

THE CONVOLUTION SUMS $\sum_{m < \frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m)$

ABSTRACT. In this paper we obtain the convolution sum formulae of

$$\sum_{m < \frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m)$$

for $n \in \mathbb{N}$ and an odd positive integer e and f . Furthermore we obtain some identities induced from $\sum_{m < \frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m)$.

1. INTRODUCTION

For $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define an necessary divisor function and the infinite product sums which also appear in many areas of number theory:

(1)

$$\sigma_s(n) = \sum_{d|n} d^s, \quad \Delta(q) := \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24},$$

$$B(q) := \sum_{n=1}^{\infty} b(n)q^n = (\Delta(q)\Delta(q^2))^{\frac{1}{3}} = q \prod_{n=1}^{\infty} (1-q^n)^8(1-q^{2n})^8,$$

$$C(q) := \sum_{n=1}^{\infty} c(n)q^n = (\Delta(q)^4\Delta(q^2))^{\frac{1}{6}} = q \prod_{n=1}^{\infty} (1-q^n)^{16}(1-q^{2n})^4,$$

$$D(q) := \sum_{n=1}^{\infty} d(n)q^n = (\Delta(q)^2\Delta(q^2)\Delta(q^4))^{\frac{1}{6}} = q^2 \prod_{n=1}^{\infty} (1-q^n)^8(1-q^{2n})^4(1-q^{4n})^8.$$

In general, it is satisfied that

$$(2) \quad b(n) = -8b\left(\frac{n}{2}\right)$$

for even n (see [6, Remark 4.3]). The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ for $q \in \mathbb{C}$ with $|q| < 1$ are

$$(3) \quad L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$(4) \quad M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

2010 *Mathematics Subject Classification.* 11A05.

Key words and phrases. Divisor functions, Convolution sums.

$$(5) \quad N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

by [2, p. 318]. Lahiri [7, p. 149] has derived the following identities from the work of Ramanujan [9] :

$$(6) \quad L^2(q) = 1 - 288 \sum_{n=1}^{\infty} n \sigma_1(n) q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$(7) \quad M^2(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(8) \quad M^3(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n + \frac{432000}{691} \sum_{n=1}^{\infty} \tau(n) q^n,$$

$$(9) \quad L(q)M(q) = 1 + 720 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$(10) \quad L(q)M^2(q) = 1 + 720 \sum_{n=1}^{\infty} n \sigma_7(n) q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

$$(11) \quad N^2(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n - \frac{762048}{691} \sum_{n=1}^{\infty} \tau(n) q^n,$$

$$(12) \quad L(q)N(q) = 1 - 1008 \sum_{n=1}^{\infty} n \sigma_5(n) q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n.$$

For $e, f, m, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we define

$$(13) \quad \begin{aligned} I_{m,e,f}^k(n) &:= \sum_{m=1}^{n-1} m^k \sigma_e(m) \sigma_f(n-m), \\ I_{m,e,f}(n) &:= I_{m,e,f}^1(n) = \sum_{m=1}^{n-1} m \sigma_e(m) \sigma_f(n-m), \\ I_{e,f}(n) &:= I_{m,e,f}^0(n) = \sum_{m=1}^{n-1} \sigma_e(m) \sigma_f(n-m). \end{aligned}$$

Ramanujan [9] and Lahiri [7], [8] have shown that $I_{e,f}$ can be expressed as :

$$\begin{aligned}
 I_{1,1}(n) &= \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma_1(n), \\
 I_{1,3}(n) &= \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n), \\
 I_{1,5}(n) &= \frac{5}{126}\sigma_7(n) + \frac{(1-2n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n), \\
 I_{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n), \\
 I_{1,7}(n) &= \frac{11}{480}\sigma_9(n) + \frac{(2-3n)}{48}\sigma_7(n) - \frac{1}{480}\sigma_1(n), \\
 I_{3,5}(n) &= \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n), \\
 I_{1,9}(n) &= \frac{455}{30404}\sigma_{11}(n) + \frac{(5-6n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(n) - \frac{36}{3455}\tau(n), \\
 I_{3,7}(n) &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n), \\
 I_{5,5}(n) &= \frac{65}{174132}\sigma_{11}(n) + \frac{1}{252}\sigma_5(n) - \frac{3}{691}\tau(n), \\
 I_{1,11}(n) &= \frac{691}{65520}\sigma_{13}(n) + \frac{(1-n)}{24}\sigma_{11}(n) - \frac{691}{65520}\tau(n), \\
 I_{3,9}(n) &= \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n), \\
 I_{5,7}(n) &= \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).
 \end{aligned}$$

And we have already obtained :

Proposition 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a) (See [10, p. 155])

$$I_{m,1,1}(n) = \frac{1}{24}n \{5\sigma_3(n) - (6n-1)\sigma_1(n)\},$$

(b) (See [10, p. 157])

$$\begin{aligned}
 I_{m,1,3}(n) &= \frac{7}{240}n\sigma_5(n) - \frac{1}{40}n^2\sigma_3(n) - \frac{1}{240}n\sigma_1(n), \\
 I_{m,3,1}(n) &= \frac{7}{120}n\sigma_5(n) + \left(\frac{1}{24}n - \frac{1}{10}n^2\right)\sigma_3(n),
 \end{aligned}$$

(c) (See [4, Theorem 3.3, Theorem 3.1], [10, p. 155])

$$\begin{aligned}
 I_{m,1,5}(n) &= \frac{1}{504}n \{5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n)\}, \\
 I_{m,3,3}(n) &= \frac{1}{240}n \{\sigma_7(n) - \sigma_3(n)\}, \\
 I_{m,5,1}(n) &= \frac{1}{168}n \{5\sigma_7(n) - (12n-7)\sigma_5(n)\},
 \end{aligned}$$

(d) (See [4, Theorem 3.8 (a), (b), Theorem 3.7, Theorem 3.5])

$$\begin{aligned} I_{m,1,7}(n) &= \frac{1}{7200} \{33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n)\}, \\ I_{m,3,5}(n) &= \frac{1}{12600} \{11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n)\}, \\ I_{m,5,3}(n) &= \frac{1}{8400} \{11n\sigma_9(n) - 35n\sigma_5(n) + 24\tau(n)\}, \\ I_{m,7,1}(n) &= \frac{1}{1800} \{33n\sigma_9(n) - 25n(4n-3)\sigma_7(n) - 8\tau(n)\}. \end{aligned}$$

Moreover, we can see that

$$(14) \quad I_{m,3,3}^2(n) = \frac{1}{2160} \{5n^2\sigma_7(n) - 9n^2\sigma_3(n) + 4\tau(n)\}$$

in [5, Theorem 1.2 (b)]. Similarly, for $e, f, m, n \in \mathbb{N}$ we set

$$(15) \quad \begin{aligned} T_{m,e,f}(n) &:= \sum_{m < \frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m), \\ T_{e,f}(n) &:= \sum_{m < \frac{n}{2}} \sigma_e(m)\sigma_f(n-2m). \end{aligned}$$

