

Original Research Article

A HYBRID OF THE NEW CONJUGATE GRADIENT METHOD AND GALERKIN THEORY FOR OPTIMIZING BEAM DEFLECTION UNDER UNIFORMLY DISTRIBUTED LOAD

ABSTRACT

A hybrid of the new Conjugate gradient method and Galerkin theory has been developed to find the maximum deflection of a beam under uniformly distributed load. Maximum deflection of a beam structure under a given pressure was found by solving a two point linear boundary value problem with homogeneous boundary conditions without evaluating the inverse of a matrix. An objective function associated with a given member of this class of boundary value problems was optimized. The numerical results obtained from tested problems are very close to the exact solutions. This method is easy to implement and automate computer-wise.

Keywords: Hybrid of the new conjugate gradient method, Two point boundary value problems, Objective function, Galerkin theory.

1. INTRODUCTION

Burden and Faires [1] have shown how physical problems that are position-dependent rather than time-dependent are described in terms of differential equations with conditions imposed at more than one point. A common problem in civil engineering is to find the maximum deflection of a beam subject to uniform loading while the ends are supported in order to avoid deflection at the two fixed ends. A mathematical model of this physical phenomenon results in the formulation of a boundary value problem (BVP). A BVP is an ordinary differential equation with specified values at the extreme points or boundaries of a given system [2]. A beam deflection model, formulated from figure (1), seeks the values of a function $y(x)$ from the following two point linear boundary value problem.

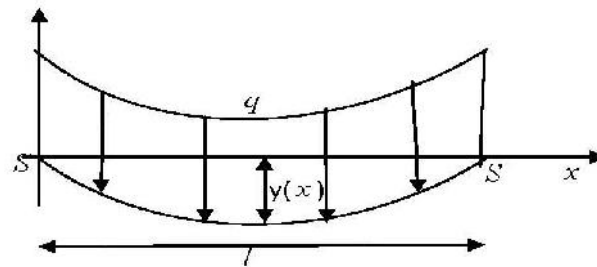


Figure 1: Simply supported beam with a uniformly distributed load.

$$\frac{d^2 y(x)}{dx^2} = \frac{S}{EI} y(x) + \frac{q}{2EI} x(l-x), 0 \leq x \leq l \quad (1)$$

$$y(0) = 0 = y(l),$$

where x is the location along the beam, l is the length of the beam, E is the young's modulus of elasticity of the beam, I is the central moment of inertia, S is the stress at the endpoints and q is the intensity of the uniform load.

Galerkin's Method is one of the best methods for solving (1) numerically. Galerkin's numerical solutions of this class of boundary value problems are very good but difficult to implement. The method leads to full matrices that must be inverted in order to obtain the required solutions [3]. Fortunately, the new conjugate gradient method [4] has been used to solve many optimization problems successfully without involving matrix inversion. We, therefore, present a hybrid of the new Conjugate Gradient Method and Galerkin theory for solving problem (1) optimally with ease and high accuracy.

2. HYBRID OF THE NEW CONJUGATE GRADIENT METHOD AND GALERKIN THEORY

The hybrid of the new Conjugate Gradient Method and Galerkin theory seeks to

$$\text{Maximize } y(x)_{0 \leq x \leq l}$$

Subject to

$$y'' - r(x)y - f(x) = 0$$

$$y(0) = 0 = y(l).$$

where

$$r(x) = \frac{S}{EI} \text{ and } f(x) = \frac{q}{2EI} x(l-x).$$

Galerkin's method is a variational technique used in solving equation (1) numerically. It is based on the fact that a function $y \in C^2[0, l]$ is the unique solution of (1) if and only if y is the unique function in $C^2[0, l]$ whenever

$$\int_0^l \{y''(x) - r(x)y(x) - f(x)\} \phi_j(x) dx = 0 \quad (2)$$

$j = 1, 2, \dots, n$ and each ϕ_j is a weight function. Equation (2) yields the Galerkin's weighted n residual equations in n unknowns. This method approximates the solution of (1) by solving the system of equations derived from equation (2) simultaneously. The hybrid of the new Conjugate Gradient Method and Galerkin theory builds on Galerkin's method by constructing and optimizing an objective function from equation (2). It uses the finite element method as a variation tool. Thus, we partition the interval $[0, l]$ into $n+1$ subintervals with n interior points such that

$$0 = x_0 < x_1 < \dots < x_n < x_{n+1} = l; h = x_{i+1} - x_i; x_i = ih; i = 0, 1, 2, \dots, n.$$

