

Original Research Article

A new approach of modified Mickens' iteration method for solving some nonlinear jerk equations

Abstract

A new approach of the modified Mickens' iteration method has been presented to obtain approximate analytic solutions of some nonlinear jerk equations. It has been shown that the partial derivatives of integral functions are valid for modified iteration method in each step of iteration. Also the solutions give more accurate result than other existing methods.

Keywords: Jerk equations; Nonlinear oscillator; Iteration method;

Mathematics subject classification: 34A34, 34B99.

1. Introduction

The subject of differential equation not only is one of the most beautiful parts of mathematics, but it is also an essential tool for modeling many physical situations such as mechanical vibration, nonlinear circuits, chemical oscillation, space dynamics and so forth. These equations have also demonstrated their usefulness in ecology, business cycle and biology. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. The differential equations are linear or nonlinear, autonomous or non-autonomous. Practically, numerous differential equations involving physical phenomena are nonlinear. Methods of solutions of linear differential equations are comparatively easy and highly developed. Whereas, very little of a general character is known about nonlinear equations.

Generally, the nonlinear problems are solved by converting into linear equations imposing some conditions; but such linearization is always not possible. In this situation there are several analytical approaches to find approximate solutions to nonlinear problems, such as: Perturbation [11,19,20], Standard and modified Linstedt-Poincaré [21-23], Harmonic Balance [1-4,6-9,13,14,16,24-29], Homotopy [5,15], Iterative [10, 12,18] methods, etc.

29 Among them the most widely used method is the perturbation method where the nonlinear
30 term is small. Another recent technique is developed by Mickens [17] and farther work has
31 been done by Wu [29], Gottlieb [7,8], Alam [1] and so forth for handling the strong
32 nonlinear problems named HB method. Recently, some authors used an iteration procedure
33 which is valid for small as well as large amplitude of oscillation to obtain the approximate
34 frequency and the corresponding periodic solution of such nonlinear problems.

35 The term “jerk” i.e. the third-order derivative of the displacement (which might also be
36 termed “triceleration”) was first introduced by Schot in 1978 [27]. We know most of the
37 efforts on dynamical systems are related to second-order differential equations, but some
38 dynamical systems can be described by nonlinear jerk (third-order) differential equations. As
39 for example oscillations in a nonlinear vacuum tube circuit [24] and third order mechanical
40 oscillators [6]. Originally jerk equations

$$41 \quad \ddot{x} = J(x, \dot{x}, \ddot{x}). \quad (1)$$

42 being of some interest in mechanics [4,26], these equations are finding increasing
43 importance in the study of chaos [4]. Many third-order nonlinear systems (three
44 simultaneous first order differential equations), both mathematically and physically
45 motivated, such as the now–classical Rössler system [25], may be recast into a single
46 nonlinear third-order differential (jerk) equation involving only one of the dependent
47 variables [7,14]. Some early investigations into nonlinear jerk equations (although not
48 termed as such) include oscillations in a nonlinear vacuum tube circuit [24], and third order
49 mechanical oscillators [6]. Other physical situations in which nonlinear jerk-type equations
50 have been investigated with more emphasis in chaotic solutions (called a periodic in earlier
51 works), include a thermo–mechanical oscillator model with thermal dissipation [16], fluid
52 dynamical convection [2], and stellar ionization zone oscillations [3]. Jerk equations, even if
53 not nearly as common as acceleration (or force) equations $\ddot{x} = f(x, \dot{x})$, are therefore of
54 direct physical interest. Moreover, simple forms of the jerk function J which lead to maybe
55 the simplest manifestation of chaos have been found by Sprott [28].

56 In this article we have investigated not chaotic solutions to jerk equations (as many of the
57 above references do), but the analytical approximate periodic solutions and corresponding
58 frequencies of some nonlinear jerk equations using the iteration method.

