

Original Research Article**Application of the $\exp(-\Phi(\xi))$ -expansion method for finding exact solutions of the (1+1)-dimensional dispersive long wave equations**

Abstract. The dispersive long wave (DLW) equations are very important coupled nonlinear partial differential equations that appear for describing nonlinear water wave model in harbor and coastal design. In this paper, the $\exp(-\Phi(\xi))$ -expansion method has been implemented to find the explicit solutions of the coupled (1+1)-dimensional DLW equations. The efficiency of the method for finding exact solutions has been demonstrated. With the help of symbolic computation, it has been shown that the method is direct, effective and can be used for many other nonlinear partial differential equations (NPDEs) in mathematical physics and engineering.

Keywords: The dispersive long wave equations, The $\exp(-\Phi(\xi))$ -expansion method, Solitary wave solutions, NPDEs.

1. Introduction

Nowadays nonlinear partial differential equations (NPDEs) have been the topic of all-embracing studies in various branches of nonlinear sciences. It is well known that the traveling wave solutions of NPDEs play an important role in the study of nonlinear wave phenomena, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NPDEs. The wave phenomena are observed in various fields, such as fluid dynamics, plasma, elastic media, optical fibres, biology, solid state physics, chemical kinematics, geochemistry, meteorology, electricity etc (see for instance Ref. [1]). In the recent decade, several methods for finding the exact solutions to NPDEs have been proposed, such as the homogeneous balance method [2-4], the Hirota method [5], the tanh method [6], the extended tanh-method [7], Modified extended tanh-function method [8], the Exp-function method [9], , the (G'/G) -expansion method [10-12], the modified simple equation method [13,14], the $\exp(-\Phi(\xi))$ -expansion method [15, 16] and others. The aim of this paper is to

implement of the $\exp(-\Phi(\xi))$ -expansion method to construct the explicit solutions and exact traveling wave solutions for the DLW equations which are very important in mathematical physics.

The paper is organized as follows: In section 2, we present the descriptions of the $\exp(-\Phi(\xi))$ -expansion method. The explicit solutions and exact solutions of the DLW equations are derived in section 3. In section 4, conclusions are given.

2. Description of the $\exp(-\Phi(\xi))$ -expansion method

This section presents the briefly descriptions of the $\exp(-\Phi(\xi))$ -expansion method.

Let us consider the nonlinear PDE with two variables x and t as follows:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \dots) = 0, \quad (1)$$

where, F is function of $u(x, t), u_t(x, t), u_x(x, t), u_{xx}(x, t), \dots$ including nonlinear terms and the subscripts denote the partial derivatives. Let $u(x, t) = u(\xi)$, $\xi = x \pm wt$, where w is an arbitrary constant, then the equation (2) reduces to an ordinary differential equation (ODE) for $u = u(\xi)$:

$$Q(u, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, \dots) = 0, \quad (2)$$

where, Q is a function of $u, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, \dots$ with nonlinear terms.

Let us consider the traveling wave solution of equation (2) as

$$u(\xi) = \sum_{i=1}^N A_i \left(e^{-\Phi(\xi)} \right)^i, \quad A_N \neq 0 \quad (3)$$

where the coefficients A_i ($0 \leq i \leq N$) are constants to be evaluated and $\Phi = \Phi(\xi)$ satisfies the first order nonlinear ODE:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda, \quad (4)$$

where λ and μ are arbitrary constants.

The value of the positive integer N can be determined by balancing the higher order derivative term with the nonlinear term of the highest order in ODE. By substituting (3) into (2) and using (4) rapidly, we obtain a system of algebraic equations for A_i ($0 \leq i \leq N$), λ, μ, w . With the help of symbolic computation, such as Maple, we can evaluate the obtaining system and find out the values A_i ($0 \leq i \leq N$), λ, μ, w . It is note that the equation (4) has the following five types of general solutions [15, 16]:

$$\Phi(\xi) = \ln \left(\frac{-\sqrt{\Theta} \tanh \left\{ 0.5 \sqrt{\Theta} (\xi + \xi_0) \right\} - \lambda}{2\mu} \right), \mu \neq 0, \Theta = \lambda^2 - 4\mu > 0, \quad (5a)$$

$$\Phi(\xi) = \ln \left(\frac{\sqrt{-\Theta} \tan \left\{ 0.5 \sqrt{-\Theta} (\xi + \xi_0) \right\} - \lambda}{2\mu} \right), \mu \neq 0, \Theta = \lambda^2 - 4\mu > 0, \quad (5b)$$

