

1 **Original Research Article**
2 **Preliminary Test Stochastic Restricted $r-k$ Class Estimator**
3 **and Preliminary Test Stochastic Restricted $r-d$ Class**
4 **Estimator in Linear Regression Model**

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7 **ABSTRACT**

In this study two new preliminary test stochastic restricted estimators, a Preliminary Test stochastic restricted $r-k$ class estimator and a Preliminary Test stochastic restricted $r-d$ class estimator, are proposed. The comparison of one estimator over the other is done in the mean square error matrix sense. Further preliminary test stochastic restricted $r-k$ class estimator is compared with $r-k$ class estimator (Baye and Parker, 1984) and stochastic restricted $r-k$ class estimator (Wu, 2014). Similarly preliminary test stochastic restricted $r-d$ class estimator is compared with $r-d$ class estimator (Kaçiranlar and Sakallıoğlu, 2001) and stochastic restricted $r-d$ class estimator (Wu, 2014). Finally a numerical example is given to illustrate the theoretical results.

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9 **Keywords:** Preliminary Test Estimator, $r-d$ class estimator, $r-k$ class estimator, mean
10 square error matrix

11 **Mathematics Subject Classification:** Primary 62J07, Secondary 62F03.

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15 **1. INTRODUCTION**

16 Instead of using the Ordinary Least Square Estimator (OLSE), the biased estimators
 17 are considered in the regression analysis in the presence of multicollinearity. Some
 18 of these are namely the Principal Component Regression Estimator (PCRE) (Massy,
 19 1965), Ridge Estimator (RE) (Hoerl and Kennard, 1970), Liu Estimator (LE) (Liu,
 20 1993), r - k class estimator (Baye and Parker, 1984), Almost Unbiased Ridge
 21 Estimator (AURE) (Sing et al., 1986), Almost Unbiased Liu Estimator (AULE)
 22 (Akdeniz and Kaçiranlar, 1995) and the r - d class estimator (Kaçiranlar and
 23 Sakallıoğlu, 2001). An alternative method to deal with multicollinearity problem is
 24 to consider parameter estimation with some restrictions on the unknown parameters,
 25 which may be exact or stochastic restrictions. In the presence of stochastic prior
 26 information in addition to the sample information, Theil and Goldberger (1961)
 27 proposed the Mixed Estimator (ME). By replacing OLSE by ME in the RE, LE,
 28 AURE and AULE respectively, the Stochastic Mixed Ridge Estimator (SMRE) (Li
 29 and Yang, 2010), Stochastic Restricted Liu Estimator (SRLE) (Hubert and
 30 Wijekoon, 2006), Stochastic Restricted Almost Unbiased Ridge Estimator
 31 (SRAURE) (Wu and Yang, 2014) and Stochastic Restricted Almost Unbiased Liu
 32 Estimator (SRAULE) (Wu and Yang, 2014) are introduced. Recently, Wu (2014)
 33 proposed a new stochastic restricted r - k class estimator which is defined by combining
 34 the ME and r - k class estimator and a new stochastic restricted r - d class estimator
 35 which is defined by combining the ME and r - d class estimator.

36 When different estimators are available to estimate the unknown parameter,
 37 preliminary test estimation procedure is adopted to select a suitable estimator, and it
 38 can be also used as another estimator with combining properties of both estimators.
 39 The preliminary test approach was first proposed by Bancroft (1944), and then has
 40 been studied by many researchers, such as Judge and Bock (1978), Wijekoon (1990)
 41 and Arumairajan and Wijekoon (2013). By combining OLSE and ME, the Ordinary
 42 Stochastic Preliminary Test Estimator (OSPE) was proposed by Wijekoon (1990).

43 Recently, Arumairajan and Wijekoon (2013) introduced the Preliminary Test
 44 Stochastic Restricted Liu Estimator (PTSRLE) by combining the Stochastic
 45 Restricted Liu Estimator and Liu Estimator.

46 In this study, we propose a new Preliminary Test stochastic restricted $r-k$ class
 47 estimator which is defined by combining the $r-k$ class estimator and stochastic
 48 restricted $r-k$ class estimator and a new Preliminary Test stochastic restricted $r-d$
 49 class estimator which is defined by combining the $r-d$ class estimator and stochastic
 50 restricted $r-d$ class estimator. Further the proposed estimators are compared with
 51 some biased estimators in the mean square error matrix sense. Also Preliminary Test
 52 stochastic restricted $r-d$ class estimator is compared with Preliminary Test stochastic
 53 restricted $r-k$ class estimator. Finally a numerical example is given to illustrate the
 54 theoretical results.

55 2. MODEL SPECIFICATION AND ESTIMATORS

56 First we consider the multiple linear regression model

$$57 \quad Y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I), \quad (2.1)$$

58 where Y is an $n \times 1$ observable random vector, X is an $n \times p$ known design matrix of
 59 rank p , β is a $p \times 1$ vector of unknown parameters and ε is an $n \times 1$ vector of
 60 disturbances.

61 In addition to sample model (2.1), let us be given some prior information about β in
 62 the form of a set of m independent stochastic linear restrictions as follows;

$$63 \quad r = R\beta + \delta + \nu, \quad \nu \sim N(0, \sigma^2 \Omega) \quad (2.2)$$

64 where r is an $m \times 1$ stochastic known vector R is a $m \times p$ of full row rank $m \leq p$ with
 65 known elements, δ is non zero $m \times 1$ unknown vector and ν is an $m \times 1$ random
 66 vector of disturbances and Ω is assumed to be known and positive definite. Further
 67 it is assumed that ν is stochastically independent of ε . i.e. $E(\varepsilon\nu') = 0$.

68 The Ordinary Least Squares Estimator (OLSE) for the model (2.1) and the mixed
 69 estimator (Theil and Goldberger, 1961) due to a stochastic prior restriction (2.2) are
 70 given by

$$71 \quad \hat{\beta} = S^{-1} X' Y \quad \text{and} \quad \hat{\beta}_m = \hat{\beta} + S^{-1} R' (\Omega + RS^{-1} R')^{-1} (r - R\hat{\beta}) \quad (2.3)$$

72 respectively. where $S = X' X$.

