

An Application of Continuous –Time Delay Differential Equations in Price Dynamics of Maize in Ghana

Abstract

This paper seeks to study the price dynamics of maize in Ghana, in context of mathematical modelling using continuous-time cobweb models derived from linear and nonlinear delay differential equations of price. The stability conditions of the two cobweb models: continuous-time linear and nonlinear models are discussed. The data is obtained from the Ministry of Food and Agriculture, Statistical Directorate Kumasi-Ghana, from 1994 to 2013,

From the results of the analysis performed on the real economic price and production data using numerical approach, the nonlinear model showed oscillations between and around two equilibrium points and would neither converge. This result seems realistic as it shows a reflection of real market price situation because no equilibrium price would be achieved due to inflation, food insufficiency and/or producers' sensitivity towards price.

The linear model on the other hand showed little oscillations and then converged towards equilibrium point (zero equilibrium price point). However, this seems unrealistic. We draw inferences from this study that researchers should rather use nonlinear models in solving most real-life economic problems to avoid misleading conclusions instead of linear models even though linear models sometimes give acceptable solutions.

Keywords: price of maize, cobweb model, delay differential equations, continuous-time modelling and price stability

Introduction

Maize is the number one crop in terms of area planted and accounts for 50-60% of total cereal production which then makes maize the second largest commodity crop in the country after cocoa and one of the most important crops for Ghana's agricultural sector and for food security (MiDA, 2010). Maize is the largest staple crop in Ghana and contributes significantly to consumer diets and according to Alderman and Higgins (1992) in a nationwide survey carried out in 1990, 94% of all households during period of the research had consumed maize. Boateng et al. (1990) also found that maize and maize based foods account for 10.8% of household food expenditures by the poor and 10.3% of food expenditures by other all income groups. The per capital consumption of maize in Ghana was estimated to be 42.5 kg in the year 2000 and 943000 Mt as national consumption in 2006 (MOFA, 2000; SRID, 2007).

Maize is a food for an estimated 50 % of the population in sub-Saharan Africa and provides 50 % of their basic calories. This staple food has great nutritional values as it contains carbohydrate, protein, iron, oil, fibre, sugar, ash, vitamin B, and minerals. People in the sub-region consume maize as a starchy base in a wide variety of porridges, pastes, grits, and beer. Green maize (fresh on the cob) is eaten parched, baked, roasted or boiled and plays an important role in filling the hunger gap after the dry season (Chaudhry, 1983). As a result of this consumption trend, price instability is raising concerns to producers and consumers in that increase in food prices have implications for food situation of the poor in Ghana.

Research has revealed that foodstuff prices exhibit high volatility with maize showing continual increasing price in recent years by twenty three percent (Kuwornu et al., 2011) and this assertion is buttressed by Wodonet al. (2008) that food prices for rice, maize and other cereals increased by 20 to 30 percent between the last few months of 2007 and beginning of 2008.

According to ISSER (2008), high food prices have radically different effects across countries and population groups such that the net food exporting countries benefit from improved terms of trade while net food importing countries however struggle to meet domestic food demand. The fact that in Africa most countries are net importers of cereals, these countries will be hard hit by rising prices. Higher food prices lead the poor limiting their food consumption and shifting to less-balanced diets, with harmful effects on health in the short and long run. Sanyang and Jones (2008) also affirm that, higher food prices result in people eating less frequently and in lesser quantities, as well as cheaper and less nutritious food.

There has been a steady upward trend in recent times in food prices at a modest rate and a major contributor to this effect has been the increase in crude oil price, since a large increase in crude oil prices stands out among numerous factors to explain most of the jump in food prices over the last decade. Indeed in a recent World Bank study, oil prices are found to be major contributing factor to food price increase than several other long-term price drivers,

including exchange rates, interest rates and income (Baffes, 2013). Oil price increases exert both direct and indirect upward pressures on aggregate prices of food (ISSER, 2008). Prices of agricultural commodities in addition are affected by higher energy and chemical input prices, more frequent than usual adverse weather conditions, and the diversion of some food commodities to the production of bio-fuels; notably maize in the United States and edible oils in Europe. These conditions led global stock to use ratios of some grains down to levels not seen since the early 1970s (Baffes, 2013).

