

STABILITY ANALYSIS AND HOPF BIFURCATION OF A SEIV EPIDEMIC MODEL WITH SATURATED INCIDENCE RATE

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ABSTRACT

In this paper, a SEIV epidemic model with saturated incidence rate that incorporates polynomial information on current and past states of the disease is investigated. The model exhibits two equilibria, disease-free equilibrium and the endemic equilibrium. It is shown that if the basic reproduction number, $R_0 < 1$, the disease free equilibrium is locally and globally asymptotically stable and in such a case, the endemic equilibrium is unstable. Moreover, if $R_0 > 1$, the endemic equilibrium is locally asymptotically stable. We discussed Hopf bifurcation of the model.

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28 1. INTRODUCTION

29 Vaccinating susceptible against disease infections is an effective measure to control and prevent the
 30 spread of the infection. Kribs-Zaleta and Velasco-Hemandez (2000) investigated SIS model with
 31 vaccination, standard incidence and no disease-induced deaths. Arino et al., (2003) formulated an SIRS
 32 model with vaccinations standard incidence and no disease-induced deaths. Li et al., (2006) studied an
 33 SIS model with vaccination, standard incidence and the disease induced death. Li and Ma (2002, 2004b)
 34 all analyzed global behaviour of simple SIS vaccination epidemic models under the condition that the
 35 vaccine is perfectly efficient. Li and Ma (2004a) introduced a vaccination compartment into an SIS model.
 36 They assumed that in vaccine takes effect in duration of length τ .

37 Long and Xiang (2011) studied an SEIV epidemic model with vaccination and vertical transmission. They
 38 carried out a bifurcation analysis and obtain the conditions ensuring that the system exhibits backward
 39 bifurcation. Chika and Ezeofor (2011) extended Cai and Li (2009) model by incorporating important
 40 practical factors as the proportion of recruited individuals that are exposed or infectious, the recovery rate
 41 of exposed individuals and the mortality rate due to the infection. Abdulrazak et al. (2012) discussed a
 42 vaccination model with non-linear incidence rate and vaccination waning period like (Cai and Li, 2009) but
 43 they numerically simulated the model to see how variations in rate of the variables.

44 In this paper, we extend the work of Cai and Li (2009) to incorporate a saturated incidence $\frac{\beta SI}{a_0 + \alpha I}$ that
 45 gives information on current and past states of the disease. We discuss also the hopf bifurcation of the
 46 model

$$\begin{aligned}
\frac{dS}{dt} &= (1-\rho)\pi - \frac{\beta SI}{a_0 + \alpha I} - \mu S + \omega V \\
\frac{dE}{dt} &= \frac{\beta SI}{a_0 + \alpha I} - (\mu + \sigma)E \\
\frac{dI}{dt} &= \sigma E - (\mu + \gamma)I \\
\frac{dV}{dt} &= \rho\pi - \mu V + \gamma I - \omega V
\end{aligned} \tag{1.1}$$

where

π = recruitment of individuals that includes new born and immigrants into the susceptible population S,
 ρ = fraction of recruited individuals who are vaccinated, β = rate at which susceptible individual become
infected, μ = natural death rate, σ = rate at which exposed individuals become infectious, γ = rate at
which infected individuals are treated or recovered, $\frac{1}{a_0 + \alpha(I)}$ = measurement of psychological or
inhibition effect of the behavioural change of the susceptible individuals when there is an increase in the
number of infective individuals, ω = rate at which vaccine wanes

2. THE BASIC REPRODUCTION NUMBER

It is easy to see that the region $\{(S, E, I, V), S > 0, I \geq 0, E \geq 0, V > 0\}$ is positively invariant for the model (1.1). Summing up the four equations in model (1.1), we have

$$\frac{d}{dt}(S + E + I + V) = \mu \left[\frac{\pi}{\mu} - (S + E + I + V) \right].$$

Then, $\limsup_{t \rightarrow \infty} (S + E + I + V) \leq \frac{\pi}{\mu}$. So, we study the dynamic behaviour of model (1.1) on the region

$$\Sigma = \{(S, E, I, V) \mid S > 0, I \geq 0, E \geq 0, V > 0, S + E + I + V \leq \frac{\pi}{\mu}\},$$

62 which is a positive invariant set for (1.1).