And in [10, p. 45–54] we can see that

$$\begin{aligned} T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1\left(\frac{n}{2}\right), \\ T_{3,1}(n) &= \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n), \\ T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5\left(\frac{n}{2}\right) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma_1\left(\frac{n}{2}\right), \\ T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-2n)}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n), \\ T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{272}b(n), \\ T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1\left(\frac{n}{2}\right) - \frac{1}{102}b(n), \\ T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9\left(\frac{n}{2}\right) + \frac{(2-3n)}{48}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{480}\sigma_1(n) \\ &\quad + \frac{1}{496}c(n) + \frac{2}{31}d(n), \\ T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_3(n) \\ &\quad - \frac{1}{496}c(n) - \frac{2}{31}d(n), \\ T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3\left(\frac{n}{2}\right) \\ &\quad + \frac{1}{248}c(n) + \frac{4}{31}d(n), \end{aligned}$$

$$\begin{aligned}
 T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9\left(\frac{n}{2}\right) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1\left(\frac{n}{2}\right) \\
 &\quad - \frac{1}{62}c(n) - \frac{16}{31}d(n), \\
 T_{9,1}(n) &= \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}\left(\frac{n}{2}\right) + \frac{(5-6n)}{120}\sigma_9\left(\frac{n}{2}\right) + \frac{1}{264}\sigma_1(n) \\
 &\quad - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau\left(\frac{n}{2}\right), \\
 T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{480}\sigma_3(n) \\
 &\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau\left(\frac{n}{2}\right), \\
 T_{5,5}(n) &= \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5\left(\frac{n}{2}\right) \\
 &\quad - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau\left(\frac{n}{2}\right), \\
 T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3\left(\frac{n}{2}\right) \\
 &\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau\left(\frac{n}{2}\right), \\
 T_{1,9}(n) &= \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}\left(\frac{n}{2}\right) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1\left(\frac{n}{2}\right) \\
 &\quad - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau\left(\frac{n}{2}\right).
 \end{aligned}$$

In this paper, we try to find some various convolution sum formulae of $T_{m,e,f}(n)$ and so obtain the following theorem :

Theorem 1.2. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
 T_{m,1,3}(n) &= \sum_{m < \frac{n}{2}} m\sigma_1(m)\sigma_3(n-2m) \\
 &= \frac{1}{1440} \left[n \left\{ 5\sigma_5(n) + 16\sigma_5\left(\frac{n}{2}\right) - 9n\sigma_3(n) - 3\sigma_1\left(\frac{n}{2}\right) \right\} + 4b(n) \right],
 \end{aligned}$$

(b)

$$\begin{aligned}
 T_{m,3,1}(n) &= \sum_{m < \frac{n}{2}} m\sigma_3(m)\sigma_1(n-2m) \\
 &= \frac{1}{720} \left[n \left\{ \sigma_5(n) + 20\sigma_5\left(\frac{n}{2}\right) - (36n-15)\sigma_3\left(\frac{n}{2}\right) \right\} - b(n) \right],
 \end{aligned}$$

(c)

$$\begin{aligned}
 T_{m,1,5}(n) &= \sum_{m < \frac{n}{2}} m\sigma_1(m)\sigma_5(n-2m) \\
 &= \frac{1}{17136} \left\{ 21n\sigma_7(n) + 64n\sigma_7\left(\frac{n}{2}\right) - 51n^2\sigma_5(n) + 17n\sigma_1\left(\frac{n}{2}\right) + 1632d(n) \right. \\
 &\quad \left. + 51c(n) - 21nb(n) \right\},
 \end{aligned}$$

(d)

$$\begin{aligned} T_{m,3,3}(n) &= \sum_{m < \frac{n}{2}} m\sigma_3(m)\sigma_3(n-2m) \\ &= \frac{1}{16320} \left\{ 2n\sigma_7(n) + 32n\sigma_7\left(\frac{n}{2}\right) - 34n\sigma_3\left(\frac{n}{2}\right) - 544d(n) - 17c(n) + 15nb(n) \right\}, \end{aligned}$$

(e)

$$\begin{aligned} T_{m,5,1}(n) &= \sum_{m < \frac{n}{2}} m\sigma_5(m)\sigma_1(n-2m) \\ &= \frac{1}{22848} \left\{ 4n\sigma_7(n) + 336n\sigma_7\left(\frac{n}{2}\right) - 816n^2\sigma_5\left(\frac{n}{2}\right) + 476n\sigma_5\left(\frac{n}{2}\right) + 544d(n) \right. \\ &\quad \left. + 17c(n) - 21nb(n) \right\}, \end{aligned}$$

(f)

$$\begin{aligned} T_{m,1,7}(n) &= \sum_{m < \frac{n}{2}} m\sigma_1(m)\sigma_7(n-2m) \\ &= \frac{1}{446400} \left\{ 255n\sigma_9(n) + 768n\sigma_9\left(\frac{n}{2}\right) - 775n^2\sigma_7(n) - 465n\sigma_1\left(\frac{n}{2}\right) + 1240\tau(n) \right. \\ &\quad \left. + 190464\tau\left(\frac{n}{2}\right) - 23040nd(n) - 720nc(n) \right\}, \end{aligned}$$

(g)

$$\begin{aligned} T_{m,3,5}(n) &= \sum_{m < \frac{n}{2}} m\sigma_3(m)\sigma_5(n-2m) \\ &= \frac{1}{781200} \left\{ 21n\sigma_9(n) + 320n\sigma_9\left(\frac{n}{2}\right) + 775n\sigma_3\left(\frac{n}{2}\right) - 651\tau(n) - 59520\tau\left(\frac{n}{2}\right) \right. \\ &\quad \left. + 20160nd(n) + 630nc(n) \right\}, \end{aligned}$$

(h)

$$\begin{aligned} T_{m,5,3}(n) &= \sum_{m < \frac{n}{2}} m\sigma_5(m)\sigma_3(n-2m) \\ &= \frac{1}{520800} \left\{ 5n\sigma_9(n) + 336n\sigma_9\left(\frac{n}{2}\right) - 1085n\sigma_5\left(\frac{n}{2}\right) + 310\tau(n) + 13888\tau\left(\frac{n}{2}\right) \right. \\ &\quad \left. - 10080nd(n) - 315nc(n) \right\}, \end{aligned}$$

(i)

$$\begin{aligned} T_{m,7,1}(n) &= \sum_{m < \frac{n}{2}} m\sigma_7(m)\sigma_1(n-2m) \\ &= \frac{1}{111600} \left\{ 3n\sigma_9(n) + 1020n\sigma_9\left(\frac{n}{2}\right) - 3100n^2\sigma_7\left(\frac{n}{2}\right) + 2325n\sigma_7\left(\frac{n}{2}\right) - 93\tau(n) \right. \\ &\quad \left. - 2480\tau\left(\frac{n}{2}\right) + 2880nd(n) + 90nc(n) \right\}. \end{aligned}$$

In addition, using Theorem 1.2 we can easily obtain some identities induced from $T_{m,e,f}(n)$, which are introduced in the following theorem :

Theorem 1.3. For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$\begin{aligned} L(q)L(q^2)M(q^2) &= 1 + \frac{96}{17} \sum_{n=1}^{\infty} \sigma_7(n)q^n + \frac{8064}{17} \sum_{n=1}^{\infty} \sigma_7(n)q^{2n} - 24 \sum_{n=1}^{\infty} n\sigma_5(n)q^n \\ &\quad - 2976 \sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} + 3456 \sum_{n=1}^{\infty} n^2\sigma_3(n)q^{2n} - \frac{96}{17} \sum_{n=1}^{\infty} b(n)q^n, \end{aligned}$$