Gottlieb [8] used the lowest order harmonic balance method to calculate approximations to the periodic solution and the angular frequencies obtained by Gottlieb were not accurate enough. But it is very difficult to construct the higher order approximation by harmonic balance method for the reason that the method requires analytical solutions of a set of complicated nonlinear algebraic equations. Wu et al. [29] and Leung et al. [13] applied, respectively, an improved harmonic balance approach and a residue harmonic balance approach to solve nonlinear jerk equations, and their higher order approximations give accurate results to a large range of initial velocity amplitudes. Ma et al. [15] and Hu [11] used, respectively, Homotopy perturbation method and parameter perturbation method to determine the high order approximate solutions of nonlinear jerk equations, and their results obtained are more accurate than those obtained by the low order harmonic balance method. Recently, Ramos [21-23] presented an order reduction method, two-level iterative procedure and a volterra integral formulation, respectively, to solve nonlinear jerk equations. He found that the second reduction method provides accurate solutions only for initial velocities close to unity, the third reduction method produces very accurate for the first and second differential equation in [21], the fourth reduction method provides as accurate results as or more accurate results obtained by parameter perturbation method. Hu et al. [12] presented the modified Mickens iteration procedure for a nonlinear jerk equation, and the second order approximate angular frequency was obtained by Newton's method. Leung and Guo [13] has established a residue harmonic balance approach for determining limit cycles to parity- and time-reversal invariant general non-linear jerk equations with cubic nonlinearities.

To obtain periodic solutions for jerk-oscillators certain restrictions must be placed on the mathematical structure of the oscillator. We only consider the following cubic nonlinear functions investigated by Gottlieb [8]:

$$(i) \ x \dot{x} \ddot{x}, (ii) \ \dot{x}^3 \text{ and } (iii) \ x^2 \dot{x}. \quad (2)$$

The most general jerk function with invariance of time- and space-reversal and which has only cubic nonlinearities may be written in the form [8]

$$\ddot{x} = \alpha x \dot{x} \ddot{x} - \beta \dot{x} \ddot{x}^2 - \gamma \dot{x} - \delta x^2 \dot{x} - \varepsilon \dot{x}^3. \quad (3)$$

where an over-dot denotes the time derivative and the parameters $\alpha, \beta, \gamma, \delta$ and ε are given real constants. The corresponding initial conditions are

$$x(0) = 0, \dot{x}(0) = A \text{ and } \ddot{x}(0) = 0. \quad (4)$$

These three initial conditions in Eq. (4) are to satisfy the periodic requirement. Here, at least one of α, β, δ and ε should be non-zero.

For $\alpha = 1$ with cubic nonlinearity includes $x\dot{x}\ddot{x}$ only. The resulting standard jerk equation, after rescaling of both x and t , is taken to be

$$\ddot{x} + \dot{x} - x\dot{x}\ddot{x} = 0, x(0) = 0, \dot{x}(0) = A, \ddot{x}(0) = 0. \quad (5)$$

For $\delta = 1$ and $\varepsilon = 1$, the resulting standard jerk equation, after rescaling of both x and t , is taken to be

$$\ddot{x} + \dot{x}^3 + x^2\dot{x} = 0, x(0) = 0, \dot{x}(0) = A, \ddot{x}(0) = 0. \quad (6)$$

In this article, we present a new approach of the Mickens' modified iteration method [16] for the determination of approximate solutions of nonlinear jerk Eqs. (5) & (6). Here we have defined the function which contains dependent variable, its derivatives of first and second order also the integrals of the dependent variable. Previously the function contains only dependent variable and its derivatives of first and second order. It is mentioned that our method is valid for second and higher order period of oscillations and show a good agreement compared to other existing solutions.

106

2. The method

2.1 The direct Mickens iteration procedure

Let us consider a nonlinear oscillator modeled by

$$\ddot{x} + f(\ddot{x}, \dot{x}, x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \quad (7)$$

where over dots denote differentiation with respect to time, t . We choose the frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (7), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x, \dot{x}, \ddot{x}) \equiv G(x, \dot{x}, \ddot{x}). \quad (8)$$

Following [18], we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \dot{x}_k, \ddot{x}_k); \quad k = 0, 1, 2, \dots, \quad (9)$$

together with

$$x_0(t) = A \cos(\Omega_0 t). \quad (10)$$

Herein x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \quad (11)$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms (see [19] for details) should not occur in the solution. This procedure gives the sequence of solutions: $x_0(t), x_1(t), \dots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [18].

125

2.2 The modified Mickens iteration procedure

Here x_1 can easily be obtained from Eq. (9) by demanding that secular terms should not occur in the solution.

