

Generalizing the Asymmetric Run-length-limited Systems

**Original research
paper**

Abstract

For $i = 1, 2$, if X_i is a synchronized system generated by $V_i = \{v^i \alpha_i : \alpha_i v^i \alpha_i \in \mathcal{B}(X_i), \alpha_i \not\subseteq v^i\}$ where α_i is a synchronizing word for X_i , then a natural generalization of an asymmetric-RLL(d_1, k_1, d_0, k_0) system is a coded system Z generated by $\{v^1 \alpha_1 v^2 \alpha_2 : v^i \alpha_i \in V_i, i = 1, 2\}$. We investigate the dynamical properties of Z . We show that Z is sofic or has specification with variable gap length (SVGL) if and only if X_1 and X_2 are sofic or have SVGL respectively. Also, if Z is SFT or AFT, then X and Y are SFT or AFT respectively and sufficient conditions for the converse will be given.

Keywords: *shift of finite type; sofic; almost-finite-type; synchronized; coded system.*

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1 Introduction

Recall that the Run-length-limited (RLL) [Lind and Marcus (1995)] and the Maximum Transition Run (MTR) constrained systems [Moon and Brickner (1996)] are used to improve timing and detection performance in storage channels. In particular, the MTR code, introduced by Moon and Brickner [Moon and Brickner (1996)] are to provide coding gain for extended partial response channels. The RLL code denoted by $X(d, k)$ limits the run of 0 to be at least d and at most k whereas the MTR(j, k) code limits the run of 0 to be at most k and the run of 1 at most j . When there is no restriction on the runs of 0, we say that $k = \infty$ and it is common then to denote such a constraint by MTR(j). For generalizing MTR codes, consider the asymmetric-RLL(d_1, k_1, d_0, k_0) constraint which is the set of binary sequences whose runs of 1's have length between d_1 and k_1 and the runs of 0's between d_0 and k_0 . In the case that $d_1 = d_0 = 1, k_1 = j$ and $k_0 = k$, this constraint coincides with MTR(j, k).

One may define an asymmetric-RLL(d_1, k_1, d_0, k_0) as follows. Let $S = \{d_0 - 1, d_0, \dots, k_0 - 1\} \subseteq \mathbb{N}_0$ and let $X = X(d_0 - 1, k_0 - 1)$ be the RLL system associated to S . Then X is the space generated by $V = \{0^s 1 : s \in S\}$, that is, the space constructed by concatenating the words in V . Now consider $S' = \{d_1 - 1, d_1, \dots, k_1 - 1\} \subseteq \mathbb{N}_0$ and the space $Y = X(d_1 - 1, k_1 - 1)$ generated

by $W = \{1^{s'}0 : s' \in S'\}$. The word $\alpha = 1$ (resp. $\beta = 0$) is a synchronizing word in X (resp. Y) and our asymmetric-RLL(d_1, k_1, d_0, k_0) is the space generated by $\{vw : v \in V, w \in W\}$. On the other hand, any synchronized system X with a synchronizing word α is generated by $\{v\alpha : \alpha v\alpha$ is a word in X and $\alpha \not\subseteq v\}$. If Y is another synchronized system with a synchronizing word β and a set of generators $W_\beta = \{w\beta : \beta w\beta \text{ a word in } Y \text{ and } \beta \not\subseteq w\}$, then a natural generalization for an asymmetric-RLL(d_1, k_1, d_0, k_0) constraint is a coded system Z denoted by $X \& Y$ and generated by $\{v\alpha w\beta : v\alpha \in V_\alpha, w\beta \in W_\beta\}$. Dynamical properties of this generalized system depend on α and β ; however, here, we are interested in those dynamical properties which are independent of the synchronized words.

In Theorem 4.1 (resp. Theorem 4.3), it is shown that X and Y are sofic (resp. SVGL) if and only if $Z = X \& Y$ is sofic (resp. SVGL). Also, If $Z = X \& Y$ is SFT, near Markov or AFT, then both X and Y are SFT, near Markov or AFT respectively (Theorem 4.4). But the converse does not hold necessarily. Then we give sufficient conditions such that the converse of Theorem 4.4 is true as well (Theorem 4.8).

2 Background and Notations

In this section we will bring the basic definitions in symbolic dynamics on finite alphabet $\mathcal{A}$. For justification of our claims see [Lind and Marcus (1995)].

Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. Then $\mathcal{A}^{\mathbb{Z}}$ is a Cantor set and $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ is called the *shift map*. A *block* (or *word*) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include ε , the sequence of no symbols which is called the *empty word*. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a word of length $j - i$ by $x_{[i,j]} = x_i x_{i+1} \dots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u , and put $u^0 = \varepsilon$.