A small set of functions which consists of linear combinations of some basis functions $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ contains the approximate solutions of (1) [5]. These functions are required to be linearly independent and satisfy the conditions given in equation (3) below.

$$\phi_i(0) = \phi_i(l) = 0; i = 1, 2, \dots, n. \quad (3)$$

In this paper, we used uniformly partitioned intervals of $[0, l]$, the simplest conventional basis functions and their derivatives as defined in equations (4) and (5) below.

$$\phi_i(x) = \begin{cases} 0; & 0 \leq x \leq x_{i-1} \\ \frac{x - x_{i-1}}{h}; & x_{i-1} < x \leq x_i \\ \frac{x_{i+1} - x}{h}; & x_i < x \leq x_{i+1} \\ 0; & x_{i+1} < x \leq 1 \end{cases} \quad (4)$$

$$\phi'_i(x) = \begin{cases} 0; & 0 < x < x_{i-1} \\ \frac{1}{h}; & x_{i-1} < x < x_i \\ -\frac{1}{h}; & x_i < x < x_{i+1} \\ 0; & x_{i+1} < x < 1 \end{cases} \quad (5)$$

for each $i = 0, 1, \dots, n$. $\phi_i(x_i) = 1$, $\phi_i(x_{i-1}) = 0$ and $\phi_i(x_{i+1}) = 0$. If we connect (x_i, c_i) by a line segment we obtain an approximation of the form

$$y_n(x) = \sum_{i=1}^n c_i \phi_i(x) \quad (6)$$

where c_i approximates the exact solution of (1) at $y(x_i)$. It follows that

$$y'_n(x) = \sum_{i=1}^n c_i \phi'_i(x) \quad (7)$$

The absolute error of approximation is $|y_n(x) - y(x)|$.

Replace y by $y_n(x)$ and y' by $y'_n(x)$ in (2), using equations (6) and (7).

$$\int_0^l \{y''(x) - r(x)y_n(x) - f(x)\} \phi_j(x) dx = 0$$

Integrate first term by parts:

$$\begin{aligned} \int_0^l \phi_j(x) y''(x) dx &= y'(x) \phi_j(x) \Big|_0^l - \int_0^l y'(x) \phi'_j(x) dx \\ &= \int_0^l -y'_n(x) \phi'_j(x) dx \end{aligned}$$

$$\int_0^l -\{y'_n(x) \phi'_j(x) + r(x)y_n(x) \phi_j(x)\} dx = \int_0^l f(x) \phi_j(x) dx$$

$$\int_0^l -\left\{ \sum_{i=1}^n c_i \phi'_i(x) \right\} \phi'_j(x) + r(x) \left\{ \sum_{i=1}^n c_i \phi_i(x) \right\} \phi_j(x) dx = \int_0^l f(x) \phi_j(x) dx$$

$$\sum_{i=1}^n \left[\int_0^l -\{\phi'_i(x) \phi'_j(x) + r(x) \phi_i(x) \phi_j(x)\} dx \right] c_i = \int_0^l f(x) \phi_j(x) dx \quad (8)$$