63 Corresponding to $E = I = 0$, model (1.1) always has a disease-free equilibrium,

64
$$P_0 = \left(\frac{(\mu + \omega - \rho\mu)\pi}{\mu(\mu + \omega)}, \quad 0, \quad 0, \quad \frac{\rho\pi}{\mu + \omega} \right).$$

65 For model (1.1), since we are concerned with the population that spread the infection, we only need to
66 model the exposed, E and infected, I classes

67
$$\begin{aligned} \frac{dE}{dt} &= \frac{\beta SI}{a_0 + \alpha I} - (\mu + \sigma)E \\ \frac{dI}{dt} &= \sigma E - (\mu + \gamma)I \end{aligned}$$

$$F = \begin{pmatrix} 0 & \frac{\beta S_0}{a_0} \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} \mu + \sigma & 0 \\ \sigma & \mu + \gamma \end{pmatrix}$$

68 where $F = \left[\frac{\partial F_i}{\partial x_j}(x_0) \right]$ and $V = \left[\frac{\partial V_i}{\partial x_j}(x_0) \right]$

69 and x_0 is the disease-free equilibrium

70
$$V^{-1} = \begin{pmatrix} \frac{1}{\mu + \gamma} & 0 \\ \frac{\sigma}{(\mu + \sigma)(\mu + \gamma)} & \frac{1}{\mu + \sigma} \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} \frac{\sigma\beta S_0}{a_0(\mu + \sigma)(\mu + \gamma)} & \frac{\sigma\beta S_0}{a_0(\mu + \sigma)} \\ 0 & 0 \end{pmatrix}$$

71
$$R_0 = \rho(FV^{-1}) = \frac{\sigma\beta S_0}{a_0(\mu + \sigma)(\mu + \gamma)}$$

72 where $S_0 = \frac{(\mu + \sigma)(1 - \rho)\pi + \omega\rho\pi}{\mu(\mu + \omega)}$

73 Therefore,

74
$$R_0 = \frac{\sigma\beta(\mu + \sigma)(1 - \rho)\pi + \omega\rho\pi}{a_0\mu(\mu + \omega)(\mu + \sigma)(\mu + \gamma)}$$

75 **Note:** The spectral radius $(\rho[FV^{-1}])$ is the dominant eigenvalue of the matrix FV^{-1} .

76

77 3. LOCAL AND GLOBAL STABILITY OF EQUILIBRIA

78 Let $x = S - S_0$, $I = I$, $E = E$, $y = V - V_0$ (3.1)

79 Then, equation (1.1) becomes

80
$$\begin{aligned} \frac{dS}{dt} &= (1 - \rho)\pi - \beta I \left(x + \frac{(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi}{\mu(\mu + \omega)} \right) (a_0 + \alpha I)^{-1} - \\ &\quad \mu \left(x + \frac{(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi}{\mu(\mu + \omega)} \right) + \omega \left(y + \frac{\rho\pi}{\mu + \omega} \right) \\ \frac{dE}{dt} &= \beta I \left(x + \frac{(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi}{\mu(\mu + \omega)} \right) (a_0 + \alpha I)^{-1} - (\mu + \sigma)E \\ \frac{dI}{dt} &= \sigma E - (\mu + \gamma)I \\ \frac{dV}{dt} &= \rho\pi - (\mu + \omega) \left(y + \frac{\rho\pi}{\mu + \omega} \right) + \gamma I \end{aligned} \quad (3.2)$$

81 The Jacobian matrix of equation (3.2) after linearization is

$$82 \quad A = \begin{pmatrix} -\mu & 0 & \frac{-\beta[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} & \omega \\ 0 & -(\mu + \sigma) & \frac{\beta[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} & 0 \\ 0 & \sigma & -(\mu + \gamma) & 0 \\ 0 & 0 & \gamma & -(\mu + \omega) \end{pmatrix}$$

$$83 \quad \text{Let } \frac{\beta[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{\mu(\mu + \omega)} = \Gamma$$

$$84 \quad |A - \lambda I| = \begin{vmatrix} -\mu & 0 & \frac{-\Gamma}{a_0} & \omega \\ 0 & -(\mu + \sigma) & \frac{\Gamma}{a_0} & 0 \\ 0 & \sigma & -(\mu + \gamma) & 0 \\ 0 & 0 & \gamma & -(\mu + \omega) \end{vmatrix} = 0$$

85 Therefore, $\lambda_1 = -\mu$, $\lambda_2 = -(\mu + \omega)$, and the roots of the quadratic

$$86 \quad \lambda^2 + (2\mu + \omega + \lambda)\lambda + \mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\Gamma}{a_0} = 0$$

87 For negative roots, we must have by Descartes' rule of signs

$$88 \quad \mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\Gamma}{a_0} > 0$$

