

Original Research Article

Traveling wave solutions of the simplified MCH equation via $\exp(-\Phi(\xi))$ -expansion method

Abstract

The $\exp(-\Phi(\xi))$ -expansion method is an efficient method for obtaining exact traveling wave solutions of nonlinear evolution equations. In this paper, the $\exp(-\Phi(\xi))$ -expansion method is applied to construct exact traveling wave solutions of the simplified MCH equation. The traveling wave solutions are expressed in terms of the hyperbolic functions, the trigonometric functions and the rational functions. It is shown that the method is straightforward and effective mathematical tool for solving nonlinear evolution equations in mathematical physics and engineering.

Keywords: The $\exp(-\Phi(\xi))$ -expansion method; the simplified MCH equation; traveling wave solutions; solitary wave solutions.

PACS: 05.45.Yv, 02.30.Jr, 02.30.Ik.

Introduction

Many complex real world problems in nature are due to nonlinear phenomena. Nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus the issue becomes more complicated and hence needs ultimate solution. Therefore, the studies of exact solutions of nonlinear evolution equations (NLEEs) play a crucial role to understand the internal mechanism of nonlinear phenomena. Advance nonlinear techniques are significant to solve inherent nonlinear problems, particularly those involving differential equations, dynamical systems and related areas. In recent years, both

the mathematicians and physicists have made significant improvement in finding the exact solutions of NLEEs. They establish many effective and powerful methods to handle the NLEEs. For example, the Jacobi elliptic function expansion method [1, 2], the Backlund transformation method [3], the F-expansion method[4, 5], the Darboux transformation method [6], the inverse scattering transform [7], the Adomian decomposition method [8, 9], the complex hyperbolic function method [10, 11], the homogeneous balance method [12-14], the (G'/G) -expansion method [15-22], the modified simple equation method [23, 24], the auxiliary equation method[25, 26], the exp-functions method[27], the $\exp(-\varphi(\eta))$ -expansion method [28] and so on.

The objective of this article is to apply the $\exp(-\varphi(\xi))$ -expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the simplified MCH equation.

Our aim in this paper is to present an application of the $\exp(-\varphi(\xi))$ -expansion method to the simplified MCH equation to be solved by this method for the first time.

The rest of the paper is organized as follows: In Section 2, we give the description of the $\exp(-\varphi(\xi))$ -expansion method. In Section 3, we apply this method to the simplified MCH equation and graphical representations of the solutions. Conclusions are given in the last section.

2. Description of the $\exp(-\Phi(\xi))$ -expansion method

Let us consider a general nonlinear PDE in the form

$$F(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \dots), \quad (1)$$

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u(x, t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. In the following, we give the main steps of this method:

49 **Step 1:** We combine the real variables x and t by a complex variable ξ

$$50 \quad u(x, t) = u(\xi), \quad \xi = x \pm V t, \quad (2)$$

51 where V is the speed of the traveling wave. The traveling wave transformation (2) converts

52 Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$53 \quad \Re(u, u', u'', u''', \dots), \quad (3)$$

54 where $\Re$ is a polynomial of u and its derivatives and the superscripts indicate the ordinary
55 derivatives with respect to ξ .

56 **Step 2.** Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$57 \quad u(\xi) = \sum_{i=0}^N \alpha_i (\exp(-\Phi(\xi)))^i, \quad (4)$$

58 where α_i ($0 \leq i \leq N$) are constants to be determined, such that $\alpha_N \neq 0$ and $\Phi = \Phi(\xi)$

59 satisfies the following ordinary differential equation:

$$60 \quad \Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda, \quad (5)$$

61 Eq. (5) gives the following solutions:

62 **Family 1:** When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$63 \quad \Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)} \tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + E)\right) - \lambda}{2\mu}\right) \quad (6)$$

64 **Family 2:** When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$65 \quad \Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)} \tan\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2}(\xi + E)\right) - \lambda}{2\mu}\right) \quad (7)$$

66 **Family 3:** When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$67 \quad \Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1}\right) \quad (8)$$

68 **Family 4:** When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi(\xi) = \ln\left(-\frac{2(\lambda(\xi + E) + 2)}{\lambda^2(\xi + E)}\right) \quad (9)$$

Family 5: When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi(\xi) = \ln(\xi + E) \quad (10)$$

$\alpha_N, \dots, V, \lambda, \mu$ are constants to be determined latter, $\alpha_N \neq 0$, the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: We substitute Eq. (4) into Eq. (3) and then we account the function $\exp(-\Phi(\xi))$. As a result of this substitution, we get a polynomial of $\exp(-\Phi(\xi))$. We equate all the coefficients of same power of $\exp(-\Phi(\xi))$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find $\alpha_N, \dots, V, \lambda, \mu$. Substituting the values of $\alpha_N, \dots, V, \lambda, \mu$ into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

3. The simplified MCH equation

Now we will bring to bear the $\exp(-\varphi(\eta))$ -expansion method to find exact solutions and then the solitary wave solutions of the simplified MCH equation in the form,

$$u_t + 2ku_x - u_{xxt} + \beta u^2 u_x = 0. \quad \text{where } k \in \Re, \beta > 0. \quad (11)$$

Details of CH and MCH equations can be found in references [29-33].

