

Complexiton solutions of nonlinear partial differential equations with a new auxiliary equation

Abstract

In this paper, a novel auxiliary equation: $u'' = a + bu + cu^3$ which interaction solutions including trigonometric and hyperbolic functions solutions are succeed to achieve are considered. Miscellaneous solutions corresponding to using this auxiliary equation for a series of partial differential equation models are easily, effective obtained. It is prime important to motivating authors to accurately analyze special phenomenon in their relevant aspects.

Keywords: Complexiton solution; Riccati function; $u'' = a + bu + cu^3$

1 Introduction

As it is well-known, many physics and nature science are usually characterized well by ubiquitous nonlinear dynamical equations. And soliton is one of significance fields in nonlinearity, travelling wave solutions of mathematical and physics nonlinear models especially the most representative equations like the KdV equation [1], Schrödinger equation manifest that they all contributing to assist people describe the nature science better. In previously, the complexiton solutions of nonlinear evolution equation attracted a numerous attention, as they are usually tended to uncovering potential meaningful application. As example, a long time ago people have no practical countermeasure when suffer from tsunami and earthquake, through investigating soliton solutions of nonlinear correlative equation people will looking for their related properties to prevent large damage.

In currently, multifarious methods for searching analytical solutions of nonlinear equation have evolved from complicated process with heavy computation to simpleness and understandability, and advanced compute technology is in high-speed

development along with our modernize modernise society. Sub-equation approach which contain the the homogeneous balance method [2], sine-cosine method [3], the sech-function method [4], the hyperbolic tangent function method [5, 6], the multiple exp-function method [7, 8], as one of widespread applications to constructing exact solutions. Ma [10], Fan [9], Chen [11, 12], Chen [13, 14], Yan [15, 16] are devoted to constructing special soliton solutions by using different or combination of auxiliary equations and get some success.

In this paper, a new sub-equation: $u'' = a + bu + cu^3$ which we can seek out the particular nonlinear interactive wave solutions (interaction solution) including trigonometric and hyperbolic function solutions is proposed with a wide concrete applications that can propagate with extreme rapidity to many branches of nonlinear areas. Then solutions of examples are calculated through mathematical techniques and used to illustrate the effective of this method.

2 The new interaction solutions of $u'' = a + bu + cu^3$

For desired equation read

$$u'' = a + bu + cu^3, \quad (1)$$

where $u'' = u''(\xi)$ In order to work out $u(\xi)$, hypothesis are taken as follow:

$$u(\xi) = a_0 + \frac{a_1 G(\xi) H(\xi) + a_2 G'(\xi) H'(\xi)}{a_3 F(\xi) + 1}, \quad (2)$$

Where $F(\xi), G(\xi), H(\xi)$ are functions satisfying the Riccati equation respectively. And a_0, a_1, a_2, a_3 are constant to be determined later.

$$\begin{aligned} F'(\xi) &= A_1 + B_1 F(\xi) + C_1 F^2(\xi) \\ G'(\xi) &= A_2 + B_2 G(\xi) + C_2 G^2(\xi) \\ H'(\xi) &= A_3 + B_3 H(\xi) + C_3 H^2(\xi), \end{aligned} \quad (3)$$

Where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$, are free constant.

Insert eqno(2) into eqno(1) with the related Riccati auxiliary equations eqno(3), then setting collecting the coefficients of $F^i(\xi)G^j(\xi)H^s(\xi)$ ($0 \leq i, j, s \leq 6$) equate to zero, we derive a system of over determined linear equations with a_0, a_1, a_2, a_3 $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$.

The precondition relationship:

$$A_1 = A_1, B_1 = B_1, C_1 = -a_3^2 A_1 + a_3 B_1, A_2 = A_2, B_2 = B_2, C_2, A_3 = 0, B_3 = -\frac{B_1}{2} + a_3 A_1, C_3 = 0, a = -\frac{(B_1^2 - 4a_3 B_1 A_1 + 4a_3^2 A_1^2)}{4} a_0, b = \frac{B_1^2 - a_3 B_1 A_1 + a_3^2 A_1^2}{4}, c = 0 \quad a_0 = a_0, a_1 = a_1, a_2 = 0, a_3 = a_3.$$

$$\text{When } A_1 = \frac{1}{2}, B_1 = 0, C_1 = \frac{1}{2}$$

$$u_1 = a_0 + \frac{a_1 (\tan(\xi) \pm \sec(\xi)) (\sinh(B_3 \xi) + \cosh(B_3 \xi))}{a_3 (\tan(\xi) \pm \sec(\xi)) + 1} \quad (4)$$

$$u_2 = a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1} \quad (5)$$

$$u_3 = a_0 + \frac{a_1(\frac{\tan(\xi)}{1 \pm \sec(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) + 1} \quad (6)$$