(b)

$$\begin{aligned} L(q)L(q^2)N(q) &= 1 - \frac{2040}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^n - \frac{6144}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^{2n} + \frac{9144}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\ &\quad + \frac{18432}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^{2n} - 864 \sum_{n=1}^{\infty} n^2\sigma_5(n)q^n + \frac{41472}{31} \sum_{n=1}^{\infty} d(n)q^n \\ &\quad + \frac{1296}{31} \sum_{n=1}^{\infty} c(n)q^n - \frac{3024}{17} \sum_{n=1}^{\infty} nb(n)q^n, \end{aligned}$$

(c)

$$\begin{aligned} L(q^2)M(q)M(q^2) &= 1 - \frac{120}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^n - \frac{8064}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^{2n} + \frac{360}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\ &\quad + \frac{11520}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^{2n} + \frac{63360}{31} \sum_{n=1}^{\infty} d(n)q^n + \frac{1980}{31} \sum_{n=1}^{\infty} c(n)q^n \\ &\quad + \frac{2700}{17} \sum_{n=1}^{\infty} nb(n)q^n, \end{aligned}$$

(d)

$$\begin{aligned} L(q)L(q^2)N(q^2) &= 1 - \frac{24}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^n - \frac{16320}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^{2n} + \frac{72}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\ &\quad + \frac{36576}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^{2n} - 3456 \sum_{n=1}^{\infty} n^2\sigma_5(n)q^{2n} \\ &\quad - \frac{5184}{31} \sum_{n=1}^{\infty} d(n)q^n - \frac{162}{31} \sum_{n=1}^{\infty} c(n)q^n - \frac{378}{17} \sum_{n=1}^{\infty} nb(n)q^n, \end{aligned}$$

(e)

$$\begin{aligned} &L(q)L(q^2)M^2(q) \\ &= 1 + \frac{16368}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{49152}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{36792}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\ &\quad - \frac{73728}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + 480 \sum_{n=1}^{\infty} n^2\sigma_7(n)q^n - \frac{115008}{3455} \sum_{n=1}^{\infty} \tau(n)q^n \\ &\quad - \frac{34209792}{3455} \sum_{n=1}^{\infty} \tau(n)q^{2n} + \frac{221184}{31} \sum_{n=1}^{\infty} nd(n)q^n + \frac{6912}{31} \sum_{n=1}^{\infty} nc(n)q^n, \end{aligned}$$

(f)

$$\begin{aligned} & L(q)M(q^2)N(q) \\ &= 1 + \frac{4080}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{61440}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{3024}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\ &\quad - \frac{18432}{31} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + \frac{244944}{3455} \sum_{n=1}^{\infty} \tau(n)q^n + \frac{16920576}{691} \sum_{n=1}^{\infty} \tau(n)q^{2n} \\ &\quad - \frac{580608}{31} \sum_{n=1}^{\infty} nd(n)q^n - \frac{18144}{31} \sum_{n=1}^{\infty} nc(n)q^n, \end{aligned}$$

(g)

$$\begin{aligned} & L(q^2)M(q)N(q^2) \\ &= 1 + \frac{240}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{65280}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{72}{31} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\ &\quad - \frac{48384}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + \frac{66096}{691} \sum_{n=1}^{\infty} \tau(n)q^n + \frac{3919104}{3455} \sum_{n=1}^{\infty} \tau(n)q^{2n} \\ &\quad + \frac{145152}{31} \sum_{n=1}^{\infty} nd(n)q^n + \frac{4536}{31} \sum_{n=1}^{\infty} nc(n)q^n, \end{aligned}$$

(h)

$$\begin{aligned} & L^2(q)M^2(q^2) \\ &= 1 + \frac{240}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{65280}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{288}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\ &\quad - \frac{39168}{31} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + 3840 \sum_{n=1}^{\infty} n^2\sigma_7(n)q^{2n} + \frac{31968}{3455} \sum_{n=1}^{\infty} \tau(n)q^n \\ &\quad + \frac{996096}{691} \sum_{n=1}^{\infty} \tau(n)q^{2n} - \frac{55296}{31} \sum_{n=1}^{\infty} nd(n)q^n - \frac{1728}{31} \sum_{n=1}^{\infty} nc(n)q^n, \end{aligned}$$

(i)

$$\begin{aligned} & L(q)M(q)M(q^2) \\ &= 1 - \frac{504}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^n - \frac{7680}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^{2n} + \frac{720}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\ &\quad + \frac{23040}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^{2n} - \frac{126720}{31} \sum_{n=1}^{\infty} d(n)q^n - \frac{3960}{31} \sum_{n=1}^{\infty} c(n)q^n \\ &\quad + \frac{5400}{17} \sum_{n=1}^{\infty} nb(n)q^n, \end{aligned}$$

(j)

$$L(q)B(q) = -16 \sum_{n=1}^{\infty} d(n)q^n - \frac{1}{2} \sum_{n=1}^{\infty} c(n)q^n + \frac{3}{2} \sum_{n=1}^{\infty} nb(n)q^n,$$

(k)

$$L(q^2)B(q) = 8 \sum_{n=1}^{\infty} d(n)q^n + \frac{1}{4} \sum_{n=1}^{\infty} c(n)q^n + \frac{3}{4} \sum_{n=1}^{\infty} nb(n)q^n,$$

(l)

$$M(q)B(q) = \sum_{n=1}^{\infty} \tau(n)q^n + 256 \sum_{n=1}^{\infty} \tau(n)q^{2n}.$$

2. SOME CONVOLUTION SUMS FORMULAE OF $T_{m,e,f}(n)$

We introduce Proposition 2.1 to obtain some convolution sum formulae related to $T_{m,e,f}(n)$.

Proposition 2.1. (See [1, Theorem 1.1]) For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$\begin{aligned} L(q)L(q^2) &= 1 - 72 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 48 \sum_{n=1}^{\infty} \sigma_3(n)q^n - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^{2n} \\ &\quad + 192 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}, \end{aligned}$$

(b)

$$\begin{aligned} L(q)L(q^4) &= 1 - 36 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 12 \sum_{n=1}^{\infty} \sigma_3(n)q^n + 36 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n} \\ &\quad - 576 \sum_{n=1}^{\infty} n\sigma_1(n)q^{4n} + 192 \sum_{n=1}^{\infty} \sigma_3(n)q^{4n}, \end{aligned}$$

(c)

$$L(q)M(q^2) = 1 - 24 \sum_{n=1}^{\infty} \sigma_5(n)q^n + 1440 \sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n},$$

(d)

$$L(q^2)M(q) = 1 + 360 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n)q^n - 384 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n},$$

(e)

$$\begin{aligned} L(q)L^2(q^2) &= 1 - 144 \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 144 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 24 \sum_{n=1}^{\infty} \sigma_5(n)q^n \\ &\quad - 2304 \sum_{n=1}^{\infty} n^2\sigma_1(n)q^{2n} + 2592 \sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}, \end{aligned}$$