Maize production is essentially performed by smallholder farmers under traditional tillage and rain-fed conditions which are increasingly erratic, that total market conditions for maize tend to follow the direct impact of these rainfall conditions on production. The situation is exacerbated frequently by the poor or non-existent post-harvest management infrastructure such that even in periods of good moisture conditions, inefficient storage systems often result in price pressures arising from glut at harvest time and non-availability towards the end of the season (Amanor-Boadu, 2012) and according to (Modern Ghana, 2012), prices of maize, rice and yam, among others, shot up astronomically, negatively affecting consumers, few weeks after the Christmas festivities.

The study by Food security Ghana (2010) found that tonnes of maize are rotting away in silos as a result of poor market; a situation so frustrating to farmers that a maxi bag of maize is selling for GH¢32 compared to GH¢55 the previous year. Changes of food price are becoming increasingly relevant to producers and consumers in the competitive food markets in that price serve as an efficient means for seeking out production possibilities and potential, as well as allocating scarce resources within an economy (Gortz and Weber, 1986; Kuwornu et. al., 2011). It is therefore a belief that production delays in agricultural commodities are responsible for generating fluctuations in economic indicators and a good mathematical setting in which to study these phenomena is provided by a delay-differential equation (DDE); whilst the dependence of demand on price is instantaneous, the supply term contains a delay (Eduardo and Gergely, 2013).

This paper seeks to study the behaviour of the price dynamics of maize in context of mathematical modelling using continuous-time cobweb models derived from linear and nonlinear delay differential equations. The models would be formulated from real economic data of maize price and production in Ashanti Region, Ghana (the same data was used in Anokye et al. (2014)). It is intended to assist stakeholders to have knowledge of the price dynamics of maize so as to devise mechanism to ensure food availability and stabilize prices. Thus having knowledge of the dynamics of food prices gives farmers the opportunity to make informed decisions concerning planting of a particular food crop in the future since in agriculture, planting decisions as regards of quantity of food to supply are taken based on the expected price during harvest.

1.1 Effects of Continuous-time Models

Cobweb models describe price dynamics in a market of a non-storable good that takes one time unit to produce. Thus cobweb model describes cyclical supply and demand in a market where the amount produced must be chosen before prices are observed (Brock and Hommes, 1997).

Cobweb model of market stability is a discrete-time model that leads to a recurrence equation which describes the sequence of prices when the initial price is not the equilibrium value. One of the implicit assumptions in the demand and supply analysis is that suppliers decide of the quantity goods to be sent to market after they know the price of the good. But in reality in most suppliers commit themselves to supply decision before they know the price of the good in question. Thus in reality time is continuous but it is assumed that it evolves discretely so that under this framework the market is cleared once per fixed period of time (Asano, 2012). It is therefore possible to use continuous-time models in this situation and these models lead to differential equations rather than recurrence equations of price given as function of the continuously varying time parameter (Anthony, 1996).

Nowman (1997) applied a continuous time dynamic models of the short-term interest rate in finance and noted that the volatility of short-term rates is highly sensitive. Lindsey and Ryan (1994) also found that the continuous-time model is preferable to the discrete and mixed-time models for the reason that it gives reasonable estimates with relatively few intervals while still making full use of the available information. Continuous-time linear models for time series for instance have been found valuable certain in applications, particularly in fitting models to irregularly spaced data (Jones, 1981; Jones and Ackerson, 1990). Several recent papers (Brockwell et al. (1991), Tong and Yeung (1991), Brockwell and Hyndman (1992) and Brockwell (1994)) have investigated the use of continuous-time models, which are especially convenient for modelling irregularly spaced data when linear models are not adequate.

1.2 Effectsof Nonlinear Models

Although a great deal of attention has been given to linear difference and differential equations, far less attention has been given to nonlinear relationships. However, this is now changing as most recent researches conducted in economics are considering nonlinearities (Shone, 2002). Finkenstadt (1995) applied linear supply and nonlinear demand functions. Hommes (1991) and Jensen and Urban (1984) used linear demand functions with nonlinear supply equations. Junhai and Lingling (2007) also established a nonlinear model based on cobweb theory, where the demand function and supply function are quadratic. These findings indicate that the nonlinear cobweb model may explain various irregular fluctuations observed in real economic data. Anokye et al. (2014) studied price dynamics of maize in Ghana using linear and nonlinear cobweb models which are constructed from real economic price and production data. It was deduced from their study that researchers rather use nonlinear models to avoid under or overestimation and make better predictions in real economic situations.