73 The expectation vector, and the mean square error matrix of $\hat{\beta}$ are given as

$$74 \quad E(\hat{\beta}) = \beta \text{ and } MSE(\hat{\beta}) = \sigma^2 S^{-1} \text{ respectively.}$$

75 The expectation vector, dispersion matrix, and the mean square error matrix of $\hat{\beta}_m$

$$76 \quad \text{are given as } E(\hat{\beta}_m) = \beta + H\delta, \quad D(\hat{\beta}_m) = \sigma^2 S^{-1} - \sigma^2 G \text{ and}$$

$$77 \quad MSE(\hat{\beta}_m) = \sigma^2 (S^{-1} - G) + H\delta\delta'H' \text{ respectively, where,}$$

$$78 \quad G = S^{-1} R' (\Omega + RS^{-1} R')^{-1} RS^{-1}, \quad H = S^{-1} R' (\Omega + RS^{-1} R')^{-1} \text{ and } \delta = E(r) - R\beta.$$

79 When different estimators are available for the same parameter vector β in the
 80 linear regression model one must solve the problem of their comparison. Usually as a
 81 simultaneous measure of covariance and bias, the mean square error matrix is used,
 82 and is defined by

$$83 \quad MSE(\hat{\beta}, \beta) = E \left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \right] = D(\hat{\beta}) + B(\hat{\beta})B'(\hat{\beta}), \quad (2.4)$$

84 where $D(\hat{\beta})$ is the dispersion matrix, and $B(\hat{\beta}) = E(\hat{\beta}) - \beta$ denotes the bias vector.

85 We recall that the Scalar Mean Square Error $SMSE(\hat{\beta}, \beta) = \text{trace} [MSE(\hat{\beta}, \beta)]$.

86 For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$
 87 under the MSEM criterion if and only if

$$88 \quad M(\hat{\beta}_1, \hat{\beta}_2) = MSE(\hat{\beta}_1, \beta) - MSE(\hat{\beta}_2, \beta) \geq 0. \quad (2.5)$$

89 Let us now turn to the question of the statistical evaluation of the compatibility of
 90 sample and stochastic information. The classical procedures is to test the hypothesis

$$H_0 : \delta = 0 \quad \text{against} \quad H_1 : \delta \neq 0 \quad (2.6)$$

under linear model (2.1) and stochastic prior information (2.2).

The Ordinary Stochastic Pre Test Estimator (OSPE) of β (Wijekoon, 1990) is defined as

$$\hat{\beta}_{OSPE} = \begin{cases} \hat{\beta}_m & \text{if } H_0 : \delta = 0 \\ \hat{\beta} & \text{if } H_1 : \delta \neq 0 \end{cases} \quad (2.7)$$

Further, we can write (2.7) as

$$\hat{\beta}_{OSPE} = \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]}(F) + \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)}(F), \quad (2.8)$$

$$\text{where, } F = \frac{(r - R\hat{\beta})' (\Omega + RS^{-1}R')^{-1} (r - R\hat{\beta})}{m\hat{\sigma}^2} \quad (2.9)$$

which has a non-central $F_{m,n-p,\lambda}$ distribution under $H_1 : \delta \neq 0$, with non-centrality parameter

$$\lambda = \frac{\delta' (\Omega + RS^{-1}R')^{-1} \delta}{2\sigma^2} \quad \text{with} \quad \hat{\sigma}^2 = \frac{(Y - X\hat{\beta})' (Y - X\hat{\beta})}{n - p}, \quad (2.10)$$

and $I_{[0, F_{m,n-p}(\alpha)]}(F)$ and $I_{[F_{m,n-p}(\alpha), \infty)}(F)$ are indicator functions which take the value one if F falls in the subscripted interval, and zero otherwise. $F_{m,n-p}(\alpha)$ is the upper α - level critical value from the central F distribution $F_{m,n-p,0}$.

The expectation vector, dispersion matrix, and the mean square error matrix of $\hat{\beta}_{OSPE}$ are derived by Wijekoon (1990), and given by

$$E(\hat{\beta}_{OSPE}) = \beta + h_\lambda(2)H\delta, \quad (2.11)$$

$$D(\hat{\beta}_{OSPE}) = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)]H\delta\delta'H', \quad (2.12)$$

and

$$MSE(\hat{\beta}_{OSPE}) = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G + (2h_\lambda(2) - h_\lambda(4))H\delta\delta'H' \quad (2.13)$$

112 respectively, where, $h_\lambda(\ell) = \Pr\left(\frac{\chi_{m+\ell, \lambda}^2}{\chi_{n-p}^2} \leq \frac{mF_{m, n-p}(\alpha)}{n-p}\right)$ for $\ell \in \mathbb{N}$.

113 Now we consider the transformation for model (2.1):

$$114 \quad y = XTT'\beta + \varepsilon = Z\alpha + \varepsilon \quad (2.14)$$

115 where $Z = XT$, $\alpha = T'\beta$ and $T = (t_1, t_2, \dots, t_p) = (T_r, T_{p-r})$ is a $p \times p$ orthogonal matrix
116 such that

$$117 \quad (T_r, T_{p-r})' XX (T_r, T_{p-r}) = \Lambda = \begin{pmatrix} \Lambda_r & 0 \\ 0 & \Lambda_{p-r} \end{pmatrix}$$

118 where $0 < k \leq p$, $T_r = (t_1, t_2, \dots, t_r)$, $T_{p-r} = (t_{r+1}, t_{r+2}, \dots, t_p)$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$,

119 $\Lambda_r = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$, $\Lambda_{p-r} = \text{diag}(\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_p)$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0$ are the

120 eigenvalues of XX . Note that $Z = XT = (z_1, z_2, \dots, z_p) = (Z_r, Z_{p-r})$ is the $n \times p$ matrix

121 of the principal components, where $z_i = Xt_i$ is the i^{th} principal component. When

122 Z_{p-r} contains principal components corresponding to near zero eigenvalues, Z can

123 be separated as Z_r and Z_{p-r} , where Z_{p-r} is to be deleted. Now we can rewrite the

124 model (2.14) as

$$125 \quad y = XTT'\beta = XT_r T_r' \beta + XT_{p-r} T_{p-r}' \beta + \varepsilon = Z_r \alpha_r + Z_{p-r} \alpha_{p-r} + \varepsilon. \quad (2.15)$$

126 By omitting Z_{p-r} , the OLSE of α_r is obtained, and $\hat{\alpha}_r = (Z_r' Z_r)^{-1} Z_r' y$. Then PCRE of
127 β is

$$128 \quad \hat{\beta}_{PCRE} = T_r (T_r' S T_r)^{-1} T_r' X' y. \quad (2.16)$$

129 Now the r - k class estimator proposed by Baye and Parker (1984) and the r - d class
130 estimator proposed by Kaçiranlar and Sakallıoğlu (2001) are defined as

$$131 \quad \hat{\beta}_{rk}(r, k) = T_r (T_r' X' X T_r + k I_r)^{-1} T_r' X' y \quad (2.17)$$

132 and

$$133 \quad \hat{\beta}_{rd}(r, d) = T_r (T_r' X' X T_r + k I_r)^{-1} (T_r' X' y + d T_r' \hat{\beta}_r) \quad (2.18)$$

134 respectively.