(f)

$$\begin{aligned} L^2(q)L(q^2) &= 1 - 576 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 648 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n) q^n \\ &\quad - 2304 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^{2n} + 2304 \sum_{n=1}^{\infty} n \sigma_3(n) q^{2n} - 384 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}, \end{aligned}$$

(g)

$$\begin{aligned} L(q)L(q^2)L(q^4) &= 1 - 72 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 54 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 6 \sum_{n=1}^{\infty} \sigma_5(n) q^n \\ &\quad - 576 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^{2n} + 684 \sum_{n=1}^{\infty} n \sigma_3(n) q^{2n} - 114 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \\ &\quad - 4608 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^{4n} + 3456 \sum_{n=1}^{\infty} n \sigma_3(n) q^{4n} - 384 \sum_{n=1}^{\infty} \sigma_5(n) q^{4n}. \end{aligned}$$

Theorem 2.2. For $q \in \mathbb{C}$ with $|q| < 1$, we have

$$\begin{aligned} L(q)L(q^2)L(q^4) &= 1 + \frac{2016}{17} \sum_{n=1}^{\infty} \sigma_7(n) q^n + \frac{6144}{17} \sum_{n=1}^{\infty} \sigma_7(n) q^{2n} - 744 \sum_{n=1}^{\infty} n \sigma_5(n) q^n \\ &\quad - 1536 \sum_{n=1}^{\infty} n \sigma_5(n) q^{2n} + 864 \sum_{n=1}^{\infty} n^2 \sigma_3(n) q^n - \frac{384}{17} \sum_{n=1}^{\infty} b(n) q^n. \end{aligned}$$

Proof. By (3) and Proposition 2.1 (d), we have

$$\begin{aligned} L(q)L(q^2)L(q^4) &= L(q) \cdot L(q^2)L(q^4) \\ &= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \right) \\ &\quad \times \left(1 + 360 \sum_{m=1}^{\infty} m \sigma_3(m) q^m - 120 \sum_{m=1}^{\infty} \sigma_5(m) q^m - 384 \sum_{m=1}^{\infty} \sigma_5(m) q^{2m} \right) \\ &= 1 + \sum_{N=1}^{\infty} \left\{ 360N \sigma_3(N) - 120 \sigma_5(N) - 384 \sigma_5\left(\frac{N}{2}\right) - 24 \sigma_1(N) \right. \\ &\quad - 24 \cdot 360 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m \sigma_3(m) + 24 \cdot 120 \sum_{m=1}^{N-1} \sigma_1(N-m) \sigma_5(m) \\ &\quad \left. + 24 \cdot 384 \sum_{m=1}^{N-1} \sigma_1(N-2m) \sigma_5(m) \right\} q^N \\ &= 1 + \sum_{N=1}^{\infty} \left\{ 360N \sigma_3(N) - 120 \sigma_5(N) - 384 \sigma_5\left(\frac{N}{2}\right) - 24 \sigma_1(N) \right. \\ &\quad \left. - 24 \cdot 360 \cdot I_{m,3,1}(N) + 24 \cdot 120 \cdot I_{1,5}(N) + 24 \cdot 384 \cdot T_{5,1}(N) \right\} q^N, \end{aligned}$$

where we use the definition of $I_{m,e,f}(n)$ and $T_{e,f}(n)$ given in (13) and (15) for the last line. Finally we refer to Proposition 1.1 (b).

□

Now we are interested in

$$(16) \quad T_{m,1,1}(n) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma_1(n) + \frac{1}{12}n\sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma_1\left(\frac{n}{2}\right)$$

(see [1, Theorem 4.1]), which is the starting point to obtain Theorem 1.2.

Proof of Theorem 1.2. (a) From (4) and Proposition 2.1 (a), we can see that

$$\begin{aligned} (17) \quad & L(q)L(q^2)M(q) = M(q) \cdot L(q)L(q^2) \\ & = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \left(1 - 72 \sum_{m=1}^{\infty} m\sigma_1(m)q^m + 48 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right. \\ & \quad \left. - 288 \sum_{m=1}^{\infty} m\sigma_1(m)q^{2m} + 192 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}\right) \\ & = 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\ & \quad - 240 \cdot 72 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m\sigma_1(m) + 240 \cdot 48 \sum_{m=1}^{N-1} \sigma_3(N-m)\sigma_3(m) \\ & \quad - 240 \cdot 288 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m\sigma_1(m) \\ & \quad \left. + 240 \cdot 192 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_3(m) \right\} q^N \\ & = 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\ & \quad - 240 \cdot 72 \cdot I_{m,1,3}(N) + 240 \cdot 48 \cdot I_{3,3}(N) \\ & \quad \left. - 240 \cdot 288 \sum_{m < \frac{N}{2}} m\sigma_1(m)\sigma_3(N-2m) + 240 \cdot 192 \cdot T_{3,3}(N) \right\} q^N. \end{aligned}$$

So using Proposition 1.1 (b) and equating (17) with Theorem 2.2, we obtain the proof.

(b) From (3) and (9) we have

$$\begin{aligned}
 (18) \quad & L(q)L(q^2)M(q^2) = L(q) \cdot L(q^2)M(q^2) \\
 & = \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n\right) \left(1 + 720 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) - 24\sigma_1(N) \right. \\
 & \quad \left. - 24 \cdot 720 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \cdot m\sigma_3(m) + 24 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_1(N-2m)\sigma_5(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) - 24\sigma_1(N) \right. \\
 & \quad \left. - 24 \cdot 720 \sum_{m < \frac{N}{2}} m\sigma_3(m)\sigma_1(N-2m) + 24 \cdot 504 \cdot T_{5,1}(N) \right\} q^N.
 \end{aligned}$$

Also, by (4) and Proposition 2.1 (a), we can obtain

$$\begin{aligned}
 (19) \quad & L(q)L(q^2)M(q^2) = L(q)L(q^2) \cdot M(q^2) \\
 &= \left(1 - 72 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 48 \sum_{n=1}^{\infty} \sigma_3(n)q^n - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^{2n} + 192 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}\right) \\
 &\quad \times \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) \right. \\
 &\quad - 72 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m)\sigma_1(N-2m)\sigma_3(m) + 48 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_3(m) \\
 &\quad - 288 \cdot 240 \sum_{m < \frac{N}{2}} \left(\frac{N}{2} - m\right)\sigma_1\left(\frac{N}{2} - m\right)\sigma_3(m) \\
 &\quad \left. + 192 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_3\left(\frac{N}{2} - m\right)\sigma_3(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 432\sigma_3\left(\frac{N}{2}\right) \right. \\
 &\quad - 72 \cdot 240N \cdot T_{3,1}(N) + 72 \cdot 240 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_3(m)\sigma_1(N-2m) \\
 &\quad + 48 \cdot 240 \cdot T_{3,3}(N) - 288 \cdot 240 \cdot \frac{N}{2} \cdot I_{1,3}\left(\frac{N}{2}\right) + 288 \cdot 240 \cdot I_{m,3,1}\left(\frac{N}{2}\right) \\
 &\quad \left. + 192 \cdot 240 \cdot I_{3,3}\left(\frac{N}{2}\right) \right\} q^N.
 \end{aligned}$$

Therefore, equating (18) with (19) and using Proposition 1.1 (b), we have the proof.

