

Gradient-Orthonormal Bases of Momentum-Entire Wavelets in Odd Dimension

Abstract

We construct gradient-orthonormal bases of momentum-entire wavelets in dimension n in the case where n is odd. The scale factor is $\sqrt[3]{2}$ and the coherence on a given scale is based on the propagator $\exp(\vec{s} \cdot \vec{\nabla} \Delta)$. This propagator can be understood in terms of two one-variable functions. One of them is the Airy function, while the other satisfies an ordinary differential equation less familiar than the Airy equation.

Keywords: Fourier transform, wavelets, complex variables

1 Introduction

As in [1] we define a *momentum-entire function* to be an integrable function of several variables such that the analytic continuation of its Fourier transform is an entire function of the several complex variables. A natural example of a momentum-entire function in n variables would be a Gaussian on $\mathbb{R}^n$.

Orthonormal bases of momentum-entire *wavelets* are not really new, at least in the Hilbert space $L^2(\mathbb{R}^n)$. Any continuous compactly supported function is momentum-entire, so the obvious examples are the Daubechies bases [2], which are L^2 -orthonormal. The N th-order Daubechies basis in n dimensions consists of compactly supported wavelets that are of class $C^{N-\epsilon}$ for arbitrarily small $\epsilon > 0$. If we denote this basis by $\mathcal{D}$, then

$$\mathcal{D} = \{2^{r\frac{n}{2}}\psi_\mu(2^r \vec{x} - \vec{s}) : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^n, \mu = 1, 2, \dots, 2^n - 1\}, \quad (1.1)$$

where

$$\mathcal{D}_0 = \{\psi_\mu : \mu = 1, 2, \dots, 2^n - 1\} \quad (1.2)$$

is the set of *mother wavelets* generating this wavelet basis.

The focus of this paper is on *gradient-orthonormal* bases of wavelets – i.e., wavelet bases that are orthonormal in the Hilbert space

$$\mathcal{H}_1(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left(\prod_{i=1}^n \int_{-\infty}^{\infty} dk_i \right) |\vec{k}|^2 |\hat{f}(\vec{k})|^2 < \infty \right\}, \quad (1.3)$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions. Since $|\vec{\nabla}|^{-1}$ is a unitary transformation from $L^2(\mathbb{R}^n)$ to $\mathcal{H}_1(\mathbb{R}^n)$, one can easily obtain such a basis by simply applying this operator to the elements of $\mathcal{D}$, but the output wavelets are not compactly supported. Indeed, $|\vec{\nabla}|^{-1}$ destroys the momentum-entire property, so another approach is needed for the construction of any gradient-orthonormal basis of momentum-entire wavelets.

Remark 1. Many years ago [3, 4], a method of constructing gradient-orthonormal bases of *exponentially localized* wavelets was found. (The operator $|\vec{\nabla}|^{-1}$ destroys exponential localization as well.) It was a constrained minimization technique – adapted to the continuum – that was originally used by Kupiainen and Gawedzki [5] in a renormalization group approach [6] to the lattice dipole gas [7]. While the Fourier transform of such a wavelet is real analytic, it is not entire.

For each order N we construct a gradient-orthonormal basis of momentum-entire wavelets in *odd* dimension n . The basic scale factor is $\sqrt[3]{2}$ instead of 2, and the coherence on each scale is based on the propagation $\exp(\vec{s} \cdot \vec{\nabla} \Delta)$ instead of on translation, which is given by $\exp(\vec{s} \cdot \vec{\nabla})$. Specifically, we derive

$$\mathcal{B} = \{\Psi_{\mu, \vec{s}, r} : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^n, \mu = 1, 2, \dots, 2^n - 1\}, \quad (1.4)$$

$$\Psi_{\mu, \vec{s}, r}(\vec{x}) = 2^{n\frac{r}{6} - \frac{r}{3}} \Psi_{\mu, \vec{s}}(2^{\frac{r}{3}} \vec{x}), \quad (1.5)$$

$$\Psi_{\mu, \vec{s}} = \exp(\vec{s} \cdot \vec{\nabla} \Delta) \Psi_{\mu}, \quad (1.6)$$

where each mother wavelet Ψ_{μ} is momentum-entire. By the nature of our construction, the 1st partial derivatives are at least square-integrable. In Section 3 we find that to guarantee the class C^1 property for Ψ_{μ} we need to choose the parameter N sufficiently large with respect to the dimension n .