In economic systems with backward bending supply curves or multiple-valued demand curves, the dynamics described by the nonlinear at equilibrium points exhibit broad spectrums of complex behaviour. However, if the supply and demand curves are monotonic and single-valued, the behaviour remains qualitatively the same as in the linear case as only the three classes of behaviour are displayed: convergence to a fixed point, period-2 cycles, and instability (Jensen and Urban, 1984).

The use of nonlinear dynamic models in economics and finance has expanded rapidly in the last two decades (Diks et al., 2008) and in studying the differences between linear and nonlinear models in engineering, SolidWorks (2013) discovered that neglecting nonlinear effects can lead to serious errors as nonlinear model can help one to avoid overdesign and build better products. SolidNotes (2009) in other words states that it takes longer time to run nonlinear solution, yet it replicates the actual physical system being analyzed and the test results will correlate very accurately.

1.3 Special Effects of DDE

In the study of economic dynamics, the classic qualitative method is frequently adopted, while the application of differential equation is used less, but it has changed into a class of important tool of the study of economics at present. In considering the dynamics of price, production, consumption of commodity, B  lar and Mackey (1989) and Mackey (1989) proposed the price fluctuations models and under relatively mild conditions, they determined the stability of equilibrium price. Then Farahari and Grove (1992) and Cheng (2005) considered naive consumer models and studied the oscillation of equilibrium price. But these models all are ordinary differential equation, which lack time lag (delay) and if we take this factor into account, it makes the models different from the realistic problem.

Thus most of the recent economic modelling has almost entirely ignored the potential role of delays in generating economic fluctuations and good a mathematical setting in which to consider this gap is provided by a delay-differential equation (Howroyd and Russel, 1984; Eduardo and Gergely, 2013). While it is assumed that the consumers base their buying decisions on the current market price, for most commodities there is a finite time τ that elapses before a change in production occurs. This time lag can be affected by several factors, such as the time necessary for increasing production, buying new machines, hiring workers or building factories; in the case of agricultural commodities, there are also natural constraints that affect the delay (Mackey, 1989).

Delay Differential Equations (DDEs) are special type of Functional Differential Equations (FDEs) in which there is the constraint that its time evolution can only depend on specific past values of the state variable at discrete or continuous times. DDEs arise in applied models when traditional simplifications are abandoned for more realistic assumptions (Engelborghs, Luzyanina and Roose, 2002).

Delay differential equations have been widely used for many years in control theory and have recently become popular in biological and economical models. Loretti et al. (2012) employed differential system with delays similar to Dibeh (2007) to describe the price dynamics of two markets that are coupled via diffusive coupling terms. They studied two different time delay cases, namely when both markets experience the same time delay, and when the time delay is different across markets to determine their equilibrium and stability theoretically. The study also used numerical illustrations to confirm the theoretical findings. Walter (1981) used DDE and realised that a broad spectrum of dynamic behaviours can be found in nonlinear delay differential equations. Meng-xing and Cheng (2008) developed a mathematic model for price cooperation with lag in which when the economic parameters satisfy some conditions, the existence and stability of periodic price are investigated.

2. Method

In order to study price dynamics of maize in Ghana, records of price and production of maize from seventeen major market centres in Ashanti Region were selected. The paper uses secondary data of maize that are gathered from the year 1994 to 2013, from the Ministry of Food and Agriculture, Statistical Directorate Kumasi-Ghana as shown in table 1. Linear and nonlinear delay differential equations of demand and supply are formulated from the data and then used them (equations) to derive cobweb models which are used for the study of price dynamics.