(c) First by (3) and (12), we observe that

$$\begin{aligned}
 L(q)L(q^2)N(q) &= L(q)N(q) \cdot L(q^2) \\
 &= \left(1 - 1008 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n\right) \left(1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -1008N\sigma_5(N) + 480\sigma_7(N) - 24\sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad \left. + 1008 \cdot 24 \sum_{m < \frac{N}{2}} (N-2m)\sigma_5(N-2m)\sigma_1(m) \right. \\
 (20) \quad &\quad \left. - 480 \cdot 24 \sum_{m < \frac{N}{2}} \sigma_7(N-2m)\sigma_1(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -1008N\sigma_5(N) + 480\sigma_7(N) - 24\sigma_1\left(\frac{N}{2}\right) + 1008 \cdot 24N \cdot T_{1,5}(N) \right. \\
 &\quad \left. - 1008 \cdot 24 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_1(m)\sigma_5(N-2m) - 480 \cdot 24 \cdot T_{1,7}(N) \right\} q^N.
 \end{aligned}$$

Second, by (5) and Proposition 2.1 (a), we can obtain the same result of (20) as

$$\begin{aligned}
 L(q)L(q^2)N(q) &= N(q) \cdot L(q)L(q^2) \\
 &= \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n\right) \left(1 - 72 \sum_{m=1}^{\infty} m\sigma_1(m)q^m + 48 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right. \\
 &\quad \left. - 288 \sum_{n=1}^{\infty} m\sigma_1(m)q^{2m} + 192 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5(N) \right. \\
 &\quad + 504 \cdot 72 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m\sigma_1(m) - 504 \cdot 48 \sum_{m=1}^{N-1} \sigma_5(N-m)\sigma_3(m) \\
 &\quad \left. + 504 \cdot 288 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m\sigma_1(m) - 504 \cdot 192 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_3(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5(N) \right. \\
 &\quad + 504 \cdot 72 \cdot I_{m,1,5}(N) - 504 \cdot 48 \cdot I_{3,5}(N) \\
 &\quad \left. + 504 \cdot 288 \sum_{m < \frac{N}{2}} m\sigma_1(m)\sigma_5(N-2m) - 504 \cdot 192 \cdot T_{3,5}(N) \right\} q^N.
 \end{aligned}$$

Thus equating (20) with (21) and so using Proposition 1.1 (c), we can complete the proof.

(d) First by (4) and (9), we have

$$\begin{aligned}
 (22) \quad & L(q^2)M(q)M(q^2) = M(q) \cdot L(q^2)M(q^2) \\
 & = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \left(1 + 720 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\
 & \quad \left. + 240 \cdot 720 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m\sigma_3(m) - 240 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_5(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\
 & \quad \left. + 240 \cdot 720 \sum_{m < \frac{N}{2}} m\sigma_3(m)\sigma_3(N-2m) - 240 \cdot 504 \cdot T_{5,3}(N) \right\} q^N.
 \end{aligned}$$

Similarly, by (4) and Proposition 2.1 (d), we can obtain

$$\begin{aligned}
 (23) \quad & L(q^2)M(q)M(q^2) = L(q^2)M(q) \cdot M(q^2) \\
 & = \left(1 + 360 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n)q^n - 384 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}\right) \\
 & \quad \times \left(1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) \right. \\
 & \quad \left. + 360 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m) \sigma_3(N-2m)\sigma_3(m) \right. \\
 & \quad \left. - 120 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_3(m) - 384 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2}-m\right)\sigma_3(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) \right. \\
 & \quad \left. + 360 \cdot 240N \cdot T_{3,3}(N) - 360 \cdot 240 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_3(m)\sigma_3(N-2m) \right. \\
 & \quad \left. - 120 \cdot 240 \cdot T_{3,5}(N) - 384 \cdot 240 \cdot I_{3,5}\left(\frac{N}{2}\right) \right\} q^N.
 \end{aligned}$$

Finally, equating (22) with (23) we can complete the proof.
 (e) By (3) and (12), we obtain

$$\begin{aligned}
 (24) \quad & L(q)L(q^2)N(q^2) = L(q) \cdot L(q^2)N(q^2) \\
 &= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n\right) \left(1 - 1008 \sum_{m=1}^{\infty} m\sigma_5(m)q^{2m} + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -504N\sigma_5\left(\frac{N}{2}\right) + 480\sigma_7\left(\frac{N}{2}\right) - 24\sigma_1(N) \right. \\
 &\quad \left. + 24 \cdot 1008 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \cdot m\sigma_5(m) - 24 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_1(N-2m)\sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -504N\sigma_5\left(\frac{N}{2}\right) + 480\sigma_7\left(\frac{N}{2}\right) - 24\sigma_1(N) \right. \\
 &\quad \left. + 24 \cdot 1008 \sum_{m < \frac{N}{2}} m\sigma_5(m)\sigma_1(N-2m) - 24 \cdot 480 \cdot T_{7,1}(N) \right\} q^N.
 \end{aligned}$$

As the same way, by (5) and Proposition 2.1 (a), we have

$$\begin{aligned}
 (25) \quad & L(q)L(q^2)N(q^2) = L(q)L(q^2) \cdot N(q^2) \\
 &= \left(1 - 72 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 48 \sum_{n=1}^{\infty} \sigma_3(n)q^n - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^{2n} + 192 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}\right) \\
 &\quad \times \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) \right. \\
 &\quad + 72 \cdot 504 \sum_{m < \frac{N}{2}} (N-2m)\sigma_1(N-2m)\sigma_5(m) - 48 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_5(m) \\
 &\quad + 288 \cdot 504 \sum_{m < \frac{N}{2}} \left(\frac{N}{2} - m\right)\sigma_1\left(\frac{N}{2} - m\right)\sigma_5(m) \\
 &\quad \left. - 192 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_3\left(\frac{N}{2} - m\right)\sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) \right. \\
 &\quad + 72 \cdot 504N \cdot T_{5,1}(N) - 72 \cdot 504 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_5(m)\sigma_1(N-2m) \\
 &\quad - 48 \cdot 504 \cdot T_{5,3}(N) + 288 \cdot 504 \cdot \frac{N}{2} \cdot I_{1,5}\left(\frac{N}{2}\right) - 288 \cdot 504 \cdot I_{m,5,1}\left(\frac{N}{2}\right) \\
 &\quad \left. - 192 \cdot 504 \cdot I_{3,5}\left(\frac{N}{2}\right) \right\} q^N.
 \end{aligned}$$

Finally, equating (24) with (25) and using Proposition 1.1 (c), we can complete the proof.