Remark 2. In [8] we constructed *gradient-projective* bases of wavelets that were not only momentum-entire, but compactly supported. In any given Hilbert space $\mathcal{H}$, a *projective basis* is just a set $\{v_k\}$ of elements for which the expansion

$$v = \sum_k \langle v, v_k \rangle v_k \quad (1.7)$$

of an arbitrary $v \in \mathcal{H}$ is valid but not unique. In particular, the set is not orthogonal, or even linearly independent - at least for infinite series combinations.

Our construction is based on a simple machine with Daubechies bases as input. The key is a unitary transformation U from the Hilbert space $L^2(\mathbb{R}^n)$ to another Hilbert space which preserves the momentum-entire condition. Our focus is on the family of Hilbert spaces considered in [1]. Recall that $\mathcal{S}_{\infty}(\mathbb{R}^n)$ denotes the topological vector space of Schwartz functions whose moments all vanish. For each real number p we considered the Hilbert space

$$\mathcal{H}_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'_{\infty}(\mathbb{R}^n) : \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk_{\iota} \right) |\hat{f}(\vec{k})|^2 |\vec{k}|^{2p} < \infty \right\}, \quad (1.8)$$

where $\mathcal{S}'_{\infty}(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}_{\infty}(\mathbb{R}^n)$. This is an enlargement of the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions designed to allow momentum-space singularities of

the form $|\vec{k}|^{-M}$ for arbitrarily large M - and therefore designed to include $\mathcal{H}_p(\mathbb{R}^n)$ regardless of how positive p happens to be.

We introduce our unitary transformation U in the next section. While the definition is essentially independent of the dimension n , the target space is actually the $p = n$ Hilbert space. If we choose the N th-order Daubechies basis in $\mathcal{H}_n(\mathbb{R}^n)$, then the image basis is orthonormal in $\mathcal{H}_n(\mathbb{R}^n)$, so the application of U must be followed by the application of a unitary transformation from $\mathcal{H}_n(\mathbb{R}^n)$ to $\mathcal{H}_1(\mathbb{R}^n)$. The operator $|\vec{\nabla}|^{n-1}$ is clearly such a transformation. On the other hand, only positive *even* powers of $|\vec{\nabla}|$ preserve the momentum-entire property, so this scheme works only for odd n - hence the dimensional restriction on our construction.

In Section 4 we investigate the propagation $\exp(\vec{s} \cdot \vec{\nabla} \triangle)$, which naturally decomposes into two parts. One of them is a directional convolution with a scaling of the Airy function [9]. The other part is an n -variable convolution which nevertheless reduces to a one-variable function $F(x)$ such that

$$xF''(x) + \left(\frac{5}{2} - \frac{1}{2}n\right)F'(x) - F(x) = 0. \quad (1.9)$$

This ordinary differential equation is obviously a little different from the Airy equation.

Remark 3. We have not yet considered vector-valued momentum-entire wavelets where the gradient is replaced by the curl. Such a construction would be interesting because it involves a gauge-fixing problem. Some time ago, such bases of *exponentially localized* wavelets were constructed by Federbush and Williamson [10].

2 The Transformation

Our unitary transformation is induced by a nonlinear transformation in the momentum coordinates. We define the homeomorphism $\Gamma_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\Gamma_n(\vec{k}) = |\vec{k}|^2 \vec{k}. \quad (2.1)$$

This formula is independent of dimension, but the important property is that the analytic continuation of Γ_n is the transformation

$$\tilde{\Gamma}_n(\vec{z}) = \left(\sum_{\iota=1}^n z_{\iota}^2 \right) \vec{z}. \quad (2.2)$$

If h is an entire function on $\mathbb{C}^n$, then so is $h \circ \tilde{\Gamma}_n$.