Table 1: Price and Production of Maize in Ashanti Region

Prices of Maize (Bag of 100 kg) in GH¢					Production of Maize (Metric Tons)				
Year	1 st Quarter	2 nd Quarter	3 rd Quarter	4 th Quarter	1 st Quarter	2 nd Quarter	3 rd Quarter	4 th Quarter	Total
1994	1.23	1.78	1.49	1.67	10198.13	14830.01	12403.34	13878.19	51309.67
1995	2.69	3.53	1.64	2.22	15947.12	20917.51	9730.17	13171.87	59766.67
1996	2.80	3.41	4.61	4.45	10152.87	12357.34	16710.75	16125.71	55346.67
1997	6.51	8.73	6.74	-	17268.13	23180.01	17886.86	-	58335.00
1998	5.06	5.62	4.43	4.26	19354.45	21514.8	16930.53	16287.52	74087.30
1999	4.64	4.94	5.76	4.60	15016.85	15978.78	18646.42	14875.28	64517.33
2000	9.32	11.05	9.90	10.39	14663.71	17381.97	15579.11	16342.87	63967.66
2001	13.87	18.76	12.98	13.04	13403.83	18127.28	12539.71	12595.85	56666.67
2002	14.83	15.38	10.97	11.36	25347.56	26300.98	18760.61	19417.51	89826.66
2003	14.03	17.06	16.32	14.49	14649.34	17818.76	17039.9	15132	64640.00
2004	17.80	21.67	23.46	23.06	12629.82	15372.77	16646.07	16362.01	61010.67
2005	29.67	45.02	34.25	28.56	11638.64	17660.19	13436.17	11203.67	53938.67
2006	25.91	27.39	20.98	18.87	15226.02	16097.1	12328.32	11090.56	54742.00
2007	25.85	32.34	26.41	26.06	13190.26	16500.28	13474.43	13296.04	56461.01
2008	32.34	56.66	56.72	49.39	10103.38	17700.44	17717.1	15428.41	60949.33
2009	60.79	71.77	55.13	52.21	15780.74	18631.95	14311.43	13552.55	62276.67
2010	53.29	55.51	52.76	46.82	21597.84	22497.63	21385.72	18976.81	84458.00
2011	55.10	75.41	81.34	89.95	10572.89	14470.08	15607.96	17260.74	57911.67
2012	110.69	124.44	90.97	76.00	18169.81	20426.27	14932.32	12475.61	66004.01
2013	77.09	76.24	74.86	81.67	17036.49	16847.17	16542.22	18047.12	68473.00

The table 1 contains the quarterly price and production of maize data at the right and left sections of the table respectively.

2.1 Continuous time linear Model

When the price of a good is p , the demand function of price is $D(p(t)) = a - \alpha p(t)$; this curve is generally negatively sloped-decreasing in mathematical term with a and α being positive constants. On the other hand supply function of price with delay τ is positively sloped-increasing and is given by $S(p(t)) = b + \beta p(t - \tau)$, where b , β and τ are positive constants.

The rate at which, the price changes is proportional to the difference between supply and demand (Mackey, 1989).

$$p'(t) = [D(p(t)) - S(p(t))] \quad (1)$$

This is a delay differential equation (DDE) that describes how the market price changes over time. It is assumed that when demand exceeds supply, price rises, and when supply exceeds demand it falls. It is also assumed that only the market price at time $t - \tau$ has an effect on the current supply price so that:

$$p'(t) = [(a - b) - \alpha p(t) - \beta p(t - \tau)] \quad , \text{where } \tau > 0, \text{ on } [0, b], b > 0 \quad (2)$$

This equation with a single delay like all DDEs are mostly solved in a stepwise fashion with a principle called the method of steps: equation (2) would have initial function (also known as history function) as $p(t) = s(t)$ defined over the interval $[-\tau, 0]$ and then its solution is mapped onto solutions of other functions. Thus the solution of this equation is going to be a mapping from functions on the interval $[t - \tau, t]$ into functions on the interval $[t, t + \tau]$, $[t + \tau, t + 2\tau]$ etc., from time points $t = 0, \tau, 2\tau, \dots$

In other words, the solutions of this dynamical system can be thought of as sequence of functions $p_0(t), p_1(t), p_2(t), \dots$ defined over contiguous time interval of length τ (Roussel, 2005). However, in practice this initial value problem is often solved numerically and in this study, MatLab solver dde23 would be used.

2.2 Continuous time non-linear Model

Considering simple nonlinear delay differential equation (of the form quadratic) for the supply function of price, $S(p(t)) = b + \beta p(t - \tau) - \delta p^2(t - \tau)$ and demand function of price by $D(p(t)) = a - \alpha p(t)$, where a, b, α, β and τ are positive constants.

$$p'(t) = [(a - b) - \alpha p(t) - \beta p(t - \tau) + \delta p^2(t - \tau)] \quad \text{where, } \tau > 0, \text{ on } [0, b], b > 0 \quad (3)$$

Like Equation (2), the nonlinear delay differential equation (3) cannot have an exact solution in practice using analytical method. Therefore numerical method with same principle of methods of steps is applied using MatLab solver dde23 with history function of $p(t) = s(t)$, on $[-\tau, 0]$.

3. Results and Discussion

This paper seeks to study the price dynamics of maize in context of mathematical modelling using real economic data of maize price and production in Ashanti Region, Ghana. Modelling of various functions and their parameter estimates would be done by the use of SPSS and then run the solution delay differential equations using MatLab.