$$\Phi(\xi) = -\ln \left(\frac{\lambda}{\exp(\lambda(\xi + \xi_0)) - 1} \right), \mu = 0, \lambda \neq 0, \Theta = \lambda^2 - 4\mu > 0, \quad (5c)$$

$$\Phi(\xi) = \ln \left(-\frac{2(\lambda(\xi + \xi_0) + 2)}{\lambda^2(\xi + \xi_0)} \right), \mu \neq 0, \lambda \neq 0, \Theta = \lambda^2 - 4\mu = 0, \quad (5d)$$

$$\Phi(\xi) = \ln(\xi + \xi_0), \lambda = 0, \mu = 0. \quad (5e)$$

where ξ_0 is the integrating constant.

Thus the multiple exact solutions of nonlinear PDE (1) are obtained by combing the equations (3) and (5).

3. Explicit and exact solutions of (1+1)-dimensional dispersive long wave (DLW) equations

In this section, we will apply the $\exp(-\Phi(\xi))$ -expansion method to find the exact solutions and then the solitary wave solutions of DLW equations[14,17].

Let us consider the (1+1)-dimensional dispersive long wave (DLW) equations as

$$\left. \begin{aligned} u_t + u u_x + v_x &= 0 \\ v_t + (u v)_x + \frac{1}{3} u_{xxx} &= 0 \end{aligned} \right\} \quad (6)$$

where $v(x, t)$ is the elevation of the water wave, $u(x, t)$ is the surface velocity of water along x-direction. The solutions of equation (6) are very helpful for coastal and civil engineer to describe nonlinear water wave model in harbor and coastal design.

The traveling wave transformation is

$$u(x, t) = u(\xi), \quad v(x, t) = v(\xi), \quad \xi = x - wt, \quad (7)$$

Using the transformation (7), the equation (6) reduces to a set of ODE as follows:

$$\left. \begin{aligned} -wu' + uu' + v' &= 0 \\ -wv' + (uv)' + \frac{1}{3} u''' &= 0 \end{aligned} \right\} \quad (8)$$

where the prime denotes differentiation with respect to ξ . Integrating the system of ODE (8) with respect to ξ , choosing constant of integration as zero, we obtain

$$\left. \begin{aligned} -wu + \frac{u^2}{2} + v &= 0 \\ -wv + uv + \frac{1}{3} u'' &= 0 \end{aligned} \right\} \quad (9)$$

From the first equation of (9), we have

$$v = wu - \frac{u^2}{2} \quad (10)$$

Substituting the equation (10) into the second equation of (9), yields

$$-w^2u + \frac{3}{2}wu^2 - \frac{1}{2}u^3 + \frac{1}{3}u'' = 0 \quad (11)$$

83 Balancing the higher order derivative u'' with nonlinear term u^3 gives $N=1$. Therefore, the exp(-
84 $\Phi(\xi)$)-expansion method allows us to use

$$85 \quad u(\xi) = A_0 + A_1 e^{-\Phi(\xi)} \quad A_1 \neq 0 \quad (12)$$

86 By substituting (12) into eq. (11) and using eq. (4) rapidly and then equating the coefficient of
87 $(e^{-\Phi(\xi)})^i$, $(i=0,1,2,3,4)$ to zero, we obtain a system of algebraic equations as follows:

$$88 \quad \left. \begin{aligned} & -\frac{1}{2}A_0^3 + \frac{3}{2}wA_0^3 + \frac{1}{3}A_1\mu\lambda - w^2A_0 = 0 \\ & -w^2A_1 - \frac{3}{2}A_0^2A_1 + 3wA_0A_1 + \frac{1}{3}A_1\lambda^2 + \frac{2}{3}A_1\mu = 0 \\ & A_1\lambda - \frac{3}{2}A_0A_1^2 + \frac{3}{2}wA_1^2 = 0 \\ & -\frac{1}{2}A_1^3 + \frac{2}{3}A_1 = 0 \end{aligned} \right\} \quad (13)$$

89 Solving the algebraic system (13) by using Maple, the following set of solution is obtained:

$$90 \quad \left\{ \lambda = \lambda, \mu = \pm \frac{\sqrt{3}}{2}A_0\lambda + \frac{3}{4}A_0^2, w = \mp \frac{\lambda}{\sqrt{3}} + A_0, A_0 = A_0, A_1 = \pm \frac{2}{\sqrt{3}} \right\}, \quad (14)$$

91 where A_0 and λ are arbitrary constants.