Now, we use the traveling wave transformation Eq. (2) into Eq. (11), which yields

$$-V u' + 2ku' + V u''' + \beta u^2 u' = 0. \quad (12)$$

where the superscripts stand for the derivatives with respect to ξ .

Integrating Eq. (12) once with respect to ξ yields:

$$(2k - V)u + V u'' + \frac{\beta}{3} u^3 + P = 0. \quad (13)$$

91 where P is an integral constant that could be determined later.

92 Taking the homogeneous balance between u^3 and u'' in Eq. (13), we obtain $N = 1$. Therefore,
93 the solution of Eq. (13) is of the form

$$94 \quad u(\eta) = \alpha_0 + \alpha_1(\exp(-\Phi(\xi))), \quad (14)$$

95 where α_0, α_1 are constants to be determined such that $\alpha_N \neq 0$, while λ, μ are arbitrary
96 constants.

97 Substituting Eq. (14) into Eq. (13) and then equating the coefficients of $\exp(-\Phi(\xi))$ to zero,
98 we get

$$99 \quad 2V\alpha_1 + \frac{1}{3}\beta\alpha_1^3 = 0, \quad (15)$$

$$100 \quad \beta\alpha_0\alpha_1^2 + 3V\alpha_1\lambda = 0, \quad (16)$$

$$101 \quad -V\alpha_1 + V\alpha_1\lambda^2 + 2V\alpha_1\mu + 2k\alpha_1 + \beta\alpha_0^2\alpha_1 = 0, \quad (17)$$

$$102 \quad 2k\alpha_0 + \frac{1}{3}\beta\alpha_0^3 - V\alpha_0 + P + V\alpha_1\mu\lambda = 0, \quad (18)$$

103

104 Solving the Eq. (15)-Eq. (18) yields

$$105 \quad P = 0, \quad V = \frac{4k}{2 + \lambda^2 - 4\mu}, \quad \alpha_0 = \pm \sqrt{\frac{6k}{(-2\beta - \beta\lambda^2 + 4\beta\mu)}}, \quad \alpha_1 = \pm 2\sqrt{\frac{6k}{(-2\beta - \beta\lambda^2 + 4\beta\mu)}}.$$

106 where λ, μ, β, k are arbitrary constants.

107 Now substituting the values of V, α_0, α_1 into Eq. (14) yields

$$108 \quad u(\xi) = \pm \sqrt{\frac{6k}{(-2\beta - \beta\lambda^2 + 4\beta\mu)}} (1 + 2(\exp(-\Phi(\xi)))), \quad (19)$$

109 where $\xi = x - (\frac{4k}{2 + \lambda^2 - 4\mu})t$.

110 Now substituting Eq. (6) - Eq. (10) into Eq. (19) respectively, we get the following five
111 traveling wave solutions of the simplified MCH equation.

112 When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$113 \quad u_1(\xi) = \pm \sqrt{\frac{6k}{(-2\beta - \beta\lambda^2 + 4\beta\mu)}} \left(1 - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E)\right) + \lambda} \right).$$

114 where $\xi = x - (\frac{4k}{2 + \lambda^2 - 4\mu})t$ and E is an arbitrary constant.

115 When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$116 \quad u_2(\xi) = \pm \sqrt{\frac{6k}{(-2\beta - \beta\lambda^2 + 4\beta\mu)}} \left(1 + \frac{4\mu}{\sqrt{4\mu - \lambda^2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)\right) - \lambda} \right).$$

117 where $\xi = x - (\frac{4k}{2 + \lambda^2 - 4\mu})t$ and E is an arbitrary constant.

118 When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$119 \quad u_3(\xi) = \pm \sqrt{\frac{6k}{(-2\beta - \beta\lambda^2)}} \left(1 + \frac{2\lambda}{\exp(\lambda(\xi + E)) - 1} \right).$$

120 where $\xi = x - (\frac{4k}{2 + \lambda^2})t$ and E is an arbitrary constant.

121 When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$122 \quad u_4(\xi) = \pm \sqrt{\frac{3k}{-\beta}} \left(1 - \frac{\lambda^2(\xi + E)}{(\lambda(\xi + E) + 2)} \right).$$

123 where $\xi = x - 2kt$ and E is an arbitrary constant.

