

Two-step iterative methods with sixth-order convergence for solving nonlinear equations

Abstract

Two parameter families of sixth-order iterative methods for finding simple zeros of nonlinear equations are developed. The new methods have the order of convergence order of five or six. Per iteration these new methods require two evaluations of the function and two evaluations of the first-order derivatives. In fact, the efficiency index of these methods is $\sqrt[4]{6}$. Several examples are given to illustrate the efficiency of these new methods and their comparisons with other sixth-order methods.

Keywords: Nonlinear equations; Order of convergence; Computational efficiency; Iterative methods.
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1 Introduction

Iterative methods for solving nonlinear equations are of great practical importance in numerical analysis. In this paper, we consider iterative methods to find a simple root α of a nonlinear equation

$$f(x) = 0, \quad (1)$$

where $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ is a scalar function on an open interval D and it is sufficiently smooth in a neighbourhood of α . In this paper, the two new families of iterative methods of sixth-order are constructed by using the classical Newton methods in the first step. In order to obtain these new six order iterative methods, we introduce a weight function in the second step of our new method. The prime motive of this study is to develop a class of efficient two-step iterative methods for solving nonlinear equations. The sixth-order iterative methods presented in this paper use four evaluations of the function per iteration. Furthermore, the new sixth-order iterative methods have a better efficiency index than the existing the two and three-step methods presented in [1,3-8] and in view of this fact, the new methods are significantly better when compared with the established methods. Consequently, we have found that the new sixth-order iterative methods are efficient and robust.

The paper is organized as follows. A new family of two-step iterative methods of sixth-order are constructed in the next section by using the classical Newton method in the first step. The total number of function evaluations is four and according to the efficiency index [2,9] is $\sqrt[4]{6}$, and given in definition 2. In section 3, we shall compare the effectiveness of the new methods with the well-known sixth-order methods [6]. Finally, in section 4, we compare the performance of the new sixth-order methods with other established sixth-order methods.

2 Derivation of the methods and convergence analysis

In this section we shall define new sixth-order iterative methods. In order to establish the order of convergence of these new iterative methods we state the three essential definitions.

Definition 1 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence m is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^m} = \rho \neq 0 \quad (2)$$

where ρ is the asymptotic error constant and $m \in \mathbb{N}^+$.

Definition 2 Let λ be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [4,18] and defined as

$$\mu^{1/\lambda} \quad (3)$$

where μ is the order of the method.

Definition 3 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence [10] may be approximated by

$$\text{COC} \approx \frac{\ln |(x_{n+1} - \alpha)(x_n - \alpha)^{-1}|}{\ln |(x_n - \alpha)(x_{n-1} - \alpha)^{-1}|}, \quad (4)$$

where $n \in \mathbb{N}$.

2.1 The new two-step iterative methods

Consider the following iteration scheme

$$\begin{cases} y_n = x_n - \left(\frac{f(x_n)}{f'(x_n)} \right) \\ x_{n+1} = y_n - \left(\frac{f(y_n)}{f'(y_n)} \right) \end{cases} \quad (5)$$

This scheme consists of two steps in which the Newton method is repeated. It is clear that the formula (5) requires four evaluations per iteration and using (3) the efficiency index of (5) is $\sqrt[4]{4} = 1.414$, which is the same as the classical Newton method. In fact, scheme (5) does not increase the computational efficiency. The purpose of this paper is to establish new iterative methods with the better efficiency fifth-order method presented in [6] and the repeated Newton method given above. To derive a higher efficiency index, we introduce weight function in the second step. First we denote the following functions as,

$$t = \frac{f(y_n)}{f(x_n)}, \quad u = \frac{f(y_n)}{f'(y_n)}, \quad v = \frac{f'(x_n)}{f'(y_n)}. \quad (6)$$