92 By combining the equations (5), (7), (10), (12) and (14), we have,

93 (1) If $\Omega = \lambda^2 - 4\mu > 0, \lambda \neq 0, \mu \neq 0$, then the DLW equation has the following explicit solutions

$$94 \quad u_1(\xi) = A_0 \mp \frac{2}{\sqrt{3}} \left(\frac{2\mu}{\sqrt{\Omega} \tanh(0.5\sqrt{\Omega}(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) + \lambda} \right) \quad (15)$$

$$v_1 = \left(\mp \frac{\lambda}{\sqrt{3}} + A_0 \right) \left\{ A_0 \mp \frac{2}{\sqrt{3}} \left(\frac{2\mu}{\sqrt{\Omega} \tanh(0.5\sqrt{\Omega}(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) + \lambda} \right) \right. \\ \left. - \frac{1}{2} \left\{ A_0 \mp \frac{2}{\sqrt{3}} \left(\frac{2\mu}{\sqrt{\Omega} \tanh(0.5\sqrt{\Omega}(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) + \lambda} \right) \right\}^2 \right\} \quad (16)$$

where $\mu = \pm \frac{\sqrt{3}}{2} A_0 \lambda + \frac{3}{4} A_0^2$ and A_0, λ, ξ_0 are arbitrary constants. The explicit solutions (15) and (16) represents the exact solitary wave solutions of kink-type for $u(x, t)$ and exact solitary wave solutions of bell-type for $v(x, t)$ respectively corresponding to the fixed values $A_0 = 0.5$, $\lambda = \sqrt{3}$, $\xi_0 = 1$, with $-10 \leq x, t \leq 10$, which are shown in Fig. 1.

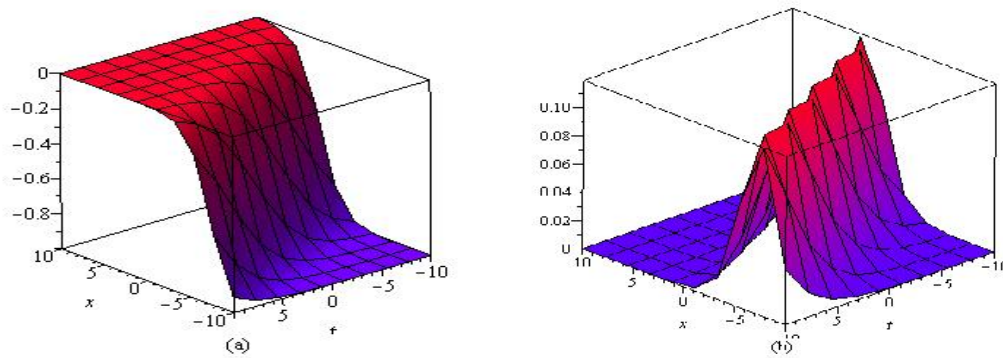


Fig. 1 Exact traveling wave solutions of (a) kink type for $u_1(x, t)$ and (b) bell type for $v_1(x, t)$.

(2) If $\Omega = \lambda^2 - 4\mu < 0, \lambda \neq 0, \mu \neq 0$, then the DLW equation has the following explicit solutions

$$u_2(\xi) = A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{2\mu}{\sqrt{-\Omega} \tanh(0.5\sqrt{-\Omega}(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) - \lambda} \right) \quad (17)$$

$$v_2 = \left(\mp \frac{\lambda}{\sqrt{3}} + A_0 \right) \left\{ A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{2\mu}{\sqrt{-\Omega} \tan(0.5\sqrt{-\Omega}(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) - \lambda} \right) \right\} - \frac{1}{2} \left\{ A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{2\mu}{\sqrt{-\Omega} \tan(0.5\sqrt{-\Omega}(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) - \lambda} \right) \right\}^2 \quad (18)$$

where $\mu = \pm \frac{\sqrt{3}}{2} A_0 \lambda + \frac{3}{4} A_0^2$ and A_0, λ, ξ_0 are arbitrary constants. The explicit solutions (17) and (18) are both represented periodic traveling wave solutions for $u(x, t)$ and $v(x, t)$ corresponding to the fixed values $A_0 = 5$, $\lambda = \sqrt{3}$, $\xi_0 = 1$, with $-\pi \leq x, t \leq \pi$, which are shown in Fig. 2.