(f) By (3) and (10), we have

$$\begin{aligned}
 L(q)L(q^2)M^2(q) &= L(q)M^2(q) \cdot L(q^2) \\
 &= \left(1 + 720 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n\right) \left(1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ 720N\sigma_7(N) - 264\sigma_9(N) - 24\sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad \left. - 720 \cdot 24 \sum_{m < \frac{N}{2}} (N-2m) \sigma_7(N-2m) \sigma_1(m) \right. \\
 (26) \quad &\quad \left. + 264 \cdot 24 \sum_{m < \frac{N}{2}} \sigma_9(N-2m) \sigma_1(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ 720N\sigma_7(N) - 264\sigma_9(N) - 24\sigma_1\left(\frac{N}{2}\right) - 720 \cdot 24N \cdot T_{1,7}(N) \right. \\
 &\quad \left. + 720 \cdot 24 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_1(m) \sigma_7(N-2m) + 264 \cdot 24 \cdot T_{1,9}(N) \right\} q^N.
 \end{aligned}$$

And by (7) and Proposition 2.1 (a), we obtain

$$\begin{aligned}
 (27) \quad L(q)L(q^2)M^2(q) &= M^2(q) \cdot L(q)L(q^2) \\
 &= \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n\right) \left(1 - 72 \sum_{m=1}^{\infty} m\sigma_1(m)q^m + 48 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right. \\
 &\quad \left. - 288 \sum_{m=1}^{\infty} m\sigma_1(m)q^{2m} + 192 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) + 480\sigma_7(N) \right. \\
 &\quad - 480 \cdot 72 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m\sigma_1(m) + 480 \cdot 48 \sum_{m=1}^{N-1} \sigma_7(N-m) \sigma_3(m) \\
 &\quad \left. - 480 \cdot 288 \sum_{m < \frac{N}{2}} \sigma_7(N-2m) \cdot m\sigma_1(m) + 480 \cdot 192 \sum_{m < \frac{N}{2}} \sigma_7(N-2m) \sigma_3(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1\left(\frac{N}{2}\right) + 192\sigma_3\left(\frac{N}{2}\right) + 480\sigma_7(N) \right. \\
 &\quad - 480 \cdot 72 \cdot I_{m,1,7}(N) + 480 \cdot 48 \cdot I_{3,7}(N) - 480 \cdot 288 \sum_{m < \frac{N}{2}} m\sigma_1(m) \sigma_7(N-2m) \\
 &\quad \left. + 480 \cdot 192 \cdot T_{3,7}(N) \right\} q^N.
 \end{aligned}$$

Therefore, we equate (26) with (27) and use Proposition 1.1 (d).

(g) First by (5) and Proposition 2.1 (c), we have

$$\begin{aligned}
 (28) \quad & L(q)M(q^2)N(q) = L(q)M(q^2) \cdot N(q) \\
 & = \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n\right) \\
 & \quad \times \left(1 - 24 \sum_{m=1}^{\infty} \sigma_5(m)q^m + 1440 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 480 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) - 504\sigma_5(N) \right. \\
 & \quad + 504 \cdot 24 \sum_{m=1}^{N-1} \sigma_5(N-m)\sigma_5(m) - 504 \cdot 1440 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m\sigma_3(m) \\
 & \quad \left. + 504 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_5(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) - 504\sigma_5(N) + 504 \cdot 24 \cdot I_{5,5}(N) \right. \\
 & \quad \left. - 504 \cdot 1440 \sum_{m < \frac{N}{2}} m\sigma_3(m)\sigma_5(N-2m) + 504 \cdot 480 \cdot T_{5,5}(N) \right\} q^N.
 \end{aligned}$$

Second by (4) and (12), we also rewrite (28) as

$$\begin{aligned}
 (29) \quad & L(q)M(q^2)N(q) = L(q)N(q) \cdot M(q^2) \\
 & = \left(1 - 1008 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n\right) \left(1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1008N\sigma_5(N) + 480\sigma_7(N) + 240\sigma_3\left(\frac{N}{2}\right) \right. \\
 & \quad - 1008 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m)\sigma_5(N-2m)\sigma_3(m) \\
 & \quad \left. + 480 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_7(N-2m)\sigma_3(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1008N\sigma_5(N) + 480\sigma_7(N) + 240\sigma_3\left(\frac{N}{2}\right) - 1008 \cdot 240N \cdot T_{3,5}(N) \right. \\
 & \quad \left. + 1008 \cdot 240 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_3(m)\sigma_5(N-2m) + 480 \cdot 240 \cdot T_{3,7}(N) \right\} q^N.
 \end{aligned}$$

Therefore, equating (28) with (29) we obtain the proof.

(h) We note that by (4) and (12),

$$\begin{aligned}
 L(q^2)M(q)N(q^2) &= M(q) \cdot L(q^2)N(q^2) \\
 &= \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \left(1 - 1008 \sum_{m=1}^{\infty} m\sigma_5(m)q^{2m} + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -504N\sigma_5\left(\frac{N}{2}\right) + 480\sigma_7\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\
 &\quad \left. - 240 \cdot 1008 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m\sigma_5(m) \right. \\
 &\quad \left. + 240 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -504N\sigma_5\left(\frac{N}{2}\right) + 480\sigma_7\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\
 &\quad \left. - 240 \cdot 1008 \sum_{m < \frac{N}{2}} m\sigma_5(m)\sigma_3(N-2m) + 240 \cdot 480 \cdot T_{7,3}(N) \right\} q^N.
 \end{aligned}
 \tag{30}$$

Also by (5) and Proposition 2.1 (d),

$$\begin{aligned}
 L(q^2)M(q)N(q^2) &= L(q^2)M(q) \cdot N(q^2) \\
 &= \left(1 + 360 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n)q^n - 384 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}\right) \\
 &\quad \times \left(1 - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) \right. \\
 &\quad \left. - 360 \cdot 504 \sum_{m < \frac{N}{2}} (N-2m) \sigma_3(N-2m)\sigma_5(m) \right. \\
 &\quad \left. + 120 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_5(m) + 384 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2}-m\right)\sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3(N) - 120\sigma_5(N) - 888\sigma_5\left(\frac{N}{2}\right) - 360 \cdot 504N \cdot T_{5,3}(N) \right. \\
 &\quad \left. + 360 \cdot 504 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_5(m)\sigma_3(N-2m) + 120 \cdot 504 \cdot T_{5,5}(N) \right. \\
 &\quad \left. + 384 \cdot 504 \cdot I_{5,5}\left(\frac{N}{2}\right) \right\} q^N.
 \end{aligned}
 \tag{31}$$

So we equate (30) with (31).

(i) Now we calculate the term $L^2(q)M^2(q^2)$ by Proposition 2.1 (c):