135 Followed by Xu and Yang (2011), the r - k class estimator and the r - d class estimator
 136 could be rewritten as follows:

$$137 \quad \hat{\beta}_{rk}(r, k) = T_r T_r' \hat{\beta}(k) = T_r T_r' W_k \hat{\beta} = R_k \hat{\beta} \quad (2.19)$$

138 and

$$139 \quad \hat{\beta}_{rd}(r, d) = T_r T_r' \hat{\beta}(d) = T_r T_r' F_d \hat{\beta} = H_d \hat{\beta}$$

140 (2.20)

141 respectively, where $R_k = T_r T_r' W_k$ and $H_d = T_r T_r' F_d$.

142 Note that when $r = p$ we may conclude that $T_r T_r' = I_p$. Hence the estimators

143 $\hat{\beta}_{rk}(r, k) = W_k \hat{\beta}$ called as RE and $\hat{\beta}_{rd}(r, d) = F_d \hat{\beta}$ named as LE.

144 The mean square error matrices of $\hat{\beta}_{rk}(r, k)$ and $\hat{\beta}_{rd}(r, d)$ can be obtained as

$$145 \quad MSE[\hat{\beta}_{rk}(r, k)] = \sigma^2 R_k S^{-1} R_k' + (R_k - I) \beta \beta' (R_k - I)' \quad (2.21)$$

146 and

$$147 \quad MSE[\hat{\beta}_{rd}(r, d)] = \sigma^2 H_d S^{-1} H_d' + (H_d - I) \beta \beta' (H_d - I)' \quad (2.22)$$

148 respectively.

149 Wu (2014) proposed a new stochastic restricted r - k class estimator which is defined
 150 by combining the ME and r - k class estimator and a new stochastic restricted r - d class
 151 estimator which is defined by combining the ME and r - d class estimator as follows:

$$152 \quad \hat{\beta}_{SRrk}(r, k) = T_r T_r' \hat{\beta}(k) = T_r T_r' W_k \hat{\beta}_m = R_k \hat{\beta}_m \quad (2.23)$$

153 and

$$154 \quad \hat{\beta}_{SRrd}(r, d) = T_r T_r' \hat{\beta}(d) = T_r T_r' F_d \hat{\beta}_m = H_d \hat{\beta}_m \quad (2.24)$$

155 respectively.

156 When $r = p$, the estimator $\hat{\beta}_{SRrk}(r, k) = W_k \hat{\beta}_m$ called as SMRE and $\hat{\beta}_{SRrd}(r, d) = F_d \hat{\beta}_m$
 157 named as SRLE.

158 The mean square error matrices of $\hat{\beta}_{SRrk}(r, k)$ and $\hat{\beta}_{SRrd}(r, d)$ can be derived as

$$\begin{aligned}
159 \quad MSE \left[\hat{\beta}_{SRrk} (r, k) \right] &= \sigma^2 R_k S^{-1} R'_k - \sigma^2 R_k G R'_k \\
&+ \left[(R_k - I) \beta + R_k H \delta \right] \left[(R_k - I) \beta + R_k H \delta \right]' \quad (2.25)
\end{aligned}$$

160 and

$$\begin{aligned}
161 \quad MSE \left[\hat{\beta}_{SRrd} (r, d) \right] &= \sigma^2 H_d S^{-1} H'_d - \sigma^2 H_d G H'_d \\
&+ \left[(H_d - I) \beta + H_d H \delta \right] \left[(H_d - I) \beta + H_d H \delta \right]' \quad (2.26)
\end{aligned}$$

162 respectively.

163

164 3. PROPOSED ESTIMATORS

165 Arumirajan and Wijekoon (2013) proposed the Preliminary Test Stochastic
 166 Restricted Liu Estimator (PTSRLE) by combining the LE and SRLE as follows:

$$167 \quad \tilde{\beta}_{PTSRLE} (d) = F_d \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + F_d \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) = F_d \hat{\beta}_{OSPE} \quad (3.1)$$

168 Following Arumirajan and Wijekoon (2013), we may write the Preliminary Test
 169 Stochastic Mixed Ridge Estimator (PTSMRE) by combining RE and SMRE as
 170 follows:

$$171 \quad \tilde{\beta}_{PTSMRE} (k) = W_k \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + W_k \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) = W_k \hat{\beta}_{OSPE} \quad (3.2)$$

172 Following Arumairajan and Wijekoon (2013), we propose a new Preliminary Test
 173 stochastic restricted r - k class estimator which is defined by combing the r - k class
 174 estimator and stochastic restricted r - k class estimator and a new Preliminary Test
 175 stochastic restricted r - d class estimator which is defined by combing the r - d class
 176 estimator and stochastic restricted r - d class estimator as follows:

$$177 \quad \tilde{\beta}_{PTSRrk} (k) = R_k \left[\hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) \right] = R_k \hat{\beta}_{OSPE}, \quad (3.3)$$

178 and

$$179 \quad \tilde{\beta}_{PTSRrd} (d) = H_d \left[\hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) \right] = H_d \hat{\beta}_{OSPE} \quad (3.4)$$

180 respectively.