89 That is,

$$(\mu + \sigma)(\mu + \gamma) > \sigma\beta \frac{[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)}$$

$$90 \quad \sigma\beta \frac{[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)(\mu + \sigma)(\mu + \gamma)} < 1$$

91 Therefore,

$$R_0 = \frac{\sigma\beta [(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)(\mu + \sigma)(\mu + \gamma)} < 1$$

Theorem 3.1: If $R_0 < a_0$, the disease free equilibrium P_0 is locally asymptotically stable; if $R_0 = a_0$, P_0 is stable; if $R_0 > a_0$, P_0 is unstable

Proof: We shall check the stability of the disease-free equilibrium P_0 , from the model, thus the linearization of model of the disease-free equilibrium P_0 gives the following characteristics equation.

$$(\mu + \lambda)(\mu + \sigma + \lambda) \left[(\mu + \gamma + \lambda)(\mu + \omega + \lambda) \frac{\sigma\beta [(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} \right] = 0 \quad (3.3)$$

98

99 Thus, we have

100

$$(\mu + \lambda)(\mu + \sigma + \lambda) \left[\lambda^2 + (2\mu + \gamma + \sigma)\lambda + \mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\beta [(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} \right] = 0 \quad (3.4)$$

102 From equation (3.4), it can be seen that $\lambda_1 = -\mu$, $\lambda_2 = -(\mu + \sigma)$ are two of the eigenvalues and they
103 are always negative. To obtain the other eigenvalues of equation (3.4) we consider the equation

$$\lambda^2 + (2\mu + \gamma + \sigma)\lambda + \mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\beta [(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} = 0 \quad (3.5)$$

105 From equation (3.5) we see that all roots have negative real parts if

$$\mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\beta [(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} > 0$$

107 That is, if $R_0 < a_0$.

108 $\therefore$ The disease-free equilibrium P_0 , is locally asymptotically stable.

109 If $R_0=a_0$, one eigenvalue of equation (3.5) is zero and it is simple. Then P_0 is stable.

110 If $R_0>a_0$, one of the roots of equation (3.5) has a positive real part, then P_0 is unstable.

111 **Theorem 3.2:** If $R_0<a_0$, the disease-free equilibrium P_0 is globally asymptotically stable.

112 **Proof:** From the above lemma, we have obtained that the disease-free equilibrium P_0 of the model is
113 locally asymptotically stable whenever $R_0<a_0$.

114 From the first equation of model (4.3), we have

115
$$S(t) \leq (1-\rho)\pi + \frac{\omega\pi}{\mu} - (\mu + \omega)x \quad \text{for } \phi(I) \geq 1$$

116 a solution of the equation

117
$$x(t) \leq (1-\rho)\pi + \frac{\omega\pi}{\mu} - (\mu + \omega)x$$

118 is a supper solution of $S(t)$. This implies that $x(t)=S(t)$ for all $t=0$.

119

120 We note that

121
$$x(t) \rightarrow \frac{[\omega + \mu(1-\rho)\pi]\pi}{\mu(\mu + \omega)} \quad \text{as } t \rightarrow \infty$$

122 It follows that for a given $E>0$, there exist a $x(t)$ such that

123
$$S^\infty \leq x(t) \leq \frac{[\omega + \mu(1-\rho)\pi]\pi}{\mu(\mu + \omega)} + E$$

124 letting $E \rightarrow \infty$, we have

125
$$S^\infty \leq \frac{[\omega + \mu(1-\rho)\pi]\pi}{\mu(\mu + \omega)}$$

126

127 Thus, the second equation of the model (4.3) can be expressed as

$$128 \quad E(t) \leq \frac{[\omega + \mu(1 - \rho)\pi]\pi + E}{\mu(\mu + \omega)} - (\mu + \sigma)e \quad \text{for } t = t_0$$

129 By applying the second and third equations of model (4.3), we obtain the following

$$130 \quad \begin{bmatrix} E \\ I \end{bmatrix} \leq Q \begin{bmatrix} E \\ I \end{bmatrix}$$

131 where

$$132 \quad Q = \begin{bmatrix} (\mu + \sigma) & \frac{\beta[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} + E \\ \sigma & (\mu + \gamma) \end{bmatrix}$$

133 Let $M \in IR^+$ such that $M > \max[(\mu + \sigma), (\mu + \gamma)]$.

134 Thus, $Q + MI_{2 \times 2}$ is a strictly positive matrix where $I_{2 \times 2}$ is the identity matrix. It is clear that if

135 λ_1 and λ_2 are the eigenvalues of $Q + MI_{2 \times 2}$. It follows from the Perron Frobenious theorem that