The proposed iteration scheme of Eq. (8) is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = & G(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1}) + G_x(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(x_k - x_{k-1}) \\ & + G_{\dot{x}}(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(\dot{x}_k - \dot{x}_{k-1}) + G_{\ddot{x}}(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(\ddot{x}_k - \ddot{x}_{k-1}) \\ & + G_{x'}(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(x'_k - x'_{k-1}); \quad k = 1, 2, 3, \dots \end{aligned} \quad (12)$$

where over dots denote differentiation with respect to time, t over dash denote integration with respect to time, t and

$$G_x = \frac{\partial G}{\partial x}, \quad G_{\dot{x}} = \frac{\partial G}{\partial \dot{x}}, \quad G_{\ddot{x}} = \frac{\partial G}{\partial \ddot{x}}, \quad G_{x'} = \frac{\partial G}{\partial x'} \quad (13)$$

At each stage of the iteration, Ω_k ; $k=1,2,3,\dots$ is determined by the requirement that secular terms should not occur in the solution. This procedure gives the sequence of solutions: $x_2(t), x_3(t), x_4(t), \dots$.

3. Example: Here we have considered two particular Jerk functions

(i) $x \dot{x} \ddot{x} - \dot{x}$ and (ii) $\dot{x}^3 + x^2 \dot{x}$

3.1 Solution of the Jerk oscillator containing displacement times velocity time's acceleration, and velocity [8] $\ddot{x} + \dot{x} = x \dot{x} \ddot{x}$

Let us consider the nonlinear jerk oscillator [8]

$$\ddot{x} + \dot{x} = x \dot{x} \ddot{x}. \quad (14)$$

Introducing the phase space variable (y, t) by the relation $\dot{x} = y$ then Eq. (14) becomes

$$\ddot{y} + y = y \dot{y} y'. \quad (15)$$

Obviously, Eq. (15) can be written as

$$\ddot{y} + \Omega^2 y = (\Omega^2 - 1 + \dot{y} y') y. \quad (16)$$

Now the iteration scheme is (according to Eq. (9))

$$\ddot{y}_{k+1} + \Omega_k^2 y_{k+1} = (\Omega_k^2 - 1 - \dot{y}_k y'_k) y_k. \quad (17)$$

Equation (10) is rewritten as

$$y_0 = y_0(t) = A \cos \theta, \quad (18)$$

where $\theta = \Omega t$. For $k = 0$, the Eq. (17) becomes

$$\ddot{y}_1 + \Omega_0^2 y_1 = (\Omega_0^2 - 1 - \dot{y}_0 y'_0) y_0. \quad (19)$$

Substituting the initial function Eq. (18) into the right hand side of Eq. (19) and expanding in a Cosine series, we obtain

$$\ddot{y}_1 + \Omega_0^2 y_1 = a_1 \cos \theta + a_3 \cos 3\theta. \quad (20)$$

157 where,

$$158 \quad a_1 = \frac{A}{4}(-4 - A^2 + 4\Omega_0^2), \quad a_3 = \frac{1}{4}A^3. \quad (21)$$

159 To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of
160 Eq. (20). Thus we have

$$161 \quad \Omega_0^2 = 1 + \frac{A^2}{4}. \quad (22)$$

162 Then solving Eq. (20) and satisfying the initial condition $x_1(0) = A$, we obtain

$$163 \quad y_1(t) = \left(A + \frac{a_3}{8\Omega_0^2} \right) \cos\theta - \frac{a_3}{8\Omega_0^2} \cos 3\theta. \quad (23)$$

164 This is the first approximate solution of Eq. (15) and the related Ω_1 is to be determined.

165 According to Eq. (12), we have

$$166 \quad \ddot{y}_{k+1} + \Omega_k^2 y_{k+1} = G(\Omega_k, y_{k-1}, \dot{y}_{k-1}, y'_{k-1}) + G_y(\Omega_k, y_{k-1}, \dot{y}_{k-1}, y'_{k-1})(y_k - y_{k-1}) \\ + G_{\dot{y}}(\Omega_k, y_{k-1}, \dot{y}_{k-1}, y'_{k-1})(\dot{y}_k - \dot{y}_{k-1}) \quad ; \quad k=1,2,3,\dots \quad (24)$$

167 Here

$$168 \quad G = y(\Omega^2 - 1 + \dot{y}y'), \quad G_y = \Omega^2 - 1 + \dot{y}y', \quad G_{\dot{y}} = y y', \quad G_{y'} = y \dot{y}. \quad (25)$$