124 When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$,

125
$$u_s(\xi) = \pm \sqrt{\frac{3k}{-\beta}} \left(1 + \frac{2}{(\xi + E)} \right).$$

126 where $\xi = x - 2kt$ and E is an arbitrary constant.

127 Graphical representation of the solutions

128 The graphical illustrations of the solutions are given below in the figures with the aid of

129 Maple.

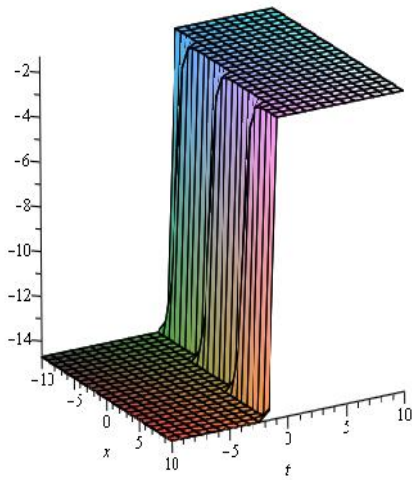


Fig. 1: Kink wave solution $u_1(\xi)$ when $\beta = 1$, $k = 2$, $\mu = 1$, $\lambda = 3$, $E = 1$ and $-10 \leq x, t \leq 10$.

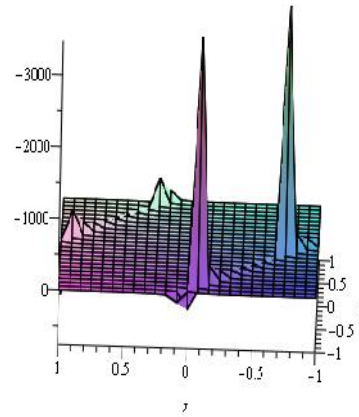


Fig. 2: Periodic solution $u_2(\xi)$ when $\beta = 1$, $k = 2$, $\mu = 3$, $\lambda = 1$, $E = 1$ and $-1 \leq x, t \leq 1$.

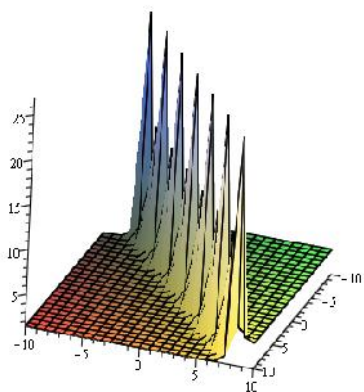


Fig. 3: Singular soliton solution $u_3(\xi)$ when $\beta=1$, $k=2$, $\mu=0$, $\lambda=2$, $E=1$ and $-10 \leq x, t \leq 10$.

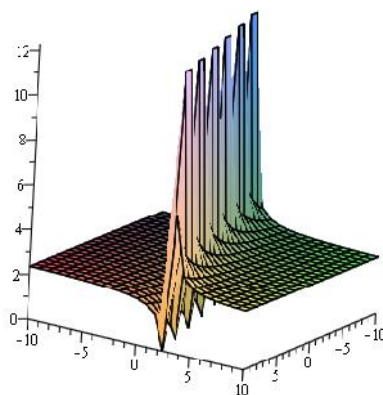


Fig. 4: Singular Kink wave solution $u_4(\eta)$ when $\beta=1$, $k=2$, $\mu=1$, $\lambda=2$, $E=1$ and $-10 \leq x, t \leq 10$.

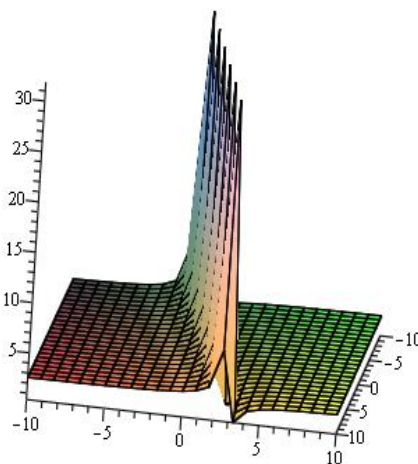


Fig. 5: Singular Kink wave solution $u_5(\eta)$ when $\beta=1$, $k=2$, $\mu=0$, $\lambda=0$, $E=1$ and $-10 \leq x, t \leq 10$.

Conclusion: In this paper, the $\exp(-\Phi(\xi))$ -expansion method is applied successfully for solving the simplified MCH equation. The procedure is simple, direct and constructive without the help of a computer algebra system. The results show that this method is efficient in finding the exact traveling wave solutions of nonlinear differential equations.

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