$$\begin{aligned}
 (32) \\
 L^2(q)M^2(q^2) &= L(q)M(q^2) \cdot L(q)M(q^2) \\
 &= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_5(n)q^n + 1440 \sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}\right) \\
 &\quad \times \left(1 - 24 \sum_{m=1}^{\infty} \sigma_5(m)q^m + 1440 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 480 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) - 24\sigma_5(N) \right. \\
 &\quad + 24 \cdot 24 \sum_{m=1}^{N-1} \sigma_5(N-m)\sigma_5(m) - 24 \cdot 1440 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m\sigma_3(m) \\
 &\quad + 24 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_5(m) + 720N\sigma_3\left(\frac{N}{2}\right) \\
 &\quad - 1440 \cdot 24 \sum_{n < \frac{N}{2}} n\sigma_3(n)\sigma_5(N-2n) \\
 &\quad + 1440 \cdot 1440 \sum_{m < \frac{N}{2}} \left(\frac{N}{2} - m\right) \sigma_3\left(\frac{N}{2} - m\right) \cdot m\sigma_3(m) \\
 &\quad - 1440 \cdot 480 \sum_{m < \frac{N}{2}} \left(\frac{N}{2} - m\right) \sigma_3\left(\frac{N}{2} - m\right) \sigma_5(m) - 480\sigma_5\left(\frac{N}{2}\right) \\
 &\quad + 480 \cdot 24 \sum_{n < \frac{N}{2}} \sigma_5(n)\sigma_5(N-2n) - 480 \cdot 1440 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right) \cdot m\sigma_3(m) \\
 &\quad \left. + 480 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right) \sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) - 24\sigma_5(N) + 24 \cdot 24 \cdot I_{5,5}(N) \right. \\
 &\quad - 24 \cdot 1440 \cdot T_{m,3,5}(N) + 24 \cdot 480 \cdot T_{5,5}(N) + 720N\sigma_3\left(\frac{N}{2}\right) \\
 &\quad - 1440 \cdot 24 \cdot T_{m,3,5}(N) + 1440 \cdot 1440 \cdot \frac{N}{2} \cdot I_{m,3,3}\left(\frac{N}{2}\right) - 1440 \cdot 1440 \cdot I_{m,3,3}^2\left(\frac{N}{2}\right) \\
 &\quad - 1440 \cdot 480 \cdot \frac{N}{2} \cdot I_{3,5}\left(\frac{N}{2}\right) + 1440 \cdot 480 \cdot I_{m,5,3}\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) \\
 &\quad \left. + 480 \cdot 24 \cdot T_{5,5}(N) - 480 \cdot 1440 \cdot I_{m,3,5}\left(\frac{N}{2}\right) + 480 \cdot 480 \cdot I_{5,5}\left(\frac{N}{2}\right) \right\} q^N
 \end{aligned}$$

so we use Proposition 1.1 (c), (d), (14), and Theorem 1.2 (g), thus we can obtain $L^2(q)M^2(q^2)$. Also the calculated value of $L^2(q)M^2(q^2)$ is introduced in Theorem 1.3 (h). To be continued we can see that by (6) and (7),

$$\begin{aligned}
 L^2(q)M^2(q^2) &= L^2(q) \cdot M^2(q^2) \\
 &= \left(1 - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \left(1 + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -288N\sigma_1(N) + 240\sigma_3(N) + 480\sigma_7\left(\frac{N}{2}\right) \right. \\
 &\quad \left. - 288 \cdot 480 \sum_{m < \frac{N}{2}} (N-2m)\sigma_1(N-2m)\sigma_7(m) \right. \\
 (33) \quad &\quad \left. + 240 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -288N\sigma_1(N) + 240\sigma_3(N) + 480\sigma_7\left(\frac{N}{2}\right) - 288 \cdot 480N \cdot T_{7,1}(N) \right. \\
 &\quad \left. - 288 \cdot 480 \cdot 2 \sum_{m < \frac{N}{2}} m\sigma_7(m)\sigma_1(N-2m) + 240 \cdot 480 \cdot T_{7,3}(N) \right\} q^N.
 \end{aligned}$$

Therefore we equate (32) with (33).

□

3. THE INDUCED IDENTITIES FROM $\sum_{m < \frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m)$

We can see the following proposition in [3, Theorem 5.1, Theorem 6.1, Theorem 7.1] :

Proposition 3.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$N(q)L(q^2) = \frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q),$$

(b)

$$M(q)M(q^2) = \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q),$$

(c)

$$\begin{aligned}
 L(q)M^2(q^2) &= 2L(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q) - \frac{342}{341}M(q^2)N(q^2) - \frac{720}{31}C(q) \\
 &\quad - \frac{23040}{31}D(q),
 \end{aligned}$$

(d)

$$M^2(q)L(q^2) = \frac{1}{2}L(q)M^2(q) - \frac{171}{682}M(q)N(q) + \frac{256}{341}M(q^2)N(q^2) + \frac{5760}{31}C(q) + \frac{184320}{31}D(q),$$

(e)

$$M(q)M^2(q^2) = \frac{11}{78}M^3(q) + \frac{196}{39}M^3(q^2) - \frac{25}{182}N^2(q) - \frac{1100}{273}N^2(q^2).$$

Now Theorem 1.2 is the based results to induce Theorem 1.3.

Proof of Theorem 1.3. (a) Insert Theorem 1.2 (b) into (18).

(b) Insert Theorem 1.2 (c) into (20).

(c) Insert Theorem 1.2 (d) into (22).

(d) Insert Theorem 1.2 (e) into (24).

(e) Insert Theorem 1.2 (f) into (26).

(f) Insert Theorem 1.2 (g) into (28).

(g) Insert Theorem 1.2 (h) into (30).

(h) It is obvious by (32).

(i) By (4) and Proposition 2.1 (c), let us expand

$$\begin{aligned} L(q)M(q)M(q^2) &= M(q) \cdot L(q)M(q^2) \\ &= \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \\ &\quad \times \left(1 - 24 \sum_{m=1}^{\infty} \sigma_5(m)q^m + 1440 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 480 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\ &= 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) + 240\sigma_3(N) \right. \\ &\quad - 240 \cdot 24 \sum_{m=1}^{N-1} \sigma_3(N-m)\sigma_5(m) + 240 \cdot 1440 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m\sigma_3(m) \\ &\quad \left. - 240 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_5(m) \right\} q^N \\ &= 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) + 240\sigma_3(N) - 240 \cdot 24 \cdot I_{3,5}(N) \right. \\ &\quad \left. + 240 \cdot 1440 \cdot T_{m,3,3}(N) - 240 \cdot 480 \cdot T_{5,3}(N) \right\} q^N \end{aligned}$$

so we refer to Theorem 1.2 (d).

(j) By Proposition 3.1 (b), we can observe that

$$\begin{aligned}
 L(q)M(q)M(q^2) &= L(q) \cdot M(q)M(q^2) \\
 &= L(q) \left(\frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q) \right) \\
 &= \frac{1}{17}L(q)M^2(q) + \frac{16}{17}L(q)M^2(q^2) + \frac{3600}{17}L(q)B(q),
 \end{aligned}$$

therefore we use (10), Proposition 3.1 (c), and Theorem 1.3 (i).

(k) From Proposition 3.1 (b), we have

$$\begin{aligned}
 L(q^2)M(q)M(q^2) &= L(q^2) \cdot M(q)M(q^2) \\
 &= L(q^2) \left(\frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q) \right) \\
 &= \frac{1}{17}L(q^2)M^2(q) + \frac{16}{17}L(q^2)M^2(q^2) + \frac{3600}{17}L(q^2)B(q).
 \end{aligned}$$

So we refer to (10), Proposition 3.1 (d), and Theorem 1.3 (c).

(l) In a similar manner to proof of Theorem 1.3 (k), we note that

$$\begin{aligned}
 M^2(q)M(q^2) &= M(q) \cdot M(q)M(q^2) \\
 &= M(q) \left(\frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q) \right) \\
 &= \frac{1}{17}M^3(q) + \frac{16}{17}M(q)M^2(q^2) + \frac{3600}{17}M(q)B(q)
 \end{aligned}$$

by Proposition 3.1 (b). So we apply (8) and Proposition 3.1 (e).