169 The value of Ω_1 will be obtained from the solution of

$$170 \quad \ddot{y}_2 + \Omega_1^2 y_2 = G(\Omega_1, y_0, \dot{y}_0, y'_0) + G_y(\Omega_1, y_0, \dot{y}_0, y'_0)(y_1 - y_0) + \\ G_{\dot{y}}(\Omega_1, y_0, \dot{y}_0, y'_0)(\dot{y}_1 - \dot{y}_0) + G_{y'}(\Omega_1, y_0, \dot{y}_0, y'_0)(y'_1 - y'_0). \quad (26)$$

171 Substituting $y_1(t)$ and $y_0(t)$ from Eq. (23) and Eq. (18) into the right hand side of Eq. (26),
172 we obtain

$$173 \quad \ddot{y}_2 + \Omega_1^2 y_2 = b_1 \cos\theta + b_3 \cos 3\theta + b_5 \cos 5\theta. \quad (27)$$

174 where,

$$\begin{aligned}
 175 \quad b_1 &= -\frac{A}{192\Omega_0^3\Omega_1}(48A^2\Omega_0^4 - 3A^4\Omega_1^2 + 3A^2\Omega_0\Omega_1(2 + A^2 - 2\Omega_1^2) - 48\Omega_0^3\Omega_1(-4 + A^2 + 4\Omega_1^2) \\
 &\quad + \Omega_0^2(A^4 + 48A^2\Omega_1^2)) \\
 176 \quad b_3 &= -\frac{A^3}{128\Omega_0^3\Omega_1}(32\Omega_0^4 - 32\Omega_0^3\Omega_1 + A^2\Omega_1^2 + \Omega_0\Omega_1(4 + 3A^2 - 4\Omega_1^2) + \Omega_0^2(A^2 + 32\Omega_1^2)) \\
 177 \quad b_5 &= -\frac{A^5}{384\Omega_0^3\Omega_1}(\Omega_0^2 + 3\Omega_0\Omega_1 + 9\Omega_1^2) \quad (28)
 \end{aligned}$$

178 Again avoiding secular terms in the solution of Eq. (27), we obtain

$$\begin{aligned}
 179 \quad \Omega_1^2 &= (49152 - 732A^6 - 126A^8 + P + 32A^2(864 + P) + A^4(2992 + 6P))^2 / (36(4 + A^2) \\
 &\quad (32 + 9A^2)^2 P) \quad (29)
 \end{aligned}$$

180 where,

$$\begin{aligned}
 181 \quad P &= (9437184A^2 + 11206656A^4 + 5405696A^6 + 1321920A^8 + 163836A^{10} + 8235A^{12} + \\
 &\quad 3\sqrt{((128 + 68A^2 + 9A^4)^2(-805306368 - 503316480A^2 + 567803904A^4 + 831356928A^6 \\
 &\quad + 452990208A^8 + 135794368A^{10} + 23789664A^{12} + 2302128A^{14} + 95769A^{16})))^{2/3} \quad (30)
 \end{aligned}$$

182 Then solving Eq. (27) and satisfying initial condition, we obtain the second approximate
183 solution,

$$184 \quad y_2(t) = \left(A + \frac{1}{\Omega_1^2} \left(\frac{b_3}{8} + \frac{b_5}{24} \right) \right) \cos \theta - \frac{1}{\Omega_1^2} \left(\frac{b_3}{8} \cos 3\theta + \frac{b_5}{24} \cos 5\theta \right). \quad (31)$$

185 This is the second approximate solution of Eq. (15).

186 The third approximation y_3 and the value of Ω_2 will be obtained from the solution of

$$\begin{aligned}
 187 \quad \ddot{y}_3 + \Omega_2^2 y_3 &= G(\Omega_2, y_1, \dot{y}_1, y_1') + G_y(\Omega_2, y_1, \dot{y}_1, y_1')(y_2 - y_1) + G_{\dot{y}}(\Omega_2, y_1, \dot{y}_1, y_1')(\dot{y}_2 - \dot{y}_1) \\
 &\quad + G_{y'}(\Omega_2, y_1, \dot{y}_1, y_1')(y_2' - y_1'). \quad (32)
 \end{aligned}$$

188 In a similar way the method can be proceeded higher order approximations and $\Omega_0, \Omega_1, \dots$
189 respectively obtained by Eqs. (22), (29), ... represent the approximation of frequencies of
190 oscillator (14). However due to growing algebraic complexity, most of the approximate
191 techniques are applied to second or third approximations.

