

On the Properties of the Set Solutions of a Class of Paratingent Equation with Delay

ABSTRACT

In this paper we will study the main properties of the set solutions of the paratingent equation (type differential inclusion) with delay of the form: $(Pt x)(t) \subset F([x]_t)$ for $t \geq 0$ with the initial condition: $x(t) = \xi(t)$ for $t \leq 0$. We will be interested particularly in the topological properties of the set solutions and the continuous dependence on initial conditions of the set solutions.

Keywords: Attainable set, compact, continuous dependence, delay, differential inclusion, emission, paratingent, set solutions.

1. INTRODUCTION

In the present study, we focus on the fundamental properties of set solutions of paratingent equation with delay. The study of these properties is justified by the fact that the main topic considered while dealing with differential inclusions is the topological properties of the set solutions. This set, which is consisted of the union of the graphs of all trajectories of the differential inclusions, is called emission or reachable or attainable set of differential inclusions. It is well know that the main properties of trajectories and set solutions of control system are equivalent to the properties of trajectories and set solutions of a certain differential inclusions. On the other hand, in control approach, the reaction of real world systems to exogenous signals is never instantaneously and it needs some delay. In general, a time-delay system arises as a result of inherent delays in the transmission of information between different parts of the system and/or as a deliberate introduction of time delay into the system for control purposes. Such systems exist in various fields of application, such as in physics, engineering, biology, medicine, and economics. In control theory, many proofs of optimal control existence depend upon showing that the set solutions is compact.

The existence of solution and the main properties of set solutions of differential inclusions have been studied by a number of authors [2, 4, 9, 11, 12] and references therein. For more details on differential inclusions see the books [1, 5, 7, 8, 10].

In [4] we have established the existence of solutions to the paratingent equation with delay of the form: $(Pt x)(t) \subset F([x]_t)$ for $t \geq 0$ with the initial condition: $x(t) = \xi(t)$ for $t \leq 0$, under weak assumptions. The main purpose of this paper is to prove the compactness and the (upper semi) continuous dependence on the initial conditions of the set solutions of the above paratingent equation under the same weak assumptions than in [4].

The paper is organized as follows: In section 2, notations, preliminary definitions are introduced and we recall an existence result. In section 3 we obtain new results. In Theorem

3.1 the compactness of the solution set is proved. In Theorem 3.2 we establish the (upper semi) continuous dependence on the initial conditions of the set solutions. Our proofs are essentially shorter.

2. PRELIMINARIES

Let (E, ρ) and (E', ρ') two metric spaces. By $CompE$, we denote the set of all the nonempty and compact subsets of E . If, moreover, E is a vector space, then $ConvE$ denotes the set of all the convex elements of $CompE$.

A set-valued map, $F: E \rightarrow CompE'$, is called upper semi-continuous in E , and denoted by u.s.c, if for any point $a \in E$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that: $x \in B(a, \delta) \Rightarrow F(x) \in B(F(a), \varepsilon)$ when $B(a, \delta) = \{x \in E / \rho(a, x) < \delta\}$ and $B(F(a), \varepsilon) = \{y \in E' / z \in F(a) \text{ and } \rho'(y, z) < \varepsilon\}$. (See [3])

We have, on the upper semi-continuity of a set-valued map, the following lemma (see [6]):

Lemma 2.1. *Let (E, ρ) and (E', ρ') two metric spaces. A set-valued map $F: E \rightarrow CompE'$, is u.s.c if and only if, for all sequences $\{x_i\} \in E$ and $\{y_i\} \in E'$ such that $\{x_i\} \rightarrow x_0$ and $\{y_i\} \in F(x_i)$, there exists a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ which is convergent to $y_0 \in F(x_0)$.*

Let C the space of all continuous functions $x: R \rightarrow R^n$ with the topology defined by an almost uniform convergence (i.e. a uniform convergence on each compact interval of R). It is well know that the almost uniform convergence in C is equivalent to the convergence by the metric defined as follows:

$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, \sup\{|x(t) - y(t)|, -i \leq t \leq i\}\}$, for $x, y \in C$, then C is a metric locally convex linear topological space. Let $\beta < 0$ be a fixed real number and let $I = [0, \infty[\subset R$. If $x \in C$, the symbol $[x]_t$ will be denote the restriction of x on the interval $[\beta, t]$ when $t \in I$ and $\|x\|_t = \max\{|x(s)|, \beta \leq s \leq t\}$ with $|x| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ for $x = (x_1, x_2, \dots, x_n) \in R^n$.

Let G denote the metric space whose elements are functions $[x]_t, [y]_t, \dots$, where $t \in I, u \in I$, the distance between two functions $[x]_t, [y]_t$, being understood as a distance of their graphs in $R \times R^n$ in the Hausdorff sense.

Having a function $x \in C$ and $t \in I$, the set of limit points $\lim_{\substack{u_i \rightarrow t \\ u_i \neq s_i}} \frac{x(u_i) - x(s_i)}{u_i - s_i}$ where $u_i \in I, s_i \in I, u_i \neq s_i, (i = 1, 2, \dots)$, and $\lim u_i = \lim s_i = t$, is called the paratingent of x at the point t and denoted by $(Pt x)(t)$. It is easy to see that $Pt x$ maps the interval I into the family of the nonempty and closed subsets of R^n .

2.1 Paratingent equation with a delay

Let a set-valued map $F: G \rightarrow CompR^n$, a relation of the form:

$$(Pt x)(t) \subset F([x]_t) \quad (1)$$

is called paratingent equation with delay where $t \in I$ and $x \in C$. Every function $x \in C$ satisfying (1) will be called the solution of the equation (1).