When $A_1 = -\frac{1}{2}, B_1 = 0, C_1 = -\frac{1}{2}$

$$u_4 = a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1} \quad (7)$$

$$u_5 = a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1} \quad (8)$$

$$u_6 = a_0 + \frac{a_1(\frac{\cot(\xi)}{1 \pm \csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) + 1} \quad (9)$$

When $A_1 = 1, B_1 = 0, C_1 = 4$

$$u_7 = a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \tan^2(\xi)}) + 1} \quad (10)$$

When $A_1 = -1, B_1 = 0, C_1 = -4$

$$u_8 = a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \cot^2(\xi)}) + 1} \quad (11)$$

Where a_0, a_2, a_3, B_2, B_3 are arbitrary constant. Typical explicit solutions are taken into consideration above, and there exist abundant interaction solutions of eqno(1) we omit in the paper. In the wake of such work, we aim at wielding the auxiliary equation to the governing equations as following.

3 Concrete applications of this sub-equation

Example 1 Consider the (2+1) dimensional KdV equation[17]

$$\begin{aligned} U_t + U_{xxx} - 3U_xV - 3UV_x &= 0 \\ U_x - V_y &= 0 \end{aligned} \quad (12)$$

Here, to start off, we have the hypothesis in the following terms is obtained:

$$U(\xi) = \sum_{p=0}^m m_p u^p(\xi),$$

$$V(\xi) = \sum_{q=0}^n n_q u^q(\xi), u(\xi) = x + ky - vt \quad (13)$$

Here k, v are constant, v represent as a the wave speed. Where m, n are positive integer equate to 2 respectively which are determined by the principle of homogeneous balance. $u(\xi)$ satisfy the sub-equation: $u'' = a + bu + cu^3$.

$$\begin{aligned} U(\xi) &= m_0 + m_1 u(\xi) + m_2 u^2(\xi), \\ V(\xi) &= n_0 + n_1 u(\xi) + n_2 u^2(\xi) \end{aligned} \quad (14)$$

$m_0, m_1, m_2, n_0, n_1, n_2$ are all acquired in the later. Hence, when we substitute eqno(14) into eqno(13) along with aid of auxiliary equation. Equating the coefficients of $u^\alpha(\xi), u'(\xi)$ ($0 \leq \alpha \leq 3$) to zero, a set of algebraic equations are yielded that unknown parameters $m_0, m_1, m_2, n_0, n_1, n_2, k, c, v$ are able to solve through using the computation of Maple.

Then analytical interaction solutions of system eqno(12): when $k = k, v = v, m_0 = \frac{1}{3}k(-v + 4b - 3n_0), m_1 = 0, m_2 = n_2, n_0 = n_0, n_1 = 0, n_2 = n_2$ with $u_1 \dots u_8$ as explicit solutions of eqno(1):

$$\begin{aligned} U_1 &= \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1})^2 \\ V_1 &= n_0 + n_2(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1})^2 \end{aligned} \quad (15)$$

$$\begin{aligned} U_2 &= \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1})^2 \\ V_2 &= n_0 + n_2(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1})^2 \end{aligned} \quad (16)$$

$$\begin{aligned} U_3 &= \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) + 1})^2 \\ V_3 &= n_0 + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) + 1})^2 \end{aligned} \quad (17)$$

$$\begin{aligned} U_4 &= \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1})^2 \\ V_4 &= n_0 + n_2(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1})^2 \end{aligned} \quad (18)$$

$$U_5 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1})^2$$

$$V_5 = n_0 + n_2(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1})^2 \quad (19)$$

$$U_6 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\cot(\xi)}{1 \pm \csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) + 1})^2$$

$$V_6 = n_0 + n_2(a_0 + \frac{a_1(\frac{\cot(\xi)}{1 \pm \csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) + 1})^2 \quad (20)$$

$$U_7 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \tan^2(\xi)}) + 1})^2$$

$$V_7 = n_0 + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \tan^2(\xi)}) + 1})^2 \quad (21)$$

$$U_8 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \cot^2(\xi)}) + 1})^2$$

$$V_8 = n_0 + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \cot^2(\xi)}) + 1})^2 \quad (22)$$

Where B_3 have the same meaning as before in eqno(3).