181 Now we will see some properties of proposed estimators.

182 1. When $r = p$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \tilde{\beta}_{PTSMRE}(k)$ and

183 $\tilde{\beta}_{PTSRrd}(d) = \tilde{\beta}_{PTSRLE}(d).$

184 2. When $r = p$, $k = 0$ and $d = 1$, we may conclude that

185 $\tilde{\beta}_{PTSRrk}(k) = \tilde{\beta}_{PTSRrd}(d) = \hat{\beta}_{OSPE}.$

186 3. When $\alpha = 0$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \hat{\beta}_{SRrk}(r, k)$ and

187 $\tilde{\beta}_{PTSRrd}(d) = \hat{\beta}_{SRrd}(r, d).$

188 4. When $\alpha = 1$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \hat{\beta}_{rk}(r, k)$ and

189 $\tilde{\beta}_{PTSRrd}(d) = \hat{\beta}_{rd}(r, d).$

190 By using (2.11), (2.12) and (2.13), the expectation vector, bias vector, dispersion
191 matrix, and the mean square error matrix of $\tilde{\beta}_{PTSRrk}(k)$ can be shown as follows:

192 $E[\tilde{\beta}_{PTSRrk}(k)] = R_k E(\hat{\beta}_{OSPE}) = R_k \beta + h_\lambda(2) R_k H \delta, \quad (3.5)$

193 $B[\tilde{\beta}_{PTSRrk}(k)] = E[\tilde{\beta}_{PTSRrk}(k)] - \beta = (R_k - I) \beta + h_\lambda(2) R_k H \delta, \quad (3.6)$

194 $D[\tilde{\beta}_{PTSRrk}(k)] = R_k D(\hat{\beta}_{OSPE}) R'_k$
 $= \sigma^2 R_k S^{-1} R'_k - \sigma^2 h_\lambda(2) R_k G R'_k + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] R_k H \delta \delta' H' R'_k, \quad (3.7)$

195 and

$MSE[\tilde{\beta}_{PTSRrk}(k)] = \sigma^2 R_k S^{-1} R'_k - \sigma^2 h_\lambda(2) R_k G R'_k$
 $+ [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] R_k H \delta \delta' H' R'_k$
 $+ R_k [(I - R_k^{-1}) \beta + h_\lambda(2) H \delta] [(I - R_k^{-1}) \beta + h_\lambda(2) H \delta]' R'_k \quad (3.8)$

197 respectively.

198 Similarly the expectation vector, bias vector, dispersion matrix, and the mean square
199 error matrix of $\tilde{\beta}_{PTSRrd}(d)$ can be shown as follows:

200 $E[\tilde{\beta}_{PTSRrd}(d)] = H_d E(\hat{\beta}_{OSPE}) = H_d \beta + h_\lambda(2) H_d H \delta, \quad (3.9)$

201 $B[\tilde{\beta}_{PTSRrd}(d)] = (H_d - I) \beta + h_\lambda(2) H_d H \delta, \quad (3.10)$

$$D[\tilde{\beta}_{PTSRrd}(d)] = \sigma^2 H'_d S^{-1} H'_d - \sigma^2 h_\lambda(2) H_d G H'_d + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] H_d H \delta \delta' H' H'_d, \quad (3.11)$$

and

$$MSE[\tilde{\beta}_{PTSRrd}(d)] = \sigma^2 H'_d S^{-1} H'_d - \sigma^2 h_\lambda(2) H_d G H'_d + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] H_d H \delta \delta' H' H'_d + H_d [(I - H_d^{-1})\beta + h_\lambda(2)H\delta] [(I - H_d^{-1})\beta + h_\lambda(2)H\delta]' H'_d \quad (3.12)$$

respectively.

206

207 4. MEAN SQUARE ERROR MATRIX COMPARISONS

208 In this section the Preliminary Test stochastic restricted r - d class will be compared
209 with r - d class estimator and stochastic restricted r - d class estimator and, the
210 Preliminary Test stochastic restricted r - k class estimator will be compared with r - k
211 class estimator and stochastic restricted r - k class estimator in the mean square error
212 matrix sense. Also Preliminary Test stochastic restricted r - d class estimator will be
213 compared with Preliminary Test stochastic restricted r - k class estimator.

214 4.1. Comparison between $\hat{\beta}_{rk}(r, k)$ and $\tilde{\beta}_{PTSRrk}(k)$

215 The mean square error matrix difference between $\tilde{\beta}_{PTSRrk}(k)$ and $\hat{\beta}_{rk}(r, k)$ is obtained
216 as

$$MSE[\hat{\beta}_{rk}(r, k)] - MSE[\tilde{\beta}_{PTSRrk}(k)] = R_k [D + d_1 d'_1 - d_2 d'_2] R'_k \quad (4.1)$$

218 where $D = \sigma^2 h_\lambda(2) G - \xi H \delta \delta' H'$, $\xi = 2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)$, $d_1 = (I - R_k^{-1})\beta$ and

219 $d_2 = (I - R_k^{-1})\beta + h_\lambda(2)H\delta$.

220 Now the following theorem can be stated.

221 **Theorem 4.1:** When $\lambda \leq \frac{1}{2[2 - h_\lambda(2)/h_\lambda(4) - h_\lambda(2)]}$, the estimator $\tilde{\beta}_{PTSRrk}(k)$ is

222 superior to $\hat{\beta}_{rk}(r, k)$ if and only if $d'_2 (D + d_1 d'_1)^{-1} d_2 \leq 1$.

223 **Proof:**

224 The mean square error matrix difference between $\tilde{\beta}_{PTSRrk}(k)$ and $\hat{\beta}_{rk}(r, k)$ is
 225 nonnegative definite matrix if and only if $D + d_1 d_1' - d_2 d_2' \geq 0$. To find the conditions
 226 that $D + d_1 d_1' - d_2 d_2' \geq 0$ by using lemma 3 (Appendix), we have to show that that D
 227 is a nonnegative definite matrix and $d_1, d_2 \in \Re(D)$, where $\Re(\cdot)$ denote the column
 228 space of the corresponding matrix.