136 $Q + MI_{2 \times 2}$ has a simple positive eigenvalue equal to spectral radius and a corresponding eigenvector

137 $e > 0$. This implies that λ_1 and λ_2 are both real. If $\lambda_1 + M$ then $\lambda_1 > \lambda_2$ and $eQ = \lambda_1 e$, that is from

138 equation (3.6) below

$$\begin{aligned} \frac{dS}{dt} &= (1 - \rho)\pi - \frac{\beta SI}{a_0 + \alpha I} - \mu S + \omega \left(\frac{\pi}{\mu} - S - E - I \right) \\ \frac{dE}{dt} &= \frac{\beta SI}{a_0 + \alpha I} - (\mu + \sigma)E \\ \frac{dI}{dt} &= \sigma E - (\mu + \gamma)I \end{aligned} \quad (3.6)$$

140 we have

$$(Q - \lambda I) = \begin{bmatrix} (\mu + \sigma) + \lambda & \frac{\beta[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} + E \\ \sigma & (\mu + \gamma) + \lambda \end{bmatrix}$$

Obviously, λ_1 and λ_2 are roots of the equation

$$\lambda^2 + (2\mu + \gamma + \sigma)\lambda + \mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \sigma\beta\left(\frac{[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} + E\right) = 0$$

Since $R_0 < a_0$, for $\varepsilon > 0$

Sufficiently small, we have

$$(\mu + \sigma)(\mu + \gamma) - \sigma\beta\left(\frac{[(\mu + \omega)(1 - \rho)\pi + \omega\rho\pi]}{a_0\mu(\mu + \omega)} + E\right) > 0$$

From the model (4.14), we have that for $t \geq t_0$

$$\frac{d}{dt}(e \cdot [EI]) \leq \lambda_1 e \cdot [EI]$$

We note that ' $\cdot$ ' is the dot product of two vectors.

Integrating the above inequality, we obtain

$$0 \leq e \cdot [EI] \leq e \cdot [EI] e^{\lambda_1(t-t_1)}$$

For $t \geq t_1 > t_0$. Since $\lambda_1 < 0$ it follows that

$$e \cdot [EI] \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

Using $e > 0$, we conclude that

$$[EI] \rightarrow (0, 0) \quad \text{as} \quad t \rightarrow \infty \quad (3.7)$$

Now, we show

$$\lim_{t \rightarrow \infty} S(t) = \frac{[\omega + \mu(1 - \rho)]\pi}{\mu(\mu + \omega)}$$

We can choose a sequence $t_n \rightarrow \infty$, $S_n \rightarrow \infty$ ($n \rightarrow \infty$), such that

$$\begin{aligned} S(S_n) &\rightarrow S^\infty \\ S(t_n) &\rightarrow S^\infty \end{aligned}$$

and

$$\begin{aligned} \dot{S}(S_n) &\rightarrow 0 \\ \dot{S}(t_n) &\rightarrow 0 \end{aligned}$$

Noting that from (3.7)

$$[EI] \rightarrow 0 \quad \text{for } t \rightarrow \infty$$

And from the first equation of the model, we have that

$$\left. \begin{aligned} (1 - \rho)\pi + \frac{\omega\pi}{\mu} - (\mu + \omega) \lim_{t \rightarrow \infty} SupS(t) \\ (1 - \rho)\pi + \frac{\omega\pi}{\mu} - (\mu + \omega) \lim_{t \rightarrow \infty} InfS(t) \end{aligned} \right\} \quad (3.8)$$

Thus from the model. It shows that

$$\lim_{t \rightarrow \infty} S(t) = \frac{[\omega + \mu(1 - \rho)]\pi}{\mu(\mu + \omega)}$$

Combining this result with the above lemma, we have that the disease-free equilibrium P_0 is globally asymptotically stable of $R_0 < a_0$, which completes the proof.

170

171 4. HOPF BIFURCATION

Re $\lambda < 1$ means that the critical points, endemic equilibrium is either a stable focus or a stable node depending on whether $(2\mu + \gamma + \sigma)^2 > 4(\mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\Gamma}{a_0})$ or not.

5. DISCUSSION

If $\phi(I) = a_0 + \dots$ where $a_0 \geq 1$, then the basic reproduction number is proportional to $\frac{1}{a_0}$.

Since the basic reproduction number R_0 of an infection is the mean number of secondary cases a typical single infected case will cause in a population with no immunity to the disease in the absence of interventions to control the infection, the information on past history of the disease is a good way to control the disease.

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