The Jacobian matrix of Γ_n will be denoted by $G(\vec{k})$. Clearly this is the $n \times n$ matrix such that

$$G_{\iota\nu}(\vec{k}) = \delta_{\iota\nu}|\vec{k}|^2 + 2k_\iota k_\nu. \quad (2.3)$$

The Jacobian determinant

$$J(\vec{k}) = \det G(\vec{k}) \quad (2.4)$$

is easy to calculate because the ν th column of $G(\vec{k})$ is just

$$\vec{g}_\nu(\vec{k}) = |\vec{k}|^2 \vec{e}_\nu + 2k_\nu \vec{k}, \quad (2.5)$$

where $\vec{e}_\nu$ denotes the ν th standard coordinate vector in $\mathbb{R}^n$. Indeed, since $\vec{k} \wedge \vec{k} = 0$, we have the determinant expansion

$$\begin{aligned} J(\vec{k}) &= \vec{g}_1(\vec{k}) \wedge \vec{g}_2(\vec{k}) \wedge \dots \wedge \vec{g}_n(\vec{k}) \\ &= |\vec{k}|^{2n} \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n + 2|\vec{k}|^{2n-2} (k_1 \vec{k} \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n \\ &\quad + \vec{e}_1 \wedge k_2 \vec{k} \wedge \vec{e}_3 \wedge \dots \wedge \vec{e}_n + \dots + \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge k_n \vec{k}). \end{aligned} \quad (2.6)$$

Since

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n = \det I_n = 1, \quad (2.7)$$

the ι th term is clearly

$$\begin{aligned} &\vec{e}_1 \wedge \dots \wedge \vec{e}_{\iota-1} \wedge k_\iota \vec{k} \wedge \vec{e}_{\iota+1} \wedge \dots \wedge \vec{e}_n \\ &= \sum_{\lambda=1}^n k_\iota k_\lambda \vec{e}_1 \wedge \dots \wedge \vec{e}_{\iota-1} \wedge \vec{e}_\lambda \wedge \vec{e}_{\iota+1} \wedge \dots \wedge \vec{e}_n \\ &= k_\iota^2 \vec{e}_1 \wedge \dots \wedge \vec{e}_{\iota-1} \wedge \vec{e}_\iota \wedge \vec{e}_{\iota+1} \wedge \dots \wedge \vec{e}_n \\ &= k_\iota^2 \det I_n = k_\iota^2. \end{aligned} \quad (2.8)$$

Hence

$$\begin{aligned} J(\vec{k}) &= |\vec{k}|^{2n} + 2|\vec{k}|^{2n-2} \sum_{\iota=1}^n k_\iota^2 \\ &= 3|\vec{k}|^{2n} \end{aligned} \quad (2.9)$$

is the Jacobian determinant of Γ_n .

We are interested in the covariant action of this coordinate transformation on the Fourier transform of a function. Consider the mapping $g \rightarrow Ug$ defined by

$$\widehat{Ug} = \sqrt{3} \hat{g} \circ \Gamma_n. \quad (2.10)$$

We establish below that U is a unitary transformation from $\mathcal{H}_p(\mathbb{R}^n)$ to $\mathcal{H}_q(\mathbb{R}^n)$ with the relation

$$q = 3p + n. \quad (2.11)$$

In particular, U is unitary from $L^2(\mathbb{R}^n)$ to $\mathcal{H}_n(\mathbb{R}^n)$. Indeed,

$$\begin{aligned} \|Ug\|_{\mathcal{H}_n}^2 &= \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk_{\iota} \right) |\widehat{Ug}(\vec{k})|^2 |\vec{k}|^{2n} \\ &= 3 \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk_{\iota} \right) |\hat{g}(\Gamma_n(\vec{k}))|^2 |\vec{k}|^{2n} \\ &\quad \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk'_{\iota} \right) |\hat{g}(\vec{k}')|^2 \\ &= \|\hat{g}\|_{L^2}^2 = \|g\|_{L^2}^2, \end{aligned} \quad (2.12)$$

where

$$\vec{k}' = \Gamma_n(\vec{k}). \quad (2.13)$$

More generally,

$$\begin{aligned} \|Ug\|_{\mathcal{H}_q}^2 &= \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk_{\iota} \right) |\widehat{Ug}(\vec{k})|^2 |\vec{k}|^{2q} \\ &= 3 \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk_{\iota} \right) |\hat{g}(\Gamma_n(\vec{k}))|^2 |\vec{k}|^{2q} \\ &= 3 \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk_{\iota} \right) |\hat{g}(\Gamma_n(\vec{k}))|^2 |\vec{k}|^{2(q-n)} |\vec{k}|^{2n} \\ &= \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk'_{\iota} \right) |\hat{g}(\vec{k}')|^2 |\Gamma_n^{-1}(\vec{k}')|^{2(q-n)}. \end{aligned} \quad (2.14)$$