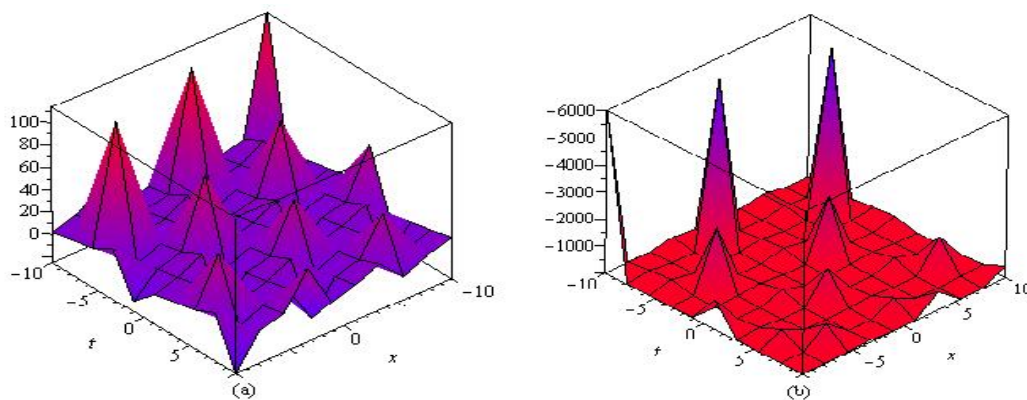


Fig. 2 Exact periodic traveling wave solution for (a) $u_2(x, t)$ and (b) $v_2(x, t)$.

(3) If $\Omega = \lambda^2 - 4\mu > 0, \lambda \neq 0, \mu = 0$, then the DLW equation has explicit solutions as

$$u_3(\xi) = A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{\lambda}{\exp(\lambda(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) - 1} \right) \quad (19)$$

$$v_3 = \left(\mp \frac{\lambda}{\sqrt{3}} + A_0 \right) \left\{ A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{\lambda}{\exp(\lambda(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) - 1} \right) \right\} - \frac{1}{2} \left\{ A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{\lambda}{\exp(\lambda(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)) - 1} \right) \right\}^2 \quad (20)$$

where $\mu = \pm \frac{\sqrt{3}}{2} A_0 \lambda + \frac{3}{4} A_0^2$ and A_0, λ, ξ_0 are arbitrary constants. The explicit solutions (19) and (20) are both represented solitary wave solutions of singular kink-type corresponding to the fixed values $A_0 = -1, \lambda = \sqrt{3}, \xi_0 = 0$, with $-10 \leq x, t \leq 10$, which are shown in Fig. 3.

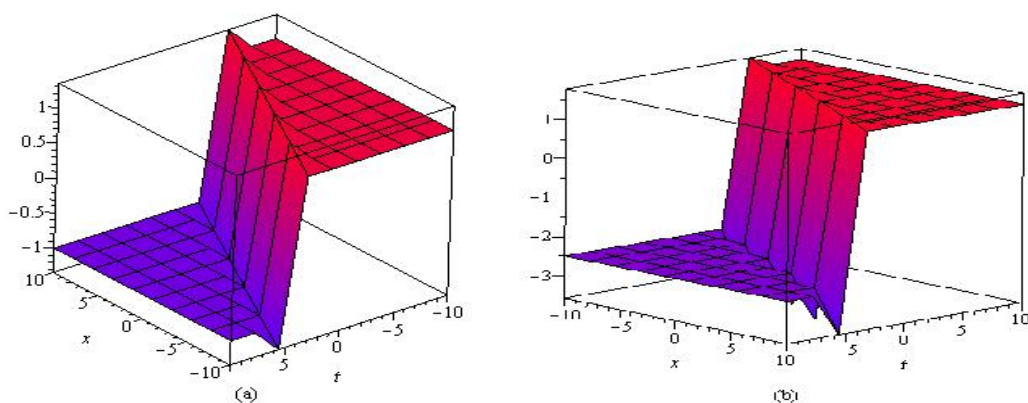


Fig. 3 Exact traveling wave solutions of singular kink type for (a) $u_3(x, t)$ and (b) $v_3(x, t)$.

(4) If $\Omega = \lambda^2 - 4\mu = 0, \lambda \neq 0, \mu \neq 0$, then the DLW equation has the following explicit solutions

$$u_4(\xi) = A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{\lambda^2(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0)}{2(\lambda(x \pm \frac{\lambda}{\sqrt{3}}t + A_0t + \xi_0) + 2)} \right) \quad (21)$$