Example 2 Consider the Hirota-Satsuma equations[18].

$$U_t + U_{xxx} + 6UU_x - 6VV_x = 0$$

$$V_t - 2V_{xxx} - 6UV_x = 0 \quad (23)$$

We handling eqno(23) as the same of eqno(12) with hypothesis eqno(13), $m = 2, n = 2$ according the homogeneous balance law.

$$U(\xi) = m_0 + m_1u(\xi) + m_2u^2(\xi),$$

$$V(\xi) = n_0 + n_1u(\xi) + n_2u^2(\xi) \quad (24)$$

Where $m_0, m_1, m_2, n_0, n_1, n_2$ are constant will be determined by the following work. Here $u(\xi)$ fulfills the ordinary sub-equation $u'' = a + bu + cu^3$ ($u(\xi) = u(x - lt)$).

With the help of solutions of $u'' = a + bu + cu^3$, collecting the coefficients of all power of $u(\xi)$ and $u'(\xi)$ equate to zero afterward taking the eqno(24) into eqno(23). Two series of free constants are obtained: Case 1: $v = v, m_0 = m_0, m_1 = 0, m_2 = \text{RootOf}(Z^2 - 2, \text{label} =_L 1)n_2, n_0 = -\frac{1}{4}\text{RootOf}(Z^2 - 2, \text{label} =_L 1)(l - 2m_0), n_1 = 0, n_2 = n_2$ Case 2: $v = \frac{-(n_1^2 + 2m_0m_2)}{m_2}, m_0 = m_0, m_1 = 0, m_2 = m_2, n_0 = n_0, n_1 = n_1, n_2 = 0$

Travelling exact solutions of eqno(23) in case 2:

$$\begin{aligned} U_1 &= m_0 + m_2(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1})^2 \\ V_1 &= n_0 + n_1(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1}) \end{aligned} \quad (25)$$

$$\begin{aligned} U_2 &= m_0 + m_2(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1})^2 \\ V_2 &= n_0 + n_1(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1}) \end{aligned} \quad (26)$$

$$\begin{aligned} U_3 &= m_0 + m_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) + 1})^2 \\ V_3 &= n_0 + n_1(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) + 1}) \end{aligned} \quad (27)$$

$$\begin{aligned} U_4 &= m_0 + m_2(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1})^2 \\ V_4 &= n_0 + n_1(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1}) \end{aligned} \quad (28)$$

$$\begin{aligned} U_5 &= m_0 + m_2(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1})^2 \\ V_5 &= n_0 + n_1(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1}) \end{aligned} \quad (29)$$

$$\begin{aligned} U_6 &= m_0 + m_2(a_0 + \frac{a_1(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) + 1})^2 \\ V_6 &= n_0 + n_1(a_0 + \frac{a_1(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) + 1}) \end{aligned} \quad (30)$$

$$U_7 = m_0 + m_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \tan^2(\xi)}) (\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \tan^2(\xi)}) + 1})^2$$

$$V_7 = n_0 + n_1(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\tan^2(\xi)}) + 1}) \quad (31)$$

$$U_8 = m_0 + m_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1})^2$$

$$V_8 = n_0 + n_1(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1}) \quad (32)$$

Where $m_0, m_2, n_0, n_1, a_0, a_2, B_2, B_3$, are arbitrary constant. $U(\xi) = U(x - vt)$, v is propagation speed of soliton waves. Particular obtained solutions $U_i, V_i (1 \leq i \leq 8)$ work in concert with solutions of solvable ordinary eqno(1) which coefficients need to satisfying $a = -ca_0^3 - ba_0, b = b, c = c$. There some more interaction solutions of eqno(23) we ignored in this paper as in case 1.

4 Conclusion and discussion

In this paper, two physics model examples response the novel auxiliary equation that carry a major contribution for searching interaction wave solutions. Trigonometric and hyperbolic function solutions are not generated in ref [17], [19]. It will draw great attention as solutions of the novel auxiliary equation themselves are interactive. And the results of the paper dedicate that obtained accurate soliton solutions using this sub-equation remain have a large application. Complicate physical phenomena in nonlinear model systems may be study well in the future apply the typical interaction solutions we get.

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