3.2 Jerk oscillator containing velocity-cube and velocity times displacement-squared

$$[8]: \ddot{x} + \dot{x}^3 + x^2 \dot{x} = 0$$

Let us consider the nonlinear jerk oscillator [8]

$$\ddot{x} + \dot{x}^3 + x^2 \dot{x} = 0. \quad (33)$$

Introducing the phase space variable (y, t) by the relation $\dot{x} = y$ then Eq. (33) becomes

$$\ddot{y} + y^3 + y y'^2 = 0. \quad (34)$$

The Eq. (34) can be written as

$$\ddot{y} + \Omega^2 y = (\Omega^2 - y^2 - y'^2) y. \quad (35)$$

According to Eq. (9), the iteration scheme of Eq. (35) will be

$$\ddot{y}_{k+1} + \Omega_k^2 y_{k+1} = (\Omega_k^2 - y_k^2 - y_k'^2) y_k. \quad (36)$$

The first approximation $y_1(t)$ and the frequency Ω_0 will be obtained from the solution

(putting $k = 0$ in Eq. (36))

$$\ddot{y}_1 + \Omega_0^2 y_1 = (\Omega_0^2 - y_0^2 - y_0'^2) y_0. \quad (37)$$

Utilizing the initial function Eq. (18) into the right hand side of Eq. (37) and expanding in a

Fourier Cosine series, we obtain

$$\ddot{y}_1 + \Omega_0^2 y_1 = c_1 \cos \theta + c_3 \cos 3\theta. \quad (38)$$

where,

$$c_1 = -\frac{A}{4\Omega_0^2} (A^2 + 3A^2\Omega_0^2 - 4\Omega_0^4), \quad c_3 = -\frac{A^3}{4\Omega_0^2} (-1 + \Omega_0^2). \quad (39)$$

To avoid secular terms in the solution, we have to remove $\cos \theta$ from the right hand side of

Eq. (38). Thus we have

$$\Omega_0^2 = \frac{A}{8} (3A + \sqrt{16 + 9A^2}). \quad (40)$$

Then solving Eq. (38) and satisfying the initial condition $x_1(0) = A$, we obtain

$$y_1(t) = \left(A + \frac{c_3}{8 \Omega_0^2} \right) \cos \theta - \frac{c_3}{8 \Omega_0^2} \cos 3\theta. \quad (41)$$

This is the first approximate solution of Eq. (34) and the related Ω_1 is to be determined.

According to Eq. (12), the second approximation $y_2(t)$ and the value of Ω_1 will be obtained from the solution of

$$\ddot{y}_2 + \Omega_1^2 y_2 = G(\Omega_1, y_0, y_0') + G_y(\Omega_1, y_0, y_0')(y_1 - y_0) + G_{y'}(\Omega_1, y_0, y_0')(y_1' - y_0'). \quad (42)$$

where

$$G = (\Omega^2 - y^2 - y'^2)y, \quad G_y = \Omega^2 - 3y^2 - y'^2, \quad G_{y'} = -2yy'. \quad (43)$$

Substituting $y_1(t)$ and $y_0(t)$ from Eq. (41) and Eq. (18) into the right hand side of Eq. (42), we obtain

$$\ddot{y}_2 + \Omega_1^2 y_2 = d_1 \cos \theta + d_3 \cos 3\theta + d_5 \cos 5\theta. \quad (44)$$