$$v_4 = \left(\mp \frac{\lambda}{\sqrt{3}} + A_0 \right) \left\{ A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{\lambda^2 (x \pm \frac{\lambda}{\sqrt{3}} t + A_0 t + \xi_0)}{2(\lambda(x \pm \frac{\lambda}{\sqrt{3}} t + A_0 t + \xi_0) + 2)} \right) \right\} - \frac{1}{2} \left\{ A_0 \pm \frac{2}{\sqrt{3}} \left(\frac{\lambda^2 (x \pm \frac{\lambda}{\sqrt{3}} t + A_0 t + \xi_0)}{2(\lambda(x \pm \frac{\lambda}{\sqrt{3}} t + A_0 t + \xi_0) + 2)} \right) \right\}^2 \quad (22)$$

where $\mu = \pm \frac{\sqrt{3}}{2} A_0 \lambda + \frac{3}{4} A_0^2$ and A_0, λ, ξ_0 are arbitrary constants. The explicit solutions (21) and (22) are both represented solitary wave of single soliton type for $u(x, t)$ and $v(x, t)$ corresponding to the fixed values $A_0 = 1$, $\lambda = (\sqrt{3} + \sqrt{15})/2$, $\xi_0 = 5$, with $-10 \leq x, t \leq 10$, which are shown in Fig. 4.

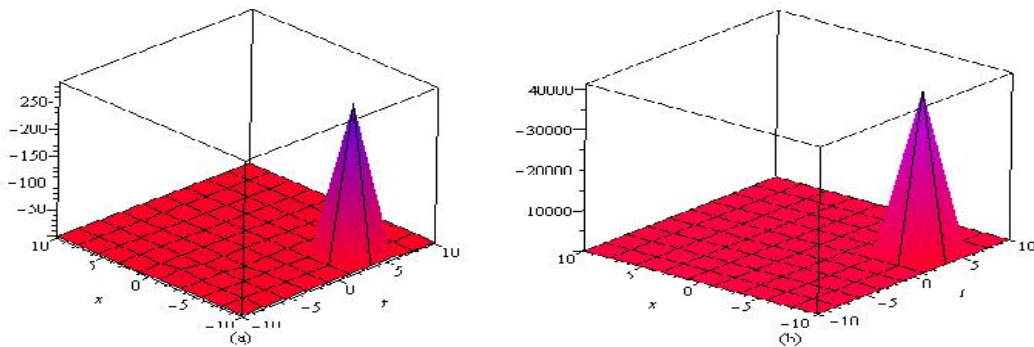


Fig. 4 Exact traveling wave solutions of single soliton type for (a) $u_4(x, t)$ and (b) $v_4(x, t)$.

(5) If $\lambda = 0, \mu = 0$, then the DLW equation has the following non-traveling wave solutions

$$u_5(\xi) = \pm \frac{2}{\sqrt{3}} \left(\frac{1}{x + \xi_0} \right) \quad (23)$$

$$v_5 = -\frac{1}{2} \left\{ \pm \frac{2}{\sqrt{3}} \left(\frac{1}{x + \xi_0} \right) \right\}^2 \quad (24)$$

The exact non-traveling wave solutions (23) and (24) are shown in Fig. 5, where we have used $\xi_0 = 0$, and $-10 \leq x, t \leq 10$.

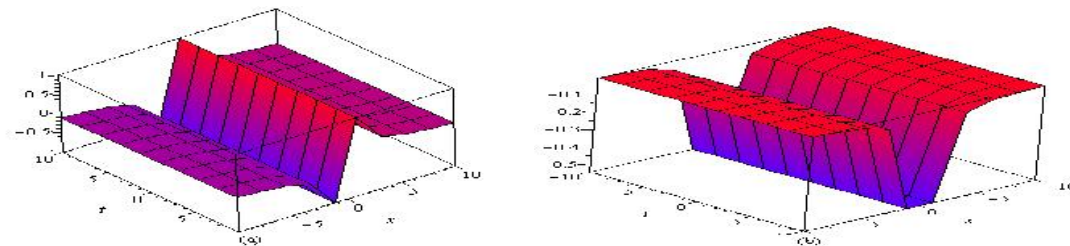


Fig. 5 Exact non-traveling wave solutions for (a) $u_5(x, t)$ and (b) $v_5(x, t)$.

4. Conclusions

The $\exp(-\Phi(\xi))$ -expansion method has been successfully implemented to obtain the explicit solutions of the DLW equations. The obtaining results indicate the DLW equations admit multiple exact traveling wave solutions with few parameters as well as non-traveling wave solutions. The type of exact solitary wave solution is different along with different value of arbitrary parameters. So we can choose appropriate parameter value to obtain solutions we need in practical problems. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equation. We hope that the traveling wave solutions of DLW equations are very helpful for coastal and civil engineer to describe nonlinear water wave model in harbor and coastal design.

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