□

Lemma 3.2. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{n-1} \sigma_1(m)b(n-m) = \frac{1}{48} \{32d(n) + c(n) - (3n-2)b(n)\},$$

(b)

$$\sum_{m < \frac{n}{2}} \sigma_1(m)b(n-2m) = -\frac{1}{96} \{32d(n) + c(n) + (3n-4)b(n)\},$$

(c)

$$\sum_{m=1}^{n-1} \sigma_3(m)b(n-m) = \frac{1}{240} \left\{ \tau(n) + 256\tau\left(\frac{n}{2}\right) - b(n) \right\}.$$

Proof. (a) By (1) and (3), we obtain

$$\begin{aligned} 24 \sum_{N=1}^{\infty} \left(\sum_{m=1}^{N-1} \sigma_1(m) b(N-m) \right) q^N &= 24 \left(\sum_{n=1}^{\infty} b(n) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_1(m) q^m \right) \\ &= B(q) (1 - L(q)) \\ &= B(q) - B(q) L(q). \end{aligned}$$

Thus we refer to Theorem 1.3 (j).

(b) In a similar manner to proof of Lemma 3.2 (a), we have

$$\begin{aligned} 24 \sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{2}} \sigma_1(m) b(N-2m) \right) q^N &= 24 \left(\sum_{n=1}^{\infty} b(n) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_1(m) q^{2m} \right) \\ &= B(q) (1 - L(q^2)) \\ &= B(q) - B(q) L(q^2). \end{aligned}$$

So we use Theorem 1.3 (k).

(c) By (1) and (4), we obtain

$$\begin{aligned} 240 \sum_{N=1}^{\infty} \left(\sum_{m=1}^{N-1} \sigma_3(m) b(N-m) \right) q^N &= 240 \left(\sum_{n=1}^{\infty} b(n) q^n \right) \left(\sum_{m=1}^{\infty} \sigma_3(m) q^m \right) \\ &= B(q) (M(q) - 1) \\ &= B(q) M(q) - B(q). \end{aligned}$$

Thus we refer to Theorem 1.3 (l). □

As an application of Theorem 1.3, we can consider the convolution sum composing with three divisor functions in the following corollary :

Corollary 3.3. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+2b+c=n}} \sigma_1(a) \sigma_1(b) \sigma_5(c) &= \sum_{k=1}^{n-1} \sum_{m < \frac{k}{2}} \sigma_1(m) \sigma_1(k-2m) \sigma_5(n-k) \\ &= \frac{1}{6374592} \left\{ 1445 \sigma_9(n) + 4352 \sigma_9\left(\frac{n}{2}\right) - 31 (381n - 424) \sigma_7(n) \right. \\ &\quad - 3968 (3n - 2) \sigma_7\left(\frac{n}{2}\right) + 1581 (12n^2 - 21n + 7) \sigma_5(n) + 1054 \sigma_3(n) \\ &\quad + 4216 \sigma_3\left(\frac{n}{2}\right) - 527 (3n - 1) \sigma_1(n) - 527 (6n - 1) \sigma_1\left(\frac{n}{2}\right) \\ &\quad \left. - 29376d(n) - 918c(n) + 1302 (3n - 2) b(n) \right\}. \end{aligned}$$

Proof. By (3) and (5), we have

$$\begin{aligned}
& 6374592 \sum_{n=1}^{\infty} \left(\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+2b+c=n}} \sigma_1(a) \sigma_1(b) \sigma_5(c) \right) q^n \\
&= 24^2 \cdot 504 \sum_{(a,b,c) \in \mathbb{N}^3} \sigma_1(a) \sigma_1(b) \sigma_5(c) q^{a+2b+c} \\
&= \left(24 \sum_{a=1}^{\infty} \sigma_1(a) q^a \right) \left(24 \sum_{b=1}^{\infty} \sigma_1(b) q^{2b} \right) \left(504 \sum_{c=1}^{\infty} \sigma_5(c) q^c \right) \\
&= (1 - L(q)) (1 - L(q^2)) (1 - N(q)) \\
&= 1 - N(q) - L(q^2) + L(q^2)N(q) - L(q) + L(q)N(q) + L(q)L(q^2) \\
&\quad - L(q)L(q^2)N(q).
\end{aligned}$$

So we refer to (3), (5), (12), Proposition 3.1 (a), Proposition 2.1 (a), and Theorem 1.3 (b).

□

4. APPENDIX

The first twenty values of $\tau(n)$ are given in the Table 1,

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	-6048	11	534612	16	987136
2	-24	7	-16744	12	-370944	17	-6905934
3	252	8	84480	13	-577738	18	2727432
4	-1472	9	-113643	14	401856	19	10661420
5	4830	10	-115920	15	1217160	20	-7109760

TABLE 1. $\tau(n)$ for n ($1 \leq n \leq 20$)

similarly the first twenty values of $b(n)$, $c(n)$ and $d(n)$ are listed in the following tables.

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	6	-96	11	1092	16	4096
2	-8	7	1016	12	768	17	14706
3	12	8	-512	13	1382	18	16344
4	64	9	-2043	14	-8128	19	-39940
5	-210	10	1680	15	-2520	20	-13440

TABLE 2. $b(n)$ for n ($1 \leq n \leq 20$)

n	$c(n)$	n	$c(n)$	n	$c(n)$	n	$c(n)$
1	1	6	2496	11	-38996	16	-65536
2	-16	7	-4536	12	39936	17	311442
3	100	8	-4096	13	37806	18	-74448
4	-256	9	23085	14	15232	19	128244
5	-154	10	-13920	15	-146472	20	-222720

TABLE 3. $c(n)$ for n ($1 \leq n \leq 20$)

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	6	-156	11	-536	16	4096
2	1	7	112	12	-2496	17	-17472
3	-8	8	256	13	4384	18	4653
4	16	9	-576	14	-952	19	5848
5	32	10	870	15	336	20	13920

TABLE 4. $d(n)$ for n ($1 \leq n \leq 20$)

REFERENCES

- [1] A. Alaca, F. Uygul, and K. S. Williams, *Some arithmetic identities involving divisor functions*, Functiones et Approximatio, **46.2**, (2012), 261–271, doi: 10.7169/facm/2012.46.2.9.
- [2] B. C. Berndt, *Ramanujan's notebooks, Part II*, Springer-Verlag, New York, 1989.
- [3] N. Cheng and K. S. Williams, *Evaluation of some convolution sums involving the sum of divisors functions*, Yokohama Mathematical J., **52** (2005), 39–57.
- [4] A. Kim, *Evaluation of convolution sums as $\sum_{m=1}^{n-1} m\sigma_i(m)\sigma_j(n-m)$* , British Journal of Mathematics & Computer Science, Vol. 4, (2014).
- [5] A. Kim, *The convolution sums $\sum_{m=1}^{n-1} m^k\sigma_i(m)\sigma_j(n-m)$* , Submitted.
- [6] A. Kim, D. Kim, and N. Y. İkikardes, *Remarks of congruent arithmetic sums of theta functions derived from divisor functions*, Honam Mathematical J., No.3, **35** (2013), 351–372.
- [7] D. B. Lahiri, *On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$* , I, Bull. Calcutta Math. Soc., **38** (1946), 193–206.
- [8] D. B. Lahiri, *On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$* , Part II, Bull. Calcutta Math. Soc., **39** (1947), 33–52.
- [9] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc., **22** (1916), 159–184.
- [10] K. S. Williams, *Number Theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, (2011).