where,

$$\begin{aligned} d_1 &= \frac{1}{192 \Omega_0^6 \Omega_1} (2 A^3 \Omega_0 (-48 \Omega_0^4 + A^2 (-1 + \Omega_0^2)) + 3 A^3 (-A^2 - 2 A^2 \Omega_0^2 + (16 + 3 A^2) \Omega_0^4 \\ &\quad - 48 \Omega_0^6) \Omega_1 + 6 A \Omega_0^2 (A^2 - A^2 \Omega_0^2 + 32 \Omega_0^4) \Omega_1^3) \\ d_3 &= \frac{A^3}{128 \Omega_0^6 \Omega_1} (2 \Omega_0 (A^2 - A^2 \Omega_0^2 + 32 \Omega_0^4) - (1 + \Omega_0^2) (32 \Omega_0^4 + 3 A^2 (-1 + \Omega_0^2)) \Omega_1 \\ &\quad + 4 \Omega_0^2 (-1 + \Omega_0^2) \Omega_1^3) \\ d_5 &= -\frac{A^5}{384 \Omega_0^6 \Omega_1} ((-1 + \Omega_0^2) (-2 \Omega_0 + (-3 + 9 \Omega_0^2) \Omega_1)) \end{aligned} \quad (45)$$

Again avoiding secular terms in the solution of Eq. (44), we obtain

$$\Omega_1^2 = (R / (18 \times 2^{1/3} \Omega_0^2 (A^2 - A^2 \Omega_0^2 + 32 \Omega_0^4)) - 3 \times 2^{1/3} A^2 (-A^2 - 2 A^2 \Omega_0^2 + (16 + 3 A^2) \Omega_0^4 - 48 \Omega_0^6) / R)^2 \quad (46)$$

where,

$$\begin{aligned}
 R = & 3 \times 2^{2/3} \times 3^{1/3} (A^2 \Omega_0^5 (6A^6 - 18A^6 \Omega_0^2 + 672A^4 \Omega_0^4 + 18A^6 \Omega_0^4 - 1344A^4 \Omega_0^6 - 6A^6 \Omega_0^6 \\
 & + 24576A^2 \Omega_0^8 + 672A^4 \Omega_0^8 - 24576A^2 \Omega_0^{10} + 294912\Omega_0^{12} + \sqrt{6}(\Omega_0^6 (6\Omega_0^4 (A^2 - A^2 \Omega_0^2 \\
 & + 48\Omega_0^4)^2 (A^3 - A^3 \Omega_0^2 + 32A\Omega_0^4)^4 - A^6 (A^2 - A^2 \Omega_0^2 + 32\Omega_0^4)^3 (A^2 + 2A^2 \Omega_0^2 - (16 \\
 & + 3A^2)\Omega_0^4 + 48\Omega_0^6))^3)^{1/2})^{1/3}
 \end{aligned} \tag{47}$$

Then solving Eq. (44) and satisfying initial condition, we obtain the second approximate solution,

$$y_2(t) = \left(A + \frac{1}{\Omega_1^2} \left(\frac{d_3}{8} + \frac{d_5}{24} \right) \right) \cos \theta - \frac{1}{\Omega_1^2} \left(\frac{d_3}{8} \cos 3\theta + \frac{d_5}{24} \cos 5\theta \right). \tag{48}$$

This is the second approximate solution of Eq. (34).

The third approximation y_3 and the value of Ω_2 will be obtained from the solution of

$$\ddot{y}_3 + \Omega_2^2 y_3 = G(\Omega_2, y_1, y_1') + G_y(\Omega_2, y_1, y_1')(y_2 - y_1) + G_{y'}(\Omega_2, y_1, y_1')(y_2' - y_1'). \tag{49}$$

In a similar way the method can be proceeded higher order approximations and $\Omega_0, \Omega_1, \dots$ respectively obtained by Eqs. (40), (46), ... represent the approximation of frequencies of oscillator (33).

4. Results and Discussions

A modified iteration method is developed based on Mickens [18] iteration method to solve a class of nonlinear jerk equations. In this section, we express the accuracy of the modified approach of iteration method by comparing with the existing results from different methods and with the exact results of the nonlinear jerk equations. To show the accuracy, we have calculated the percentage errors (denoted by Er(%)) by the definitions $|100(T_e - T_i)/T_e|$, where $T_i = 2\pi/\Omega_i; i = 0, 1, 2, \dots$ represents the approximate periods obtained by the present method and T_e represents the corresponding exact period of the oscillator.

4.1 First of all we demonstrate the comparison of results for examples 3.1. *i.e., For the Jerk function containing displacement time's velocity time's acceleration, and velocity $x \dot{x} \ddot{x} - \dot{x}$.*

Recently, Gottlieb [8], Ma *et al.* [15], Ramos [21] and Leung & Guo [13] has found approximate solutions, frequencies and as well as approximate periods of nonlinear jerk oscillator (given by Eq. (14)) by different methods other than Iteration method.