The solution of equation (8) will produce $n \times n$ linear system of equations in c_i , $i = 1, 2, \dots, n$, unknown constants. So, we must form and solve a matrix equation. Define

$$a_{i,j} = -\int_0^l \{\phi'_i(x)\phi'_j(x) + r(x)\phi_i(x)\phi_j(x)\}dx \quad (9)$$

$$b_j = \int_0^l f(x)\phi_j(x)dx \quad (10)$$

93

94 From equation (9),

$$\begin{aligned} a_{i,i} &= -\int_0^l \{\phi'_i(x)^2 + r(x)[\phi_i(x)]^2\}dx \\ &= -\int_{x_{i-1}}^{x_i} \{\phi'_i(x)^2 + r(x)[\phi_i(x)]^2\}dx - \int_{x_i}^{x_{i+1}} \{\phi'_i(x)^2 + r(x)[\phi_i(x)]^2\}dx \\ &= -\frac{1}{h^2} \int_{x_{i-1}}^{x_i} \{1 + r(x)[x - x_{i-1}]^2\}dx - \frac{1}{h^2} \int_{x_i}^{x_{i+1}} \{1 + r(x)[x_{i+1} - x]^2\}dx \end{aligned} \quad (11)$$

99

100 $i = 1, 2, \dots, n.$

$$a_{i,i+1} = -\int_{x_i}^{x_{i+1}} \{\phi'_i(x)\phi'_{i+1}(x) + r(x)\phi_i(x)\phi_{i+1}(x)\}dx \quad (12)$$

102

$$= -\frac{1}{h^2} \int_{x_i}^{x_{i+1}} \{1 + r(x)[x - x_i][x_{i+1} - x]\}dx$$

104

105 $i = 1, 2, \dots, n-1.$

$$a_{i,i-1} = -\int_{x_{i-1}}^{x_i} \{\phi'_i(x)\phi'_{i-1}(x) + r(x)\phi_i(x)\phi_{i-1}(x)\}dx \quad (13)$$

$$= -\frac{1}{h^2} \int_{x_{i-1}}^{x_i} \{1 + r(x)[x - x_{i-1}][x_i - x]\}dx$$

108 $i = 2, \dots, n.$

109 From equation (10),

$$b_j = \int_{x_{j-1}}^{x_j} \{f(x)\phi_j(x)\}dx + \int_{x_j}^{x_{j+1}} \{f(x)\phi_j(x)\}dx \quad (14)$$

$$= \frac{1}{h} \int_{x_{j-1}}^{x_j} \{f(x)(x - x_{j-1})\}dx + \frac{1}{h} \int_{x_j}^{x_{j+1}} \{f(x)(x_{j+1} - x)\}dx$$

112 $i = 1, 2, \dots, n.$

113 From the above equations, we obtain the matrix equation for the system:

$$AC = B \quad (15)$$

115 where

$$A = a_{ij}, \quad B = b_j, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad j = 1, 2, \dots, n \text{ and } i = 1, 2, \dots, n.$$

117

118 By Galerkin method, the unknown constants, $c_i, i = 1, 2, \dots, n$ will be found from equation
119 (15):

$$120 \quad C = A^{-1} B \quad (16)$$

121 The absolute error is

$$122 \quad \text{Error} = |y_n(x) - y(x)|$$

123 where $y(x)$ is the exact solution of (1).

124 By our method, we require the matrix A to be symmetric otherwise we replace A with an
125 equivalent symmetric matrix M. That is,

$$126 \quad A^T A C = A^T B$$

127 or

$$128 \quad MC = Z \quad (17)$$

129 where $M = A^T A$ and $Z = A^T B$. By our assumption of uniqueness of solution of (1), we
130 state that the matrix M is also positive definite. It follows that we are guaranteed to form an
131 objective function from equation (17) as given below.

$$132 \quad F(C) = \frac{1}{2} C^T MC - C^T Z \quad (18)$$

133 The gradient function is

$$134 \quad G(C) = MC - Z \quad (19)$$

135 Next, we seek to solve the following optimization problem.

$$136 \quad \text{Minimize } (-F) \text{ over } C \quad (20)$$

137 Since our objective function F in equation (18) is a quadratic functional, we used the
138 conventional Conjugate Gradient Method to solve (20). This technique solves the
139 optimization method through an iterative procedure

$$140 \quad C_{k+1} = C_k + \alpha_k d_k \quad (21)$$

141 $k = 0, 1, 2, \dots, d_k$ is a direction vector and α_k is the line search step length at iteration k .

