{x}, \hat{y}) < 0$. If $\beta_i(\hat{x}) > 0$ then $\hat{x} \in V(p_i)$
 455 so that

$$(p_i, \hat{x}) - \sup_{y \in S(\hat{x})} (p_i, y) > 0$$

456 from which it follows that

$$(p_i, \hat{x}) > \sup_{y \in S(\hat{x})} (p_i, y) > (p_i, \hat{y})$$

457 which implies that $(p_i, \hat{y} - \hat{x}) < 0$. Thus,

$$g(\hat{x}, \hat{y}) = \beta_0(\hat{x})f(\hat{x}, \hat{y}) + \sum_{i=1}^n \beta_i(\hat{x})(p_i, \hat{y} - \hat{x}) < 0$$

458 which contradicts (9) when $y = \hat{y}$. This contradiction proves the theorem.

459

460 The following result on the existence of solution of the quasi-variational inequality problem is
 461 a direct consequence of the above theorem.

462

463 **5.2 Corollary.** Let X be a locally convex Hausdorff topological vector space with dual X^* , K a
 464 nonempty compact convex subset of X , $T: K \rightarrow X^*$ an operator and $S: K \rightarrow 2^K$ a point-to-set
 465 upper semi-continuous map such that the set $\{x \in K : \inf_{y \in S(x)} (Tx, y - x) < 0\}$ is empty or
 466 open in K . Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$ and $(Tx_0, y - x_0) \geq 0$ for all $y \in S(x_0)$.

467

468 *Proof.* Follows directly from Theorem 5.1 with $f(x, y) = (Tx, y - x)$.

469

470

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