We have used a modified iteration procedure to obtaining approximate solutions of the oscillator. The procedure is very simple. It has been shown that, in most of the cases our solution gives significantly better result than other existing results and sometimes it is almost similar to other existing results.

Herein we have calculated the first second and third approximate frequencies which are denoted by Ω_1 , Ω_2 and Ω_3 respectively and corresponding periods are T_1 , T_2 and T_3 . All the results are given in Table 5.1a and Table 5.1b, to compare the approximate frequencies we have also given the existing results determined separately by Gottlieb [8], Ma *et al.* [15], Ramos [21] and Leung & Guo [13] respectively.

4.2 Now, we demonstrate the comparison of results for examples 3.2.

i.e., For the Jerk-function containing velocity-cube and velocity times displacement-squared

$$\ddot{x}^3 + x^2 \dot{x}.$$

Recently, Wu *et al.* [29], Ma *et al.* [15], Ramos [21], Hu [11] and Leung & Guo [13] has found approximate solutions, frequencies and as well as approximate periods of nonlinear jerk oscillator (given by Eq. (33)) by different methods other than Iteration.

We have used a modified iteration procedure to obtaining approximate solutions of the oscillator. The procedure is very simple. It has been shown that, in most of the cases our solution gives significantly better result than other existing results and sometimes it is almost similar to other existing results.

Herein we have calculated the first second and third approximate frequencies which are denoted by Ω_1 , Ω_2 and Ω_3 respectively and corresponding periods are T_1 , T_2 and T_3 . All the results are given in Table 5.2a and Table 5.2b, to compare the approximate frequencies we have also given the existing results determined separately by Wu *et al.* [29], Ma *et al.* [15], Ramos [21], Hu [11] and Leung & Guo [13] respectively.

5. Tables

Table 5.1a

Comparison of the approximate periods with exact periods T_e [8] of $\ddot{x} + \dot{x} = x \dot{x} \ddot{x}$:

A	T_e	Modified T_0 Er(%)	Modified T_1 Er(%)	Modified T_2 Er(%)
0.1	6.275347	6.275346 1.56 e ⁻⁵	6.275347 2.65 e ⁻⁶	6.275346 2.60 e ⁻⁶
0.2	6.252016	6.252003 2.07 e ⁻⁴	6.252016 3.11 e ⁻⁶	6.252016 1.49 e ⁻⁷
0.5	6.096061	6.095585 7.8 e ⁻³	6.096021 6.5 e ⁻⁴	6.096059 3.1 e ⁻⁵
1	5.626007	5.619852 1.09 e ⁻¹	5.624426 2.8 e ⁻¹	5.625802 3.6 e ⁻³
2	4.491214	4.442883 1.08	4.464823 5.9 e ⁻¹	4.482164 2.0 e ⁻¹

T_0 , T_1 and T_2 respectively denote initial, first and second modified approximate periods. Er(%) denotes percentage error.

Table 5.1b

Comparison of the approximate periods obtained by our method and other existing result with exact periods T_e [8] of $\ddot{x} + \dot{x} = x \dot{x} \ddot{x}$:

A	T_e	Modified T_2 Er(%)	Gottlieb T_{G2} (2004) [8] Er(%)	Ma et al. T_{M2} (2008) [15] Er(%)	Ramos T_{R2} (2010) [21] Er(%)	Leung-Guo T_{LG2} (2011) [13] Er(%)
0.1	6.275347	6.275346 2.60 e ⁻⁶	6.275346 1.3 e ⁻⁵	6.275347 2.5 e ⁻⁶	6.275347 9.0 e ⁻⁸	6.275347 2.6 e ⁻⁶
0.2	6.252016	6.252016 1.49 e ⁻⁷	6.252003 2.11 e ⁻⁴	6.252016 1.6 e ⁻⁷	6.252016 6.4 e ⁻⁶	6.252016 1.6 e ⁻⁷
0.5	6.096061	6.096059 3.1 e ⁻⁵	6.095585 7.8 e ⁻³	6.096059 3.21 e ⁻⁵	6.096025 6.06 e ⁻⁴	6.096061 8.2 e ⁻⁶
1	5.626007	5.625802 3.6 e ⁻³	5.619852 1.09 e ⁻¹	5.625795 3.8 e ⁻³	5.624539 2.6 e ⁻²	5.625993 2.5 e ⁻⁴
2	4.491214	4.482164 2.0 e ⁻¹	4.442883 1.08	4.482081 2.03	4.466144 5.6 e ⁻¹	4.490125 2.4 e ⁻²

290 T_2 denotes second modified approximate periods; T_{G2}, T_{M2}, T_{R2} and T_{LG2} respectively denote second
 291 approximate periods obtained by Gottlieb, Ma et al., Ramos and Leung-Guo. Er(%) denotes percentage error.