In order to achieve the sixth-order convergence in the second step, the weight functions is expressed as

$$K(t) = (1 - t^2 + \beta t^4)^{-1}, \quad \beta \in \mathbb{R}^+, \quad (7)$$

$$L(t, u, v) = K(t) - 2t - 2tuv(t-1)^2 - 2t^3v^{-1}. \quad (8)$$

The weight functions (7) and (8) are formed to improve the order of convergence of (5). Then the iteration scheme (5) in its final form is given as

$$y_n = x_n - \left(\frac{f(x_n)}{f'(x_n)} \right) \quad (9)$$

$$x_{n+1} = y_n - L(t, u, v) \left(\frac{f(y_n)}{f'(y_n)} \right) \quad (10)$$

where $n \in \mathbb{N}$ and provided that the denominators in (9) and (10) are not equal to zero. Thus the scheme (10) defines a new multipoint method with suitable weight functions. To obtain the solution of (1) by the new sixth-order methods, we must set a particular initial approximation x_0 , ideally close to the simple root. In numerical mathematics it is very useful and essential to know the behaviour of an approximate method. Therefore, we shall prove the order of convergence of the new sixth-order method.

Theorem 1

Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ in an open interval D . If x_0 is sufficiently close to α , then the order of convergence of the new iterative method defined by (10) is six.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as $e = x - \alpha$. (11)

Using the Taylor expansion, we have

$$f(x_n) = f(\alpha) + f'(\alpha)e_n + \frac{1}{2}f''(\alpha)e_n^2 + \frac{1}{6}f'''(\alpha)e_n^3 + \frac{1}{24}f^{(iv)}(\alpha)e_n^4 + \dots \quad (12)$$

Taking $f(\alpha) = 0$ and simplifying, expression (12) becomes

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \dots \quad (13)$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)f'(\alpha)} \quad \text{for } k = 2, 3, 4, \dots \quad (14)$$

Furthermore, we have

$$f'(x_n) = f'(\alpha) \left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \dots \right] \quad (15)$$

Dividing (13) by (15), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \dots \quad (16)$$

and hence, we have

$$y_n = e_n - \frac{f(x_n)}{f'(x_n)} = c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 - (7c_2 c_3 - 4c_2^3 - 3c_4) e_n^4 + \dots \quad (17)$$

The expansion of the terms used in the second step is given as

$$f(y_n) = f'(\alpha) \left[e_n + c_2^2 e_n^2 - 4c_2(c_2^2 - c_3) e_n^3 + \dots \right], \quad (18)$$

$$f'(y_n) = f'(\alpha) \left[1 + 2c_2^2 e_n^2 - 4c_2(c_2^2 - c_3) e_n^3 + \dots \right], \quad (19)$$

$$K(t) = 1 + c_2^2 e_n^2 - 2c_2(3c_2^2 - 2c_3) e_n^3, \quad (20)$$

$$L(t, u, v) = 1 + c_2^2 e_n^2 + 2c_2(c_2^2 - c_3) e_n^3, \quad (21)$$

Since from (10) we have

$$x_{n+1} - \alpha = y_n - \alpha - L(t, u, v) \left(\frac{f(y_n)}{f'(y_n)} \right), \quad (22)$$

Collecting and Simplifying the appropriate terms used in (22), we obtain the error equation

$$e_{n+1} = c_2^2 (3c_4 - 5c_2 c_3 + 3c_2^3) e_n^6 \quad (23)$$

The expression (23) establishes the asymptotic error constant for the six order of convergence for the new iterative method defined by (10). $\square$

2.2 Method 2:

The method given below is actually the first improvement of the repeated Newton method. The improvement is achieved by using a single weight function $K(t)$ in the second step of (5). Consequently, we have found that the efficiency index and the order of convergence of the repeated Newton method are improved. The iteration scheme of the new method with single weight function is given as

$$x_{n+1} = y_n - K(t) \left(\frac{f(y_n)}{f'(y_n)} \right) \quad (24)$$

where $n \in \mathbb{N}$, y_n is given by (9) and provided that the denominators in (24) are not equal to zero. It is established that the above method converges to the order five.

Theorem 2

Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ in an open interval D . If x_0 is sufficiently close to α , then the order of convergence of the new iterative method defined by (24) is five.