229 Note that

$$230 \quad D = \sigma^2 h_\lambda(2) G - \xi H \delta \delta' H' = \xi \left(\frac{\sigma^2 h_\lambda(2)}{\xi} G - H \delta \delta' H' \right) = \xi D_1$$

$$231 \quad \text{where } D_1 = \frac{\sigma^2 h_\lambda(2)}{\xi} G - H \delta \delta' H'.$$

$$232 \quad \text{Set } \gamma = \frac{\sigma^2 h_\lambda(2)}{\xi}, \quad B = G \quad \text{and } a = H \delta.$$

233 Note that $G \geq 0$, and the generalized inverse of G is $G^- = SR^+ (RS^{-1}R' + \Omega)(R')^+ S$.

234 Hence $GG^-H\delta = H\delta$. This implies $H\delta \in \Re(G)$. Then according to lemma 1
 235 (Appendix)

$$236 \quad \delta' H' G^- H \delta \leq \frac{\sigma^2 h_\lambda(2)}{\xi} \quad (4.2)$$

237 After some straightforward calculations we can easily now show that

$$238 \quad \delta' H' G^- H \delta = \delta' (RS^{-1}R' + \Omega)^{-1} \delta \quad (4.3)$$

$$239 \quad \text{By substituting this result to (4.2) we can obtain } \frac{\delta' (RS^{-1}R' + \Omega)^{-1} \delta}{2\sigma^2} \leq \frac{h_\lambda(2)}{2\xi}. \quad (4.4)$$

240 Using (2.10), this inequality can be rewritten as

$$241 \quad \lambda \leq \frac{1}{2[2 - h_\lambda(2)/h_\lambda(4) - h_\lambda(2)]}. \quad (4.5)$$

242 Then according to lemma 1, D_1 is a nonnegative definite matrix, and therefore

$$243 \quad D = \xi D_1 \geq 0 \text{ since } \xi \geq 0.$$

Now the Moore Penrose inverse of D is obtained by using lemma 2 (Appendix), and is given by

$$D^+ = \frac{1}{\sigma^2 h_\lambda(2)} \times \left[G^+ + \frac{\xi}{\sigma^2 h_\lambda(2) - \xi \delta' H' G^+ H \delta} G^+ H \delta \delta' H' G^+ \right]. \quad (4.6)$$

After some straightforward calculations we can show that

$$\delta' H' G^+ H \delta = 2\sigma^2 \lambda. \quad (4.7)$$

Using (4.6) and (4.7) we can easily prove that $DD^+ = I_p$, where I_p is an identity matrix with order $(p \times p)$. This implies that $DD^+ d_1 = d_1$ and $DD^+ d_2 = d_2$. Then we have $d_1 \in \mathfrak{R}(D)$ and $d_2 \in \mathfrak{R}(D)$. To establish condition (a) in the lemma 3, we find

$f_{ij} = d_i' D^- d_j$ for $i = 1, 2, j = 1, 2$ such that

$$f_{11} = \beta' (I - R_k^{-1})' D^+ (I - R_k^{-1}) \beta,$$

$$f_{22} = \left[(I - R_k^{-1}) \beta + h_\lambda(2) H \delta \right]' D^+ \left[(I - R_k^{-1}) \beta + h_\lambda(2) H \delta \right] \text{ and}$$

$$f_{12} = \beta' (I - R_k^{-1})' D^+ \left[(I - R_k^{-1}) \beta + h_\lambda(2) H \delta \right].$$

Note that, instead of D^- , the Moore Penrose inverse D^+ of D is used, since f_{ij} is invariant to the choice of D^- .

Now according to lemma 3 in appendix, $MSE[\hat{\beta}_{rk}(r, k)] - MSE[\tilde{\beta}_{PTSRrk}(k)] \geq 0$ if and only if

$$(d_1' D^+ d_1 + 1)(d_2' D^+ d_2 - 1) \leq (d_1' D^+ d_2)^2.$$

4.2. Comparison between $\hat{\beta}_{SRrk}(r, k)$ and $\tilde{\beta}_{PTSRrk}(k)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRrk}(k)$ and $\hat{\beta}_{SRrk}(r, k)$ is obtained as

$$MSE[\tilde{\beta}_{PTSRrk}(k)] - MSE[\hat{\beta}_{SRrk}(k)] = D_2 + d_3 d_3' - d_4 d_4' \quad (4.8)$$

where $D_2 = \sigma^2 [1 - h_\lambda(2)] R_k G R_k' + \xi R_k H \delta \delta' H' R_k'$, $d_3 = (R_k - I) \beta + h_\lambda(2) R_k H \delta$ and

$$d_4 = (R_k - I)\beta + R_k H \delta.$$

Now we can state the following theorem.

Theorem 4.2: The estimator $\hat{\beta}_{SRrk}(r, k)$ is superior to $\tilde{\beta}_{PTSRrk}(k)$ if and only if

$$d'_4 (D_2 + d_3 d'_3)^{-1} d_4 \leq 1.$$

Proof: First we consider the matrix

$$D_2 = \sigma^2 [1 - h_\lambda(2)] R_k G R'_k + \xi R_k H \delta \delta' H' R'_k.$$

The matrix D_2 can be rewritten as $D_2 = R_k M_\ell R'_k$, where

$$M_\ell = \sigma^2 [1 - h_\lambda(2)] G + \xi H \delta \delta' H'.$$

The matrix $\sigma^2 [1 - h_\lambda(2)] G$ is nonnegative definite matrix since $G \geq 0$ and $0 \leq h_\lambda(2) \leq 1$, and the matrix $\xi H \delta \delta' H'$ is positive

definite matrix since $H \delta \delta' H'$ is positive definite matrix and $\xi \geq 0$. Therefore the

matrix M_ℓ is positive definite matrix, which leads the matrix D_2 is positive definite

matrix since $R_k > 0$. Now according to lemma 4 (Appendix), the matrix

$$D_2 + d_3 d'_3 - d_4 d'_4 \geq 0 \text{ if and only if } d'_4 (D_2 + d_3 d'_3)^{-1} d_4 \leq 1. \text{ This completes the proof.}$$

4.3. Comparison between $\hat{\beta}_{rd}(r, d)$ and $\tilde{\beta}_{PTSRrd}(d)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRrd}(d)$ and $\hat{\beta}_{rd}(r, d)$ is

obtained as

$$MSE[\hat{\beta}_{rd}(r, d)] - MSE[\tilde{\beta}_{PTSRrd}(d)] = H_d (D + b_1 b'_1 - b_2 b'_2) H'_d \quad (4.9)$$

where, $b_1 = (I - H_d^{-1})\beta$ and $b_2 = (I - H_d^{-1})\beta + h_\lambda(2)H\delta$.

Theorem 4.3: When $\lambda \leq \frac{1}{2[2 - h_\lambda(2)/h_\lambda(4) - h_\lambda(2)]}$, the estimator $\tilde{\beta}_{PTSRrd}(d)$ is

superior to $\hat{\beta}_{rd}(r, d)$ if and only if $b'_2 (D + b_1 b'_1)^{-1} b_2 \leq 1$.