On the other hand, the obvious mapping property

$$|\Gamma_n(\vec{k})| = |\vec{k}|^3 \quad (2.15)$$

implies

$$|\Gamma_n^{-1}(\vec{k}')| = |\vec{k}'|^{\frac{1}{3}}, \quad (2.16)$$

so it follows that

$$\begin{aligned} \| U g \|_{\mathcal{H}_q}^2 &= \left(\prod_{\iota=1}^n \int_{-\infty}^{\infty} dk'_\iota \right) |\hat{g}(\vec{k}')|^2 |\vec{k}'|^{\frac{2}{3}(q-n)} \\ &= \| g \|_{\mathcal{H}_p}^2 \end{aligned} \quad (2.17)$$

with $p = \frac{1}{3}(q - n)$. the inversion of this relation yields (2.11). Note that the Hilbert space $\mathcal{H}_{-\frac{1}{2}n}$ is special because it is fixed under U .

In the context of this paper, the important property of U is that if g is a momentum-entire function, then so is Ug . In particular, the application of U to a Daubechies wavelet yields a momentum-entire function.

3 The Wavelet Basis for Order N

If we apply U to the wavelet basis $\mathcal{D}$ described in the Introduction, we obtain a wavelet basis that is orthonormal in $\mathcal{H}_n(\mathbb{R}^n)$. Moreover,

$$\begin{aligned} \widehat{U\psi_{\mu,\vec{s},r}}(\vec{k}) &= \sqrt{3} \widehat{\psi_{\mu,\vec{s},r}}(\Gamma_n(\vec{k})) \\ &= 2^{-r\frac{n}{2}} \sqrt{3} \widehat{\psi_{\mu,\vec{s}}}(2^{-r}\Gamma_n(\vec{k})) \\ &= 2^{-r\frac{n}{2}} \sqrt{3} \widehat{\psi_{\mu,\vec{s}}}(\Gamma_n(2^{-\frac{r}{3}}\vec{k})) \\ &= 2^{-r\frac{n}{2}} \widehat{U\psi_{\mu,\vec{s}}}(2^{-\frac{r}{3}}\vec{k}), \end{aligned} \quad (3.1)$$

and so

$$(U\psi_{\mu,\vec{s},r})(\vec{x}) = 2^{-r\frac{n}{6}} (U\psi_{\mu,\vec{s}})(2^{\frac{r}{3}}\vec{x}), \quad r \in \mathbb{Z}. \quad (3.2)$$

The scale coherence is based on the scale factor $\sqrt[3]{2}$ instead of 2.

What kind of coherence do we have on the unit scale? It we denote the translation $\psi_\mu(\cdot - \vec{s})$ by $\psi_{\mu,\vec{s}}$, we have

$$\widehat{\psi_{\mu,\vec{s}}}(\vec{k}') = \exp(-i\vec{s} \cdot \vec{k}') \hat{\psi}_\mu(\vec{k}'). \quad (3.3)$$

Thus

$$\begin{aligned} \widehat{U\psi_{\mu,\vec{s}}}(\vec{k}) &= \sqrt{3} \widehat{\psi_{\mu,\vec{s}}}(\Gamma_n(\vec{k})) \\ &= \sqrt{3} \exp(-i\vec{s} \cdot \Gamma_n(\vec{k})) \hat{\psi}_\mu(\Gamma_n(\vec{k})) \\ &= \exp(-i\vec{s} \cdot \vec{k} |\vec{k}|^2) \widehat{U\psi_\mu}(\vec{k}), \end{aligned} \quad (3.4)$$

and so

$$U\psi_{\mu,\vec{s}} = \exp(\vec{s} \cdot \vec{\nabla} \triangle) U\psi_{\mu}, \quad \vec{s} \in \mathbb{Z}^n. \quad (3.5)$$

Now consider the unitary transformation $|\vec{\nabla}|^{n-1}$ from $\mathcal{H}_n(\mathbb{R}^n)$ to $\mathcal{H}_1(\mathbb{R}^n)$. Clearly, if we apply it to the orthonormal basis UD of $\mathcal{H}_n(\mathbb{R}^n)$ we obtain an orthonormal basis of $\mathcal{H}_1(\mathbb{R}^n)$ - namely,

$$\mathcal{B} = \{\Psi_{\mu,\vec{s},r} : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^n, \quad \mu = 1, 2, \dots, 2^n - 1\}, \quad (3.6)$$

$$\Psi_{\mu,\vec{s},r} = |\vec{\nabla}|^{n-1} U\psi_{\mu,\vec{s},r}. \quad (3.7)$$