3.1 Preliminary Analysis of Price and Production Data

The data are checked to correct any errors and then used SPSS and MatLab to verify the stationary status for both price and production data sets before formulating the demand and supply functions of price using regression analysis. The following table contains the coefficients of the parameters estimated from the data analyzed.

Table 2: Coefficients of demand and supply functions

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Model	Unstandardized Coefficients		Standardized Coefficients			
	B	Std. Error	Beta	t	sig.	
2. Price	-96.162	20.257	-0.488	-4.747	0.000	Demand Function
a. Dependent Variable: Production	b. Linear Regression through the Origin					
3. Price	354.28	36.887	1.715	9.604	0.000	Supply Function
Price**2	-2.782	0.453	-1.096	-6.14	0.000	
4. Price	167.994	43.229	0.419	3.886	0.000	

a. Dependent Variable: Production_Lag

b. Linear Regression through the Origin

3.2 Demand Function of Price

The demand function of price from the table 2 is given below with its parameter estimates checked to be statistically significant.

$$D(p(t)) = -96.162 p(t) \quad (4)$$

3.3 Supply Function of Price

Similarly supply functions with time delay τ are given by:

$$S(p(t)) = 167.994 p(t - \tau) \quad (5)$$

$$S(p(t)) = 354.28 p(t - \tau) - 2.782 p^2(t - \tau) \quad (6)$$

The delay τ expresses time that is needed to realise change of supply in dependence on trend of price. Thus current production depends on the past price.

3.3.1 Analysis of Linear Model

From equations (4) and (5), the rate of change of price is given by (7) if $\tau=1$:

$$\frac{1}{96.162} P'(t) = -P(t) - 1.75 P(t - 1) \quad (7)$$

This is delay differential equation which can be solved using MatLab solver dde23. (7) is divided by 96.162 so as to make the solution very smooth when it is run. The history function is set at $P(t)=1.23$ (initial price from table 1), when $t \leq 0$, (7) on the interval $[0, 100]$. By using numerical approach, the solution of equation (7) is presented in graphical (see figure 1) below.

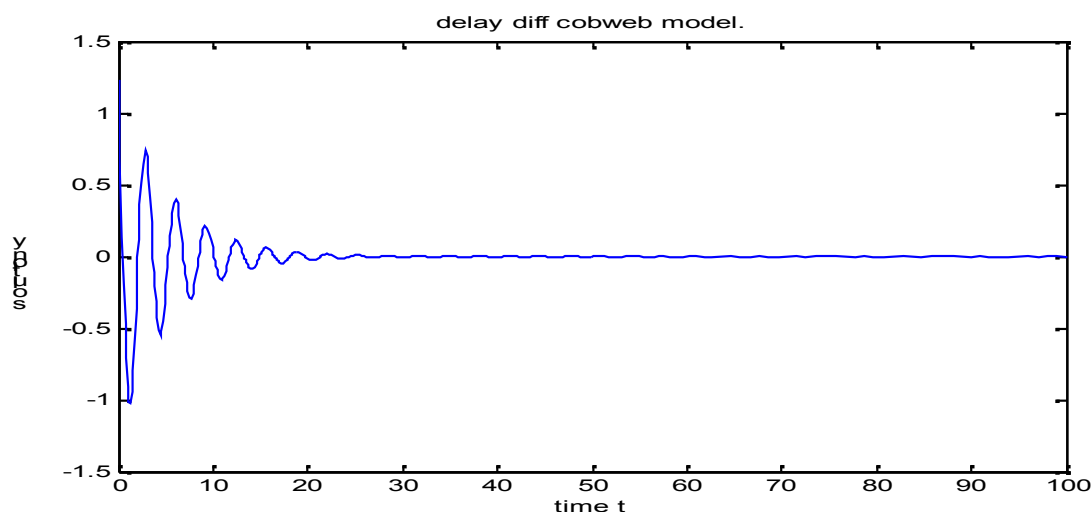


Figure 1: Oscillation of price around equilibrium

It is clear from the figure 1, above that the solution of equation (7) oscillates and tends to equilibrium price of zero with time. However, this equilibrium price is unrealistic, the fact that producers are sensitive towards price and there is insufficient of food supply.