Proof: The mean square error difference between $\tilde{\beta}_{PTSRrd}(d)$ and $\hat{\beta}_{rd}(r, d)$ is

nonnegative definite matrix if and only if $D + b_1 b'_1 - b_2 b'_2 \geq 0$. We have already

proved that $D \geq 0$ and $DD^+ = I_p$. Therefore $DD^+ b_1 = b_1$ and $DD^+ b_2 = b_2$, which

implies that $b_1, b_2 \in \mathfrak{R}(D)$. Now according to lemma 3, $D + b_1 b_1' - b_2 b_2' \geq 0$ if and only if $(b_1' D^+ b_1 + 1)(b_2' D^+ b_2 - 1) \leq (b_1' D_1^+ b_2)^2$. This completes the proof.

4.4. Comparison between $\hat{\beta}_{SRrd}(r, d)$ and $\tilde{\beta}_{PTSRrd}(d)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRrd}(d)$ and $\hat{\beta}_{SRrd}(r, d)$ is obtained as

$$MSE[\tilde{\beta}_{PTSRrd}(d)] - MSE[\hat{\beta}_{SRrd}(d)] = B_1 + b_3 b_3' - b_4 b_4' \quad (4.10)$$

where $B_1 = \sigma^2 [1 - h_\lambda(2)] H_d G H_d' + \xi H_d H \delta \delta' H' H_d'$, $b_3 = (H_d - I) \beta + h_\lambda(2) H_d H \delta$

and

$$b_4 = (H_d - I) \beta + H_d H \delta.$$

Now the following theorem can be stated.

Theorem 4.4: The estimator $\hat{\beta}_{SRrd}(r, d)$ is superior to $\tilde{\beta}_{PTSRrd}(d)$ if and only if $b_4'(B_1 + b_3 b_3')^{-1} b_4 \leq 1$.

Proof: The matrix B_1 can be rewritten as $B_1 = H_d M_\ell H_d'$. We have already proved that M_ℓ is positive definite matrix. Therefore the matrix B_1 is positive definite matrix since $H_d > 0$. Now according to lemma 1, the matrix $B_1 + b_3 b_3' - b_4 b_4' \geq 0$ if and only if $b_4'(B_1 + b_3 b_3')^{-1} b_4 \leq 1$. This completes the proof.

4.5. Comparison between $\tilde{\beta}_{PTSRrd}(d)$ and $\tilde{\beta}_{PTSRrk}(k)$

The mean square error matrix between $\tilde{\beta}_{PTSRrd}(d)$ and $\tilde{\beta}_{PTSRrk}(k)$ is given as

$$MSE[\tilde{\beta}_{PTSRrk}(k)] - MSE[\tilde{\beta}_{PTSRrd}(d)] = \mathcal{G} + d_3 d_3' - b_3 b_3' \quad (4.11)$$

where $\mathcal{G} = R_k M_x R_k' - H_d M_x H_d'$ and $M_x = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2) G + \xi H \delta \delta' H'$.

Theorem 4.5: When maximum eigenvalues of $(H_d M_x H_d')(R_k M_x R_k')^{-1}$ is less than one, the estimator $\tilde{\beta}_{PTSRrd}(d)$ is superior to $\tilde{\beta}_{PTSRrk}(k)$ if and only if $b_3'(\mathcal{G} + d_3 d_3')^{-1} b_3 \leq 1$.

312 **Proof:** First we consider $M_x = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G + \xi H \delta \delta' H'$. Note that

313 $\sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G \geq \sigma^2 (S^{-1} - G)$ since $0 \leq h_\lambda(2) \leq 1$. But

314 $S^{-1} - G = (S + R\Omega^{-1}R')^{-1} \geq 0$.

315 Therefore $\sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G \geq 0$. Since $\sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G \geq 0$ and $\xi H \delta \delta' H' > 0$,

316 the matrix $M_x > 0$. Therefore $H_d M_x H_d'$ and $R_k M_x R_k'$ are positive definite matrices

317 since $H_d > 0$ and $R_k > 0$ respectively. So that, according to lemma 5,

318 $\mathcal{G} = R_k M_x R_k' - H_d M_x H_d' > 0$ if and only if maximum eigenvalues of

319 $(H_d M_x H_d')(R_k M_x R_k')^{-1}$ is less than one. Applying lemma 4, we can easily prove that

320 the estimator $\tilde{\beta}_{PTSRrd}(d)$ is superior to $\tilde{\beta}_{PTSRrk}(k)$ if and only if $b_3'(\mathcal{G} + d_3 d_3')^{-1} b_3 \leq 1$.

321 This completes the proof.

322 **Note**

323 In the above theorem, when $\alpha = 0$ and $\alpha = 1$, we can obtain the superiority

324 conditions of the Stochastic Restricted r-k class estimator over the Stochastic

325 Restricted r-d class estimator, and the superiority conditions of the r-k class estimator

326 over the r-d class estimator respectively.

327

328 5. NUMERICAL ILLUSTRATION

329 To illustrate our theoretical results in this section we consider the data set which was

330 discussed in Gruber (1998) and later considered by Akdeniz and Erol (2003), Li and

331 Yang (2011), Alheety and Kibria (2013), Toker and Kaçiranlar (2013), Wu (2013)

332 and, Wu and Yang (2013). Table 5.1 gives Total National Research and

333 Development Expenditures-as a Percent of Gross National Product by Country:

334 1972-1986. It represents the relationship between the dependent variable y the
335 percentage spent by the United States and the dependent variables x_1, x_2, x_3 and x_4 .

336 The variable x_1 represents the percent spent by France, x_2 that spent by West

337 Germany, x_3 that spent by Japan, and x_4 that spent by the former Soviet Union.

338

Table 5.1: Total National Research and Development Expenditures-as a Percent of Gross National Product by Country: 1972-1986.