Now from (3.1) we see that

$$\begin{aligned} \widehat{\Psi_{\mu,\vec{s},r}}(\vec{k}) &= |\vec{k}|^{n-1} \widehat{U\psi_{\mu,\vec{s},r}}(\vec{k}) \\ &= 2^{-r\frac{n}{2}} |\vec{k}|^{n-1} \widehat{U\psi_{\mu,\vec{s}}}(2^{-\frac{r}{3}} \vec{k}) \\ &= 2^{-r\frac{n}{6} - \frac{r}{3}} |2^{-\frac{r}{3}} \vec{k}|^{n-1} \widehat{U\psi_{\mu,\vec{s}}}(2^{-\frac{r}{3}} \vec{k}) \\ &= 2^{-r\frac{n}{6} - \frac{r}{3}} \widehat{\Psi_{\mu,\vec{s}}}(2^{-\frac{r}{3}} \vec{k}), \end{aligned} \quad (3.8)$$

where

$$\Psi_{\mu,\vec{s}} = |\vec{\nabla}|^{n-1} U\psi_{\mu,\vec{s}}. \quad (3.9)$$

Thus

$$\Psi_{\mu,\vec{s},r}(\vec{x}) = 2^{r\frac{n}{6} - \frac{r}{3}} \Psi_{\mu,\vec{s}}(2^{\frac{r}{3}} \vec{x}), \quad r \in \mathbb{Z}. \quad (3.10)$$

On the unit scale we have the coherence

$$\Psi_{\mu,\vec{s}} = \exp(\vec{s} \cdot \vec{\nabla} \triangle) \Psi_{\mu}, \quad \vec{s} \in \mathbb{Z}^n, \quad (3.11)$$

$$\Psi_{\mu} = |\vec{\nabla}|^{n-1} U\psi_{\mu}, \quad (3.12)$$

as an immediate consequence of (3.5).

We have yet to remark on the order of smoothness. The mother wavelets ψ_{μ} of the Daubechies basis are class $C^{N-\epsilon}$ functions, but the spherically symmetric estimate in momentum space is

$$|\hat{\psi}_{\mu}(\vec{k}')| \leq c |\vec{k}'|^{-N-1}, \quad |\vec{k}'| \geq 1. \quad (3.13)$$

This estimate implies

$$|\widehat{U\psi_\mu}(\vec{k})| \leq c|\vec{k}|^{-3N-3}, \quad |\vec{k}| \geq 1, \quad (3.14)$$

and so

$$|\hat{\Psi}_\mu(\vec{k})| \leq c|\vec{k}|^{n-3N-4}, \quad |\vec{k}| \geq 1. \quad (3.15)$$

This insures that the smoothness of Ψ_μ has order $3N + 4 - 2n - \epsilon$ for arbitrarily small $\epsilon > 0$. This is the reason why we must choose

$$N > \frac{2}{3}n - 1 \quad (3.16)$$

if we expect to obtain C^1 wavelets.

Finally, it is clear that n must be odd for Ψ_μ to be a momentum-entire function. Indeed,

$$\begin{aligned} \hat{\Psi}_\mu(\vec{k}) &= |\vec{k}|^{n-1} \widehat{U\psi_\mu}(\vec{k}) \\ &= \left(\sum_{\iota=1}^n k_\iota^2 \right)^{\frac{n}{2} - \frac{1}{2}} \widehat{U\psi_\mu}(\vec{k}), \end{aligned} \quad (3.17)$$

where we have already observed that $U\psi_\mu$ is momentum-entire.

4 The Propagator

It is worthwhile to examine the new type of coherence that has replaced the translational coherence. We are interested in

$$\exp(\vec{s} \cdot \vec{\nabla} \Delta) = \exp(s \partial_{\vec{v}} \Delta), \quad (4.1)$$

where $\vec{s} = s\vec{v}$ now ranges over $\mathbb{R}^n$ instead of just $\mathbb{Z}^n$ and $\partial_{\vec{v}}$ is $\vec{v}$ -directional differentiation. We fix the unit vector $\vec{v}$ and consider the propagation

$$u(\vec{x}, s) = [\exp(s \partial_{\vec{v}} \Delta) f](\vec{x}) \quad (4.2)$$

for an arbitrary initial configuration $f(\vec{x})$.