142 Usually, we find α_k such that

$$143 \quad \alpha_k = \min_{\alpha > 0} F(C_k + \alpha d_k)$$

144 The algorithm is given below.

145

146 **3. ALGORITHM** (Hybrid of the new Conjugate gradient method and Galerkin theory)

147 Let C_k, D_k and G_k be vectors. $G_k = G(C_k)$ and $\|G_k\| = (G_k^T G_k)^{\frac{1}{2}}$. Take

148 n and $h = \frac{1}{n+1}$. Use Galerkin method to obtain equations (15) and (18) and

149 $G(C_k) = MC_k - Z$ from equation (19). Use the following initial values:

$$150 \quad C_0 = (0, 0, \dots, 0)^T, \quad k = 0, \quad D_0 = -G_0.$$

151 Repeat the following steps of the new Conjugate gradient method.

152 a. Find step length α_k such that
153 $F(C_k + \alpha_k D_k) = \min_{\alpha > 0} F(C_k + \alpha D_k)$

154 b. Compute new point:

$$155 \quad C_{k+1} = C_k + \alpha_k D_k$$

156 c. Update search direction:

$$D_{k+1} = -G_{k+1} + \beta_k D_k,$$

$$G_{k+1} = \frac{1}{2!} [g(C_k + 2\Delta C_k) + g(C_k)]$$

$$\beta_k = \frac{\|G_{k+1}\|^2}{D_k^T y_k}$$

$$y_k = G_{k+1} - G_k$$

d. Check for optimality of G :

Terminate iteration at step m when $\|G_m\|$ is so small that C_m is an acceptable estimate of the optimal point C^* of F . If not optimal set $k = k + 1$.

4. NUMERICAL EXAMPLES

We used the hybrid of the new Conjugate Gradient Method and Galerkin theory to find the maximal deflection of a simply supported beam under uniformly distributed load from the following boundary value problems. Each boundary value problem governs the deflection of a structured beam. Nine interior points were used in each case. The driving program (MatLab) terminates on an optimal point.

Problem 1:

$$\frac{d^2 y(x)}{dx^2} = \frac{S}{EI} y(x) + \frac{q}{2EI} x(120 - x), \quad 0 \leq x \leq 120$$

$$y(0) = 0 = y(120),$$

$$S = 1000N, \quad l = 120m, \quad q = 8\frac{1}{3}N/m, \quad E = 3 \times 10^7 N/m^2 \text{ and } I = 625m^4.$$

Compare your results with the exact solution

$$y(x) = C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}} + \frac{b}{a} x^2 - \frac{120b}{a} x + \frac{2b}{a^2}$$

$$C_1 = -77042.53752198143, \quad C_2 = -79207.46247801857, \quad a = 5.33333 \times 10^{-8}$$

$$b = 2.22222222 \times 10^{-10}$$

Problem 2:

$$\frac{d^2 y(x)}{dx^2} = \frac{q}{2EI} x(20 - x), \quad 0 \leq x \leq 20$$

$$y(0) = 0 = y(20),$$

$$l = 20m, \quad q = 5000N/m, \quad EI = 133.333 \times 10^6 Nm^2.$$

Compare your results with the exact solution

$$y(x) = \frac{b}{12} (40x^3 - x^4 - 8000x)$$

$$b = \frac{q}{2EI}$$

191 **Problem 3:**

192
$$\frac{d^2 y(x)}{dx^2} = \frac{S}{D} y(x) + \frac{q}{2D} x(50 - x), \quad 0 \leq x \leq 50$$

193 $y(0) = 0 = y(50),$

194 $S = 100N, \quad l = 50m, \quad q = 200N/m, \quad D = 8.8 \times 10^7 Nm^2.$

195 Compare your results with the exact solution

196
$$y(x) = C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}} + x^2 - 50x + \frac{2}{a}$$

197 $C_1 = -856553.471726025, \quad C_2 = -903446.5282739757,$

198 $a = 1.1363636363636 \times 10^{-6}$

199

200 **5. NUMERICAL SOLUTIONS**

201 We used MatLab to solve the given boundary value problems. The exact solution is
202 $y(x_i)$ at each x_i .

203

204 **Table 1:** Solution of Problem 1 (Maximal deflection is 0.12cm downwards)