Year	y	x_1	x_2	x_3	x_4
1972	2.3	1.9	2.2	1.9	3.7
1975	2.2	1.8	2.2	2.0	3.8
1979	2.2	1.8	2.4	2.1	3.6
1980	2.3	1.8	2.4	2.2	3.8
1981	2.4	2.0	2.5	2.3	3.8
1982	2.5	2.1	2.6	2.4	3.7
1983	2.6	2.1	2.6	2.6	3.8
1984	2.6	2.2	2.6	2.6	4.0
1985	2.7	2.3	2.8	2.8	3.7
1986	2.7	2.3	2.7	2.8	3.8

339

340 We assemble the data in the matrix form as follows:

$$341 \quad X = \begin{pmatrix} 1.9 & 2.2 & 1.9 & 3.7 \\ 1.8 & 2.2 & 2.0 & 3.8 \\ 1.8 & 2.4 & 2.1 & 3.6 \\ 1.8 & 2.4 & 2.2 & 3.8 \\ 2.0 & 2.5 & 2.3 & 3.8 \\ 2.1 & 2.6 & 2.4 & 3.7 \\ 2.1 & 2.6 & 2.6 & 3.8 \\ 2.2 & 2.6 & 2.6 & 4.0 \\ 2.3 & 2.8 & 2.8 & 3.7 \\ 2.3 & 2.7 & 2.8 & 3.8 \end{pmatrix}, y = \begin{pmatrix} 2.3 \\ 2.2 \\ 2.2 \\ 2.3 \\ 2.4 \\ 2.5 \\ 2.6 \\ 2.6 \\ 2.7 \\ 2.7 \end{pmatrix}$$

342

343 The four column of the 10×4 matrix X comprise the data on x_1 , x_2 , x_3 and x_4

344 respectively, and y is the response variable. Note that the eigen values of S are

345 $\lambda_1 = 302.9626$, $\lambda_2 = 0.7283$, $\lambda_3 = 0.0447$ and $\lambda_4 = 0.0345$ and the condition number of

346 X is approximately 8781.53. This implies the existence of multicollinearity in the

347 data set. The OLSE is given by

348 $\hat{\beta}_{OLSE} = S^{-1}X'y = (0.6455, 0.0896, 0.1436, 0.1526)'$

349 with $MSE(\hat{\beta}_{OLSE}, \beta) = 0.0808$ and $\hat{\sigma}^2 = 0.0015$.

350 Consider the following stochastic restrictions

351 $r = R\beta + \delta + \nu$ where $R = (1, -2, -2, -2)'$, $r = 1$ and

352 $\nu \sim N(0, \hat{\sigma}^2 = 0.0015)$. We select the significance level $\alpha = 0.05$. Following Chang

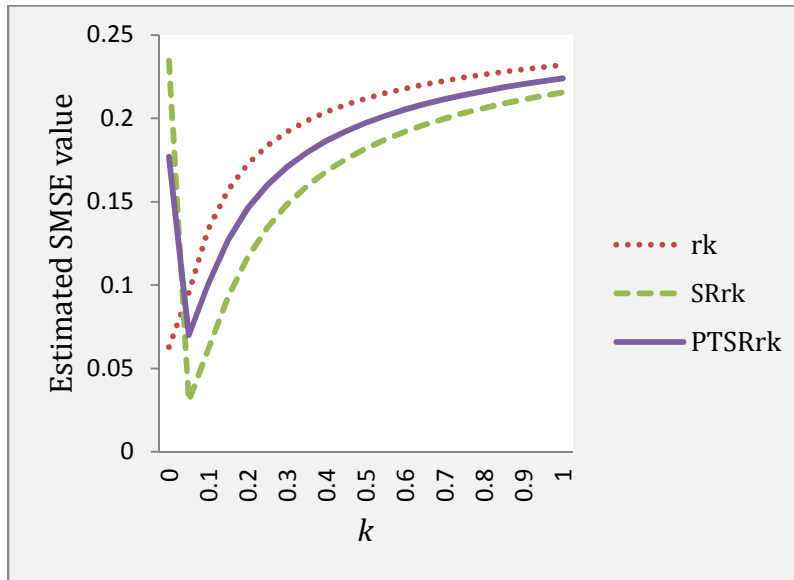
353 and Yang (2012) we choose the number of the principal components $r = 3$. Fig.1,

354 Fig.2 and Fig.3 are obtained by using the equations (2.21), (2.22), (2.25), (2.26),

355 (3.8) and (3.12) for different shrinkage parameters d and k values selected from the

356 interval $(0, 1)$.

357



358

359 **Fig.1:** Estimated SMSE value of rk, SRrk and PTSRrk.

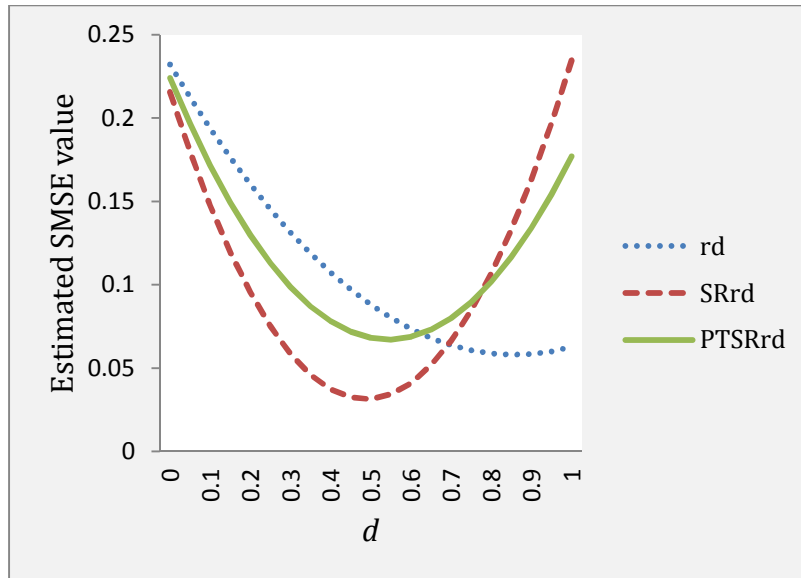


Fig.2: Estimated SMSE value of rd, SRrd and PTSRrd.