Let $\tilde{f}$ denote the $\vec{v}$ -directional Fourier transform of f . To fix our notation we write

$$\tilde{f}(k, \vec{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp(-ikx) f(x\vec{v} + \vec{y}) \quad (4.3)$$

for $\vec{y} \in \vec{v}^\perp$. It follows from (4.2) that

$$u(\vec{x}, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \exp(ik\vec{x} \cdot \vec{v}) [\exp(-isk(k^2 + \Delta')) \tilde{f}(k, \cdot)](\vec{x}'), \quad (4.4)$$

$$\vec{x}' = \vec{x} - (\vec{x} \cdot \vec{v})\vec{v}, \quad (4.5)$$

$$\Delta' = \Delta - \partial_v^2. \quad (4.6)$$

We can write

$$u(\vec{x}, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx h(\vec{x} \cdot \vec{v} - x, s) (T_s f)(x\vec{v} + \vec{x}'), \quad (4.7)$$

$$h(w, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \exp(ikw) \exp(-isk^3), \quad (4.8)$$

where T_s is the operator given by

$$(T_s f)(\vec{q}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \exp(ik\vec{v} \cdot \vec{q}) [\exp(-isk\Delta') \tilde{f}(k, \cdot)](\vec{q}'). \quad (4.9)$$

Note that h is essentially an Airy function:

$$\begin{aligned} h(w, s) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} dk \cos(kw - sk^3) \\ &= \sqrt{\frac{2}{\pi}} \sqrt[3]{\frac{3}{s}} \int_0^{\infty} dk \cos\left(\frac{1}{3}k^3 - \sqrt[3]{\frac{3}{s}}kw\right) \\ &= \sqrt{2\pi} \sqrt[3]{\frac{3}{s}} \text{Ai}\left(-\sqrt[3]{\frac{3}{s}}w\right). \end{aligned} \quad (4.10)$$

To describe T_s we insert the closed calculation

$$[\exp(-isk\Delta') \tilde{f}(k, \cdot)](\vec{q}') = (-2\pi isk)^{-\frac{n}{2} + \frac{1}{2}} \int_{\vec{v}^\perp} d\vec{y} \exp\left(-i \frac{|\vec{q}' - \vec{y}|^2}{4sk}\right) \tilde{f}(k, \vec{y}), \quad (4.11)$$

where $-\frac{1}{2}n + \frac{1}{2}$ is an integer for odd dimension n . If we define

$$\eta_{\vec{v}}(\vec{x}, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \exp(ik\vec{v} \cdot \vec{x}) k^{-\frac{n}{2} + \frac{1}{2}} \exp\left(-i \frac{|\vec{x}'|^2}{4sk}\right), \quad (4.12)$$

then

$$T_s f = (2\pi)^{-\frac{n}{2}} (-is)^{-\frac{n}{2} + \frac{1}{2}} \eta_{\vec{v}}(\cdot, s) * f \quad (4.13)$$

because

$$(\eta_{\vec{v}}(\cdot, s) * f)(\vec{q}) = \int_{-\infty}^{\infty} dz \int_{\vec{v}^\perp} d\vec{y} \eta_{\vec{v}}(\vec{q} - z\vec{v} - \vec{y}, s) f(z\vec{v} + \vec{y}). \quad (4.14)$$

Thus $\eta_{\vec{v}}$ is the function of interest in our study of T_s .

An understanding of $\eta_{\vec{v}}$ is actually a problem in ordinary differential equations. Indeed, if we introduce the change

$$p = \frac{|\vec{x}'|^2}{4s} k \quad (4.15)$$

of integration variable in (4.12), we see that

$$\eta_{\vec{v}}(\vec{x}, s) = \left(\frac{|\vec{x}'|^2}{4s} \right)^{-\frac{n}{2} + \frac{3}{2}} F \left(\frac{|\vec{x}'|^2}{4s} \vec{v} \cdot \vec{x} \right), \quad (4.16)$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ipx) p^{-\frac{n}{2} + \frac{1}{2}} \exp \left(-i \frac{1}{p} \right). \quad (4.17)$$

Since this integral diverges at $p = 0$, it is not immediately clear that $F(x)$ is defined – even as a tempered distribution. We give a rigorous definition in Section 5. The point here is that

$$\begin{aligned} \frac{d^2}{dx^2}(xF(x)) &= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ipx) p^2 \frac{d}{dp} \left[p^{-\frac{n}{2} + \frac{1}{2}} \exp \left(-i \frac{1}{p} \right) \right] \\ &= \left(-\frac{1}{2}n + \frac{1}{2} \right) \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ipx) p^{-\frac{n}{2} + \frac{3}{2}} \exp \left(-i \frac{1}{p} \right) \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ipx) p^{-\frac{n}{2} + \frac{1}{2}} \exp \left(-i \frac{1}{p} \right). \end{aligned} \quad (4.18)$$

Combining the factor ip with $\exp(ipx)$ in the first contribution, we obtain

$$\frac{d^2}{dx^2}(xF(x)) = \left(\frac{1}{2}n - \frac{1}{2} \right) F'(x) + F(x). \quad (4.19)$$

This may be written as (1.9).