The generalized problem of Cauchy for the equation (1) consists in the search for a solution of the equation (1) which will satisfy the initial condition:

$$x(t) = \xi(t) \quad (2)$$

89 for $t \in [\beta, 0]$ where the function $\xi \in \mathcal{C}$, called the initial function is given in advance.

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91 2.2 Existence of solutions

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93 Assume the following hypothesis:

94 **Hypothesis (H):** The set-valued mapping $F: G \rightarrow \text{Conv}R^n$ is upper semi-continuous and
 95 satisfies the condition: $F([x]_t) \subset B(0, w(t, \|x\|_t))$ for $t \geq 0$ where $B(0, r)$ denote the closed
 96 ball of center 0 of R^n and radius r , $w(t, y)$ is a continuous function from $I \times I$ to I , increasing
 97 in y and such that the ordinary differential equation: $y' = w(t, y)$, with the initial condition
 98 $y(0) = A$ (A an arbitrary real positive number) has a maximal solution, denoted $M(t)$, on all
 99 intervals I and for all A .

100 Let D a compact subset of $\mathcal{C}$ and Φ the class of all the functions $x \in \mathcal{C}$ satisfying the
 101 inequality $|x(t)| \leq M(T)$ for $t \in [0, T]$, T a real positive number arbitrarily fixed. In [4] we
 102 have established the following theorem:

103 **Theorem 2.2.** [4] *Under the hypothesis (H), in the class Φ and for any $\xi \in D$, there is at least*
 104 *a solution of the problem (1)-(2).*

105 From the proof of the theorem 2.2, it follows:

106
 107 **Corollary 2.3.** If $y \in \Phi$ and $(Pty)(t) \subset B(0, w(t, \|y\|_t))$, we have: $|y(t) - y(t')| \leq \Omega_T |t -$
 108 $t'|$ for $t, t' \in [0, T]$, where $\Omega_T = \max\{w(s, M(T)): s \in [0, T]\}$.

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111 3. MAIN RESULTS

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113 **Definitions.** A function x is called a trajectory of $F([y]_t)$, if $(Ptx)(t) \subset F([x]_t)$ for $t \geq 0$. We
 114 call emission of the function ξ , the set of all trajectories x of $F([y]_t)$ such that $x \in \Phi$ and
 115 $x(t) = \xi(t)$ for all $t \in [\beta, 0]$. We denote this set by $E(F, \xi)$.

116 **Theorem 3.1** *If the application $F: G \rightarrow \text{Conv}R^n$ verifies the assumption H, then the emission*
 117 *$E(F, \xi)$ of the function ξ is a nonempty and compact set of $\mathcal{C}$.*

118 **Proof.** In view of theorem 2.2, it following that the set $E(F, \xi)$ is nonempty. Let $x \in E(F, \xi)$
 119 then we have: $\|x\|_t \leq M(T)$ for $t \in [0, T]$, $x(t) = \xi(t)$ for $t \in [\beta, 0]$ and
 120 $|x(t) - x(t')| \leq \Omega_T |t - t'|$ for $t, t' \in [0, T]$, where $\Omega_T = \max\{w(s, M(T)): s \in [0, T]\}$. The
 121 functions belonging to $E(F, \xi)$ are uniformly bonded and equicontinuous. Therefore, $E(F, \xi)$ is
 122 a compact subset of space $\mathcal{C}$, according to the theorem of Arzela.

123 **Theorem 3.2.** *Consider the set-valued application $E: D \rightarrow \text{Comp}\mathcal{C}$ defined by: $E(\xi) =$*
 124 *$E(F, \xi)$ where $\xi \in D$. If the application F verifies the assumption H, then the application E is*
 125 *upper semi-continuous in D .*

126 **Proof.** Let $\{\xi_i\}, i = 1, 2, \dots$, a sequence of functions of D convergent to a function $\xi \in D$ and
 127 $\{x_i\}, i = 1, 2, \dots$, a sequence of functions of Φ such that $x_i \in E(\xi_i), i = 1, 2, \dots$. We have
 128 $x_i \in E(\xi_i)$ from where: $|x_i(t)| \leq M(T)$ for $t \in [0, T]$ and $x_i(t) = \xi_i(t)$ for $t \in [\beta, 0]$. From
 129 corollary 2.3 we have: $|x_i(t) - x_i(t')| \leq \Omega_T |t - t'|$ for $t, t' \in [0, T]$, where $\Omega_T =$
 130 $\max\{w(s, M(T)): s \in [0, T]\}$. By theorem 3.1 $E(F, \xi)$ is compact, then we can extract a
 131 subsequence $\{x_{ij}\}$ of $\{x_i\}$ which converges to a certain function x . It is obvious that the
 132 function x satisfies the following conditions: $|x(t)| \leq M(T)$ for $t \in [0, T]$ and $x(t) = \xi(t)$ for
 133 $t \in [\beta, 0]$. Otherwise we have: $(Ptx_{ij})(t) \subset F([x_i]_t)$ for $t \geq 0$, it follows, according to

lemma 3.7 in [4] that: $(Pt x)(t) \subset F([x]_t)$, for $t \geq 0$, and thus $x \in E(\xi)$. Finally, we prove the upper semi-continuity of the application E by using lemma 2.1.

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137 4. CONCLUSION

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139 In this work we have established the compactness of the solution set and proved the (upper
140 semi) continuous dependence on the initial conditions of the set solutions.

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