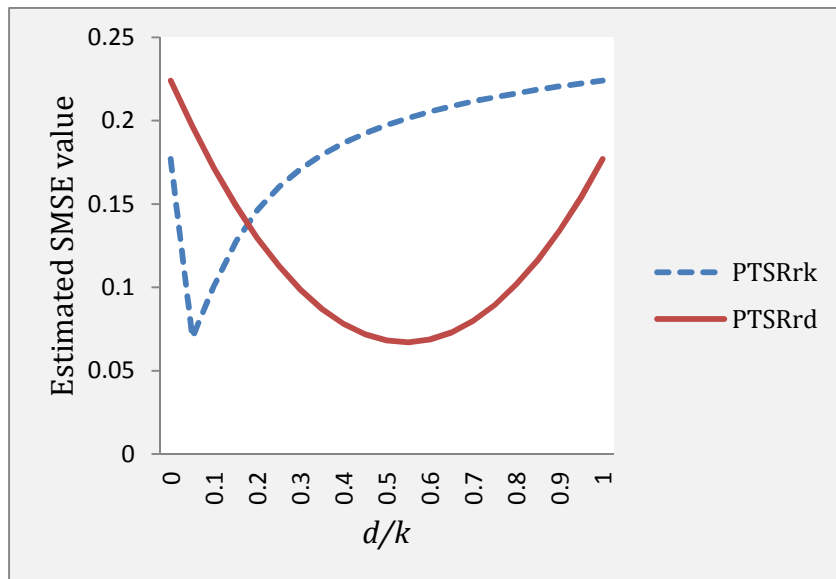


Fig.3: Estimated SMSE value of PTSRrk and PTSRrd

From Fig.1, when k is small the PTSRrk has the smallest SMSE than SRrk. When k is large, the PTSRrk has the smallest SMSE than rk. From Fig.2, when d is small the PTSRrd has the smallest SMSE than rd. When d is large, the PTSRrd has the smallest SMSE than SRrd. From Fig. 3, when d/k is small, PTSRrk has the smallest

SMSE than PTSRrd, the situation is reversed when d/k is large. From Fig.1, Fig.2 and Fig.3, we can say that no estimator is always superior to other estimators.

Remark:

Arumairajan and Wijekoon (2014) proposed the Generalized Preliminary Test Stochastic Restricted Estimator (GPTSRE) as follows:

$$\tilde{\beta}_{GPTSRE} = A_{(j)} \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]}(F) + A_{(j)} \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)}(F) = A_{(j)} \hat{\beta}_{OSPE}$$

where $A_{(j)}$ is a positive definite matrix.

Note that when $A_{(j)} = R_k$, $\tilde{\beta}_{GPTSRE}$ gives the Preliminary Test stochastic restricted r - k class estimator (PTSRrk), and when $A_{(j)} = H_d$, it gives the Preliminary Test stochastic restricted r - d class estimator (PTSRrd).

6. CONCLUSION

In this study, we propose a new Preliminary Test stochastic restricted r - k class estimator which is defined by combining the r - k class estimator and stochastic restricted r - k class estimator and a new Preliminary Test stochastic restricted r - d class estimator which is defined by combining the r - d class estimator and stochastic restricted r - d class estimator. Further the proposed estimators are compared with some biased estimators in the mean square error matrix sense. Also Preliminary Test stochastic restricted r - d class estimator is compared with Preliminary Test stochastic restricted r - k class estimator. Finally a numerical example is given to illustrate the theoretical results. Based on the theoretical findings and numerical illustration, we can conclude that no estimator is always superior to other estimators.

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470

471 **APPENDIX**472 **Lemma 1:** (Baksalary and Kala, 1983)

473 Suppose B is a symmetric real $(n \times n)$ matrix, a is an $(n \times 1)$ real vector and γ is a
 474 positive real number. Then the following two properties are equivalent

475 a) $\gamma B - aa'$ is nonnegative definite (n.n.d)476 b) B is n.n.d, $a \in \mathcal{R}(B)$ and $a'B^-a \leq \gamma$.477 **Lemma 2:** (Trenkler, 1985)

478 Let A be a symmetric $(n \times n)$ matrix, and let a , a_1 , and a_2 be $(n \times 1)$ vectors. Suppose
 479 that

480 a) $a \in \mathcal{R}(A)$, and the real numbers ϕ and ψ satisfy $\phi \neq 0$ and $\phi + \psi a'A^+a \neq 0$.

481 Then we have the identity

$$482 \quad [\phi A + \psi aa']^+ = \frac{1}{\phi} \left[A^+ - \frac{\psi}{\phi + \psi a'A^+a} A^+ aa' A^+ \right]$$

483 b) $a_j \in \mathcal{R}(A)$, $j = 1, 2$, and the real number ρ satisfies $1 + \rho a_1'A^+a_1 \neq 0$. Then484 we have $a_2 \in \mathcal{R}(A + \rho a_1 a_1')$.

485 **Lemma 3:** (Baksalary and Trenkler, 1991)

486 Let C be a nonnegative definite matrix and c_1, c_2 be linearly independent vectors.

487 Furthermore for some generalized inverse C^- of C , let $f_{ij} = c_i' C^- c_j$; $i = 1, 2, j = 1, 2$

488 and let

$$489 \quad s = \frac{c_2'(I - CC^-)'(I - CC^-)c_2}{c_1'(I - CC^-)(I - CC^-)c_1}$$

490 where $c_1 \in \mathfrak{R}(C)$ and $\mathfrak{R}(\cdot)$ denote the column space of the corresponding matrix.

491 Then we have $C + c_1 c_1' - c_2 c_2' \geq 0$ if and only if

492 a) $c_1 \in \mathfrak{R}(C), c_2 \in \mathfrak{R}(C)$ and $(f_{11} + 1)(f_{22} - 1) \leq f_{12}^2$ or

493 b) $c_1 \notin \mathfrak{R}(C), c_2 \in \mathfrak{R}(C, c_1)$ and $(c_2 - s c_1)' C^- (c_2 - s c_1) \leq 1 - s^2$

494 and all expressions in (a) and (b) are independent of the choice of C^- .

495 **Lemma 4:** (Trenkler and Toutenburg, 1990)

496 Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be two linear estimator of β . Suppose that $D = D(\hat{\beta}_1) - D(\hat{\beta}_2)$ is

497 positive definite then $\Delta = MSE(\hat{\beta}_1) - MSE(\hat{\beta}_2)$ is nonnegative definite if and only if

498 $b_2'(D + b_1 b_1')^{-1} b_2 \leq 1$, where b_j denotes the bias vector of $\hat{\beta}_j, j = 1, 2$.

499 **Lemma 5:** (Wang et al., 2006)

500 Let $n \times n$ matrices $M > 0$, $N > 0$ (or $N \geq 0$), then $M > N$ if and only if

501 $\lambda_1(NM^{-1}) < 1$. where $\lambda_1(NM^{-1})$ is the largest eigenvalue of the matrix NM^{-1} .

502