3.3.2 Analysis of Nonlinear Model

From equations (4) and (6), the rate of change of price is given by (8) if $\tau=1$:

$$\frac{1}{96.162} P'(t) = -P(t) - 3.68P(t - \tau) + 0.029P^2(t - \tau) \quad (8)$$

Based on same assumption for smoothness of the solution using MatLab solver dde23, equation (8) divided by 96.162 and history function set at $P(t)=1.23$, when $t \leq 0$, and (8) on the interval $[0, 100]$ just like the linear solution. The solution of equation (7) in numerical form is presented graphically as follows (see figure 2 below):

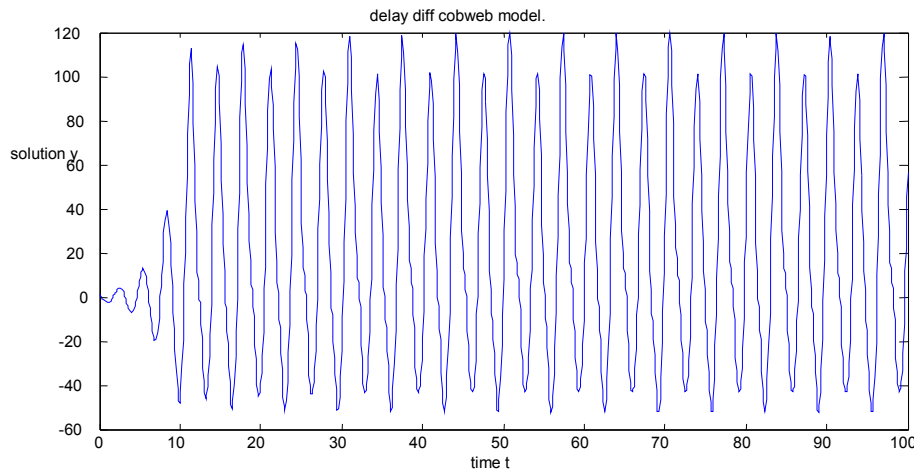


Figure 2: Oscillations of price around equilibrium

It is clear from the figure 2, above that the solution of equation (8) oscillates between two (2) equilibrium points (prices) and would never converge to either of the equilibrium price due to inflation, food insufficiency and/or producers' sensitivity towards price.

4. Conclusion

This paper seeks to study the price dynamics of maize in Ghana, in context of mathematical modelling using continuous-time cobweb model derived from linear and nonlinear delay differential functions of price. The stability conditions of the two cobweb models: continuous-time linear and nonlinear models are discussed.

From the results of the analysis performed on the real economic price and production data, using MatLab dde23 solver, the nonlinear model showed oscillations between and around two equilibrium price points and would neither converge with time. The oscillations observed from this solution are realistic and a reflection of real market price situation in Ghana. Therefore no equilibrium price would be achieved due to producers' sensitivity towards price. Inflation and food insufficiency can also contribute to this price instability.

The linear model on the other hand showed little oscillations and converged to zero equilibrium price point which is stable but unrealistic. Inferences can be made from this study that researchers would rather use nonlinear approach in solving most real-life economic problems to avoid misleading conclusions although linear models sometimes give acceptable solutions.

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 376 **DDE Code**
 377 `function ddeprice1`
 378 `%DDEPRICE excercise for DDE23.`
 379 `% This is cobweb model with delay differential equations (DDEs)`
 380 `% technigue.`
 381 `% ddeprice are solved on [0, 100] with history y_1(t) = 5, for t <= 0.`
 382 `% The lags are specified as a vector [1], the delay differential`
 383 `% equations are coded in the subfunction DDEPRICE, and the history is`
 384 `% evaluated by the function histprice. Because the history is constant it`
 385 `% could be supplied as a vector:`
 386 `% sol = dde23(@ddeprice,[1],5,[0, 100]);`
 387 `sol = dde23(@ddeprice,1,@histprice,[0, 100]);`
 388 `figure;`
 389 `plot(sol.x,sol.y)`
 390 `title('delay diff cobweb model.');`
 391 `xlabel('time t');`
 392 `ylabel('solution y');`
 393 `% -----`
 394 `function s = histprice(t)`
 395 `% Constant history function for ddeprice.`
 396 `s = (1.23);`
 397 `% -----`
 398 `function dydt = ddeprice(t,y,Z)`
 399 `% Delay Differential equations function for DDEPRICE.`
 400 `ylag1 = Z(:,1);`
 401 `dydt = -1*y(1)-1.75*ylag1(1);`
 402 `*The same for the nonlinear with few changes in code`