205

$n = 9, h = \frac{1}{10}, x_i \in [0, 120]$				
i	x_i	$y_9(x_i) = c_i$	$y(x_i)$	ERROR = $ y_9(x_i) - y(x_i) $
1	12	-0.000376675239629	-0.000376675016014	2.23615×10^{-10}
2	24	-0.000712649320008	-0.000712648849003	4.71005×10^{-10}
3	36	-0.000975668780120	-0.000975668110186	6.69934×10^{-10}
4	48	-0.001142695610452	-0.001142694818554	7.91898×10^{-10}
5	60	-0.001199907076070	-0.001199906197144	8.78927×10^{-10}

206

207 **Table 2:** Solution of Problem 2 (Maximal deflection is 7.81cm downwards)

$n = 9, h = \frac{1}{10}, x_i \in [0, 20]$				
i	x_i	$y_9(x_i) = c_i$	$y(x_i)$	ERROR = $ y_9(x_i) - y(x_i) $
1	2	-0.024525000061357	-0.024525000061312	4.4×10^{-14}
2	4	-0.046400000116089	-0.046400000116000	8.9×10^{-14}
3	6	-0.063525000158946	-0.063525000158812	1.33×10^{-13}
4	8	-0.074400000186200	-0.074400000186000	1.78×10^{-13}
5	10	-0.078125000195535	-0.078125000195312	2.22×10^{-13}

208

209

210

211 **Table 3:** Solution of Problem 3 (Maximal deflection is 18.49cm downwards).

$n = 9, h = \frac{1}{10}, x_i \in [0, 50]$				
i	x_i	$y_9(x_i) = c_i$	$y(x_i)$	ERROR = $ y_9(x_i) - y(x_i) $
1	5	-0.058044685436589	-0.058044549310580	$1.36126009 \times 10^{-7}$
2	10	-0.109817315930269	-0.109817057382315	$2.58547953 \times 10^{-7}$
3	15	-0.150347747852438	-0.150347393937409	$3.53915029 \times 10^{-7}$
4	20	-0.176085979463466	-0.176085565239191	$4.14224275 \times 10^{-7}$
5	25	-0.18490205005989	-0.184901615371928	$4.34687963 \times 10^{-7}$

6. DISCUSSION OF NUMERICAL RESULTS

The data shown in tables 1 to 3 reveal that results obtained by the hybrid of the new Conjugate Gradient Method and Galerkin theory are very close to the exact solutions of the given boundary value problems. The symmetry of the Hessian matrix, involved in each objective function, accelerated the convergence of this method. We, also, observed that errors on numerical solutions decrease as interior points increase.

7. CONCLUSION

We have used this hybrid of a new Conjugate Gradient Method and Galerkin theory to find the maximal deflection value of a simply supported beam under uniformly distributed load. The characterized two point linear boundary value problems with homogeneous boundary conditions were solved, numerically. Simple basis functions were used as trial functions in our approximations. Results from tables 1 to 3 confirm that our method is accurate and reliable.

Herein, we present the hybrid of a new Conjugate Gradient Method and Galerkin theory to engineers and scientists who wish to solve real life problems in this class of boundary value problems.

REFERENCES

- [1] Burden R. L., Faires J. D. Numerical Analysis, seventh edition, Published by Wadsworth Group publishers, USA; 2001.
- [2] Chapra S. C. Applied Numerical Methods with MatLab for Engineers and Scientists, Third edition, published by McGraw-Hill Companies, Inc, New York; 2012.
- [3] Kharab A. K., Guenther R. B. An Introduction to Numerical Methods. A matLab Approach, published by CRC Press Taylor and Francis Group, USA; 2012
- [4] Nwaeze E., Bamigbola O.M. A New Conjugate Gradient Method For Solving Unconstrained Optimization Problems. International Journal of Advanced Engineering Sciences and Technologies. 2011; 22 (2):187-192.
- [5] Grewal B. S. Numerical Methods in Engineering and Science with Programs in C and C++, published by Khanna publishers, New Delhi; 2010.