Let $\mathcal{F}$ be a collection of some words over $\mathcal{A}$. Let $X_{\mathcal{F}}$ be a non-empty closed subset of $\mathcal{A}^{\mathbb{Z}}$ and so that $X_{\mathcal{F}}$ does not contain any word in $\mathcal{F}$. This set $\mathcal{F}$ is called the set of *forbidden blocks* over $\mathcal{A}$. Then any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a $X_{\mathcal{F}}$ for some collection of forbidden blocks. If $\mathcal{F}$ is finite, then $X_{\mathcal{F}}$ is called *shift of finite type* (SFT).

Let $B_n(X)$ denote the set of all admissible n words. The *Language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} B_n(X)$. A shift space X is *irreducible* if for every ordered pair of words $u, v \in \mathcal{B}(X)$ there is a word $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever uv and vw are in $\mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$. An irreducible shift space X is a *synchronized system* if it has a synchronizing word [Blanchard and Hansel (1986)].

Fix integers m and n with $m \leq n$ and let $\mathcal{A}$ and $\mathcal{D}$ be alphabets and X a shift space over $\mathcal{A}$. Define the $(m+n+1)$ -block map $\Phi : \mathcal{B}_{m+n+1}(X) \rightarrow \mathcal{D}$ by

$$y_i = \Phi(x_{i-m}x_{i-m+1}\dots x_{i+n}) = \Phi(x_{[i-m, i+n]}) \quad (2.1)$$

where $y_i \in \mathcal{D}$. This Φ induces a map $\Phi_{\infty} = \Phi_{\infty}^{[-m, n]} : X \rightarrow \mathcal{D}^{\mathbb{Z}}$ called the *sliding block code* with *memory* m and *anticipation* n defined by $y = \Phi_{\infty}(x)$ with y_i given by (1.1). An onto sliding block code $\Phi_{\infty} : X \rightarrow Y$ is called a *factor code*. In this case, we say that Y is a factor of X . The map Φ_{∞} is a *conjugacy*, if it is invertible.

An *edge shift*, denoted by X_G , is a shift space which consist of all bi-infinite walks in a directed graph G . Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \rightarrow \mathcal{A}$. Then a subshift $X_{\mathcal{G}}$ is induced by $\mathcal{L}_{\infty}$ is the set of sequences obtained by reading the labels of walks on G ,

$$X_{\mathcal{G}} = \overline{\{\mathcal{L}_{\infty}(\xi) : \xi \in X_G\}} = \overline{\mathcal{L}_{\infty}(X_G)}. \quad (2.2)$$

We say $\mathcal{G}$ is a *presentation* of $X_{\mathcal{G}}$. If G is finite, then $X_{\mathcal{G}}$ is called *sofic* and $X_{\mathcal{G}} = \mathcal{L}_{\infty}(X_G)$.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels. A *minimal right-resolving presentation* of a sofic shift X is a right-resolving

presentation of X having the fewest vertices among all right-resolving presentations of X . A minimal right-resolving presentations of an irreducible sofic shift is unique up to conjugacy and called the *Fischer cover* of X .

Let X be a shift space and $w \in \mathcal{B}(X)$. The *follower set* $F(w)$ of w is defined by $F(w) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}$. A shift space X is sofic if and only if it has a finite number of follower sets (Lind and Marcus, 1995, Theorem 3.2.10).

A labeled graph is *right-closing* with delay D if whenever two paths of length $D + 1$ start at the same vertex and have the same label, then they must have the same initial edge. Similarly, left-closing will be defined. A labeled graph is bi-closing, if it is simultaneously right-closing and left-closing.

An irreducible sofic shift is called *almost-finite-type* (AFT) if it has a bi-closing presentation [Lind and Marcus (1995)]. Nasu [Nasu (1985)] showed that an irreducible sofic shift is AFT if and only if its Fischer cover is left-closing.

Now we review the concept of Fischer cover for a not necessarily sofic system [Fiebig and Fiebig (1992)]. Let $x \in \mathcal{B}(X)$. Then $x_+ = (x_i)_{i \in \mathbb{Z}^+}$ (resp. $x_- = (x_i)_{i < 0}$) is called *right (resp. left) infinite X-ray*. For a left infinite X -ray, say x_- , its follower set is $\omega_+(x_-) = \{x_+ \in X^+ : x_-x_+ \text{ is a point in } X\}$. Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph X^+ . There is an edge from I_1 to I_2 labeled a if and only if there is an X -ray x_- such that x_-a is an X -ray and $I_1 = \omega_+(x_-)$, $I_2 = \omega_+(x_-a)$. This labeled graph is called the *Krieger graph* for X . If X is a synchronized system with synchronizing word α , the irreducible component of the Krieger graph containing the vertex $\omega_+(\alpha)$ is called the *right Fischer cover* of X .

The *entropy* of a shift space X is defined by $h(X) = \lim_{n \rightarrow \infty} (1/n) \log |\mathcal{B}_n(X)|$.

3 Coded Systems

A shift space that can be presented by an irreducible countable labeled graph is called a *coded system*. Equivalently, a coded system X is the closure of the set of sequences obtained by freely concatenating the words in a list of words, called the set of generators, over a finite alphabet [Lind and Marcus (1995)]. A coded system is irreducible and has a dense set