Proof

Using appropriate expressions in the proof of the theorem 1 and substituting them into (24), we obtain the asymptotic error constant

$$e_{n+1} = 2c_2^2(2c_2^2 - c_3)e_n^5, \quad (25)$$

this means that the new method defined by (24) is fifth-order convergent. $\square$

2.3 The improved Fang et al method

In this section, we improve the fifth-order method presented in [1]. In order to improve the two-step fifth-order method, we have introduced a weight function in the second step of the Fang et al method. Hence we have improved the order of convergence and the efficiency of the Fang et al method. The iteration scheme of Fang et al [1] is given as

$$x_{n+1} = y_n - \left(\frac{5f'(x_n)^2 + 3f'(y_n)^2}{f'(x_n)^2 + 7f'(y_n)^2} \right) \left(\frac{f(y_n)}{f'(x_n)} \right), \quad (26)$$

and the improve scheme is

$$x_{n+1} = y_n - \left(\frac{5f'(x_n)^2 + 3f'(y_n)^2}{f'(x_n)^2 + 7f'(y_n)^2} \right) \left(\frac{f(y_n)}{f'(y_n)} \right) + M(t, u, v) \quad (27)$$

where

$$M(t, u, v) = tu - tuv(1-t)^2 - \left(\frac{5}{2}\right)t^3 f(y_n) f'(x_n)^{-1} \quad (28)$$

where $n \in \mathbb{N}$, $\{t, u, v\}$ is given by (6) and provided that the denominators in (27) are not equal to zero. By simply introducing a weight function in the second of the Fang et al method, we have formed a new method with better efficiency index and higher order of convergence.

Theorem 3

Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ in an open interval D . If x_0 is sufficiently close to α , then the order of convergence of the new iterative method defined by (27) is six.

Proof

Using appropriate expressions in the proof of the theorem 1 and substituting them into (27), we obtain the asymptotic error constant

$$e_{n+1} = 2^{-2}c_2(75c_2^4 - 33c_2^2c_3 + 4c_2c_4 - c_3^2)e_n^6 \quad (29)$$

The expression (29) establishes the asymptotic error constant for the sixth-order of convergence for the new iterative method defined by (27). $\square$

3 Kou et al three-step sixth-order methods

For the purpose of comparison, we consider two sixth-order methods presented in [6]. Since these methods are well established, we state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new sixth order iterative methods. The first of the Kou et al. method [6] is given as

$$\left. \begin{aligned} z_n &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \\ x_{n+1} &= z_n - \left(\frac{f'(y_n) + f'(x_n)}{3f'(y_n) - f'(x_n)} \right) \left(\frac{f(z_n)}{f'(x_n)} \right) \end{aligned} \right\} \quad (30)$$

The second of the sixth-order method considered is given in [6] and is expressed as

$$\left. \begin{aligned} z_n &= x_n - \left(\frac{f(x_n)}{2} \right) \left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right) \\ x_{n+1} &= z_n - \frac{2f'(y_n)f(z_n)}{f'(y_n)^2 + 2f'(y_n)f'(x_n) - f'(x_n)^2} \end{aligned} \right\} \quad (31)$$

where x_0 is the initial approximation and provided that the denominators of (30) and (31) are not equal to zero. In addition, further details and expressions of the recently introduced sixth-order method are given [6].

4 Application of the new sixth-order iterative methods

The new iterative methods given by (10), (24), (27), are employed to solve nonlinear equations and compare with the Kou et al sixth-order methods, given by (30) and (31). To demonstrate the performance of the new sixth-order methods, we use ten particular nonlinear equations. We shall determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of the sixth-order methods for determining the simple root of a nonlinear equation. Consequently, we shall give estimates of the approximate solution produced by the sixth-order methods and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

Remark 1

The new two-step iterative methods require four function evaluations and have the order of convergence six. To determine the efficiency index of these new methods, we shall use the definition 2. Hence, the efficiency index of the sixth-order iterative methods given is $\sqrt[4]{6}$, which are better than the other two and three point iterative methods given in [1,3-8]. Furthermore, the efficiency index of the fifth-order iterative methods given is $\sqrt[4]{5}$.

Remark 2

The test functions and their exact root α are displayed in table 1. The difference between the root α and the approximation x_n for test functions with initial approximation x_0 are displayed in Table 2. In fact, x_n is calculated by using the same total number of function

evaluations (TNFE) for all methods. In the calculations, 12 TNFE are used by each method. Furthermore, the computational order of convergence (COC) is displayed in Table 3.

Table 1

Test functions and their roots.