5 Appendix

Let K_0 be the tempered distribution over $\mathbb{R}$ defined for each Schwartz function f by

$$K_0(f) = \int_{-\infty}^{\infty} dp \hat{f}(p) \exp \left(-\frac{i}{p} \right). \quad (5.1)$$

The tempered distributions

$$K_m(x) = x^m K_0(x) \quad (5.2)$$

are given by

$$K_m(f) = (-i)^m \int_{-\infty}^{\infty} dp \hat{f}^{(m)}(p) \exp \left(-\frac{i}{p} \right). \quad (5.3)$$

How do we make sense of the tempered distribution H_m formally defined by the integral

$$H_m(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ipx) p^{-m} \exp\left(-\frac{i}{p}\right), \quad (5.4)$$

which is obviously divergent at $p = 0$?

Even with a test function f , the divergence is still in the integral formula

$$H_m(f) = \int_{-\infty}^{\infty} dp \hat{f}(p) p^{-m} \exp\left(-\frac{i}{p}\right). \quad (5.5)$$

However, we can define H_m inductively by using the distributions K_m . We first observe that

$$p^{-2} \exp\left(-\frac{i}{p}\right) = -i \frac{d}{dp} \exp\left(-\frac{i}{p}\right), \quad (5.6)$$

so we may regard the *lhs* as a distributional derivative. Accordingly, we define H_2 by the formula

$$\begin{aligned} H_2(f) &= i \int_{-\infty}^{\infty} dp \hat{f}'(p) \exp\left(-\frac{i}{p}\right) \\ &= -K_1(f), \end{aligned} \quad (5.7)$$

and simultaneously define H_1 by the formula

$$\begin{aligned} H_1(f) &= i \int_{-\infty}^{\infty} dp \frac{d}{dp} (p \hat{f}(p)) \exp\left(-\frac{i}{p}\right) \\ &= \int_{-\infty}^{\infty} dp \hat{f}'(p) \exp\left(-\frac{i}{p}\right) \\ &= iK_1(f') = -iK_1'(f). \end{aligned} \quad (5.8)$$

Thus $H_1 = -iK_1'$ and $H_2 = -K_1$.

To address the induction step we observe that

$$\frac{d}{dp} \left(p^{-m+1} \exp\left(-\frac{i}{p}\right) \right) = (-m+1) p^{-m} \exp\left(-\frac{i}{p}\right) + i p^{-m-1} \exp\left(-\frac{i}{p}\right) \quad (5.9)$$

and therefore

$$p^{-m-1} \exp\left(-\frac{i}{p}\right) = i(-m+1) p^{-m} \exp\left(-\frac{i}{p}\right) - i \frac{d}{dp} \left(p^{-m+1} \exp\left(-\frac{i}{p}\right) \right). \quad (5.10)$$

We can define H_{m+1} by the formula

$$\begin{aligned} H_{m+1}(f) &= i(-m+1) \int_{-\infty}^{\infty} dp \hat{f}(p) p^{-m} \exp\left(-\frac{i}{p}\right) + i \int_{-\infty}^{\infty} dp \hat{f}'(p) p^{-m+1} \exp\left(-\frac{i}{p}\right) \\ &= i(-m+1) H_m(f) - \int_{-\infty}^{\infty} dp \widehat{xf(x)}(p) p^{-m+1} \exp\left(-\frac{i}{p}\right) \\ &= i(-m+1) H_m(f) - H_{m-1}(xf(x)). \end{aligned} \quad (5.11)$$

The formal relation is

$$H_{m+1}(x) = i(-m+1)H_m(x) - xH_{m-1}(x). \quad (5.12)$$

In the context of (4.17) we have

$$F = H_{\frac{n}{2}-\frac{1}{2}} \quad (5.13)$$

with n odd.

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