292 **Table 5.2a**

293 Comparison of the approximate periods with exact periods T_e [8] of $\ddot{x} + \dot{x}^3 + x^2 \dot{x} = 0$:

294

A	T_e	Modified T_1 Er(%)	Modified T_2 Er(%)	Modified T_3 Er(%)
0.1	25.359725	25.665523 1.206	25.407289 1.876 e^{-1}	25.367539 3.081 e^{-2}
0.2	17.495410	17.6627 9.564 e^{-1}	17.52 1.404 e^{-1}	17.4992 2.184 e^{-2}
0.5	10.210761	10.250448 3.887 e^{-1}	10.215983 5.115 e^{-2}	10.211439 6.641 e^{-3}
1	6.283185	6.2831852 0.000	6.2831852 0.000	6.2831852 0.000
2	3.508793	3.50639 6.8 e^{-2}	3.50881 4.487 e^{-4}	3.50881 4.214 e^{-4}
5	1.468638	1.46862 9.5 e^{-4}	1.46888 1.65 e^{-2}	1.46867 2.4 e^{-3}
10	0.739762	0.739902 1.886 e^{-2}	0.739913 2.036 e^{-2}	0.739783 2.89 e^{-3}
20	0.370580	0.370671 2.45 e^{-2}	0.370660 2.15 e^{-2}	0.370591 1.1 e^{-3}

295 T_1, T_2 and T_3 respectively denote first, second and third modified approximate. Er(%) denotes percentage
 296 error.

297

298 **Table 5.2b**

299 Comparison of the approximate periods obtained by our method and other existing result
 300 with exact periods T_e [8] of $\ddot{x} + \dot{x}^3 + x^2 \dot{x} = 0$:

A	T_e	Modified T_2 Er(%)	Wu <i>et al.</i> [29] T_{W2} Er(%)	Ma <i>et al.</i> [15] T_{M2} Er(%)	Ramos [21] T_{R2} Er(%)	Hu [11] T_{H2} Er(%)	Leung& Guo [13] T_{LG2} Er(%)
0.1	25.35972	25.40729 1.876 e^{-1}	24.6677 2.72	24.517938 3.32	25.56407 8.06 e^{-1}	25.612713 9.98 e^{-1}	25.946122 2.312
0.5	10.21076	10.21598 5.115 e^{-2}	10.1884 2.2 e^{-1}	10.146603 2.27	10.24328 3.18 e^{-1}	10.244058	10.213682

						3.26 e^{-1}	2.86 e^{-2}
1	2π	2π 0	2π 0	2π 0	2π 0	2π 0	2π 0
5	1.468638	1.46888 1.65 e^{-2}	1.46650 1.45 e^{-1}	1.4679554 4.6 e^{-3}	1.46715 1.01 e^{-1}	1.466631 1.37 e^{-1}	1.468570 4.6 e^{-3}
10	0.739762	0.73991 2.036 e^{-2}	0.738632 1.53 e^{-1}	0.739299 6.3 e^{-3}	0.73907 9.4 e^{-2}	0.738763 1.35 e^{-1}	0.73974 3.7 e^{-3}
20	0.370580	0.37066 2.15 e^{-2}	0.370008 1.54 e^{-1}	0.37033 6.7 e^{-3}	0.37024 9.2 e^{-2}	0.370082 1.34 e^{-1}	0.37057 3.3 e^{-3}

301 T_2 denotes second modified approximate periods; T_{W2} , T_{M2} , T_{R2} , T_{H2} and T_{LG2} respectively denote
302 second approximate periods obtained by Wu *et al.* Gottlieb, Ma *et al.*, Ramos, Hu and Leung-Guo. Er(%)
303 denotes percentage error.

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305 6. References

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