<i>Functions</i>	<i>Roots</i>	<i>Initial Estimate</i>
$f_1(x) = \exp(x)\sin(x) + \ln(1+x^2)$	$\alpha = 0$	$x_0 = 0.25$
$f_2(x) = (x^2 - 1)^{-1} - 1$	$\alpha = \sqrt{2}$	$x_0 = 1.2$
$f_3(x) = (x-2)(x^{10} + x + 1)\exp(-x-1)$	$\alpha = 2$	$x_0 = 2.1$
$f_4(x) = (x+1)\exp(\sin(x)) - x^2 \exp(\cos(x)) - 1$	$\alpha = 0$	$x_0 = 0.5$
$f_5(x) = \sin(x)^2 - x^2 + 1$	$\alpha = 1.40449165\dots$	$x_0 = 1.2$
$f_6(x) = \exp(-x) - \cos(x)$,	$\alpha = 0$	$x_0 = -0.1666\dots$
$f_7(x) = \ln(x^2 + x + 2) - x + 1$	$\alpha = 4.15259074\dots$	$x_0 = 4.5$
$f_8(x) = x^{12} - 2x^3 - x + 1$	$\alpha = 0.59033443\dots$	$x_0 = 0.5$
$f_9(x) = \cos(x)^2 - 5^{-1}x$	$\alpha = 2.32020427\dots$	$x_0 = 2.5$
$f_{10}(x) = x^{20} + x^4 + 4x^2 - 15$	$\alpha = -1.113005\dots$	$x_0 = -1.2$

Table 2

Comparison of various iterative methods

f_i	(10)	(24)	(26)	(27)	(30)	(31)
f_1	0.591e-50	0.803e-85	0.107e-47	0.517e-68	0.133e-103	0.152e-157
f_2	0.550e-33	0.766e-123	0.387e-27	0.128e-37	0.266e-42	0.173e-17
f_3	0.596e-57	0.192e-105	0.308e-51	0.888e-74	0.137e-159	0.510e-121
f_4	0.109e-120	0.106e-185	0.145e-108	0.441e-149	0.248e-120	0.992e-125
f_5	0.603e-58	0.178e-109	0.687e-73	0.897e-88	0.447e-168	0.350e-221
f_6	0.109e-92	0.786e-163	0.430e-90	0.186e-135	0.226e-188	0.107e-230
f_7	0.386e-185	0.254e312	0.169e-189	0.441e-299	0.165e-386	0.264e-339
f_8	0.604e-99	0.103e-190	0.715e-106	0.461e-152	0.726e-222	0.461e-294
f_9	0.216e-130	0.350e-225	0.237e-138	0.148e-232	0.137e-195	0.359e-195
f_{10}	0.535e-35	0.110e-59	0.234e-27	0.206e-37	0.104e-71	0.715e-48

Table 3

COC of various iterative methods

f_i	(10)	(24)	(26)	(27)	(30)	(31)
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f_1	4.996	6.000	4.997	5.996	6.000	5.977
f_2	4.992	5.962	4.996	5.984	6.002	5.892
f_3	4.998	6.007	4.999	5.998	5.977	6.000
f_4	5.000	6.000	5.000	5.974	6.000	6.000
f_5	5.002	5.952	5.000	5.999	6.000	5.988
f_6	5.000	6.000	5.000	6.000	6.000	6.000
f_7	5.000	6.000	5.000	6.000	6.000	6.000
f_8	5.000	5.984	5.000	6.000	6.000	6.000
f_9	5.000	5.989	5.000	6.000	5.985	5.985
f_{10}	4.981	5.819	4.983	5.971	6.001	5.998

5 Conclusion

In this study, we have constructed new sixth-order iterative methods for solving nonlinear equations. Convergence analysis proves that these new iterative methods preserve their order of convergence. From the results in the tables and a number of numerical experiments, it can be concluded that the convergence of the new multipoint methods of the sixth-order is remarkably fast. The major advantages of the new sixth-order iterative methods are, the new sixth-order methods have a better efficiency index than the established fifth-order method [1], we have improved the existing fifth-order method and the precision of new sixth-order method is better than the improved sixth-order method. We have examined the effectiveness of the new sixth-order methods by showing the accuracy of the simple root of a nonlinear equation. The main purpose of demonstrating the new sixth-order iterative methods for many different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative methods. Finally, we conclude that the new iterative methods may be considered a very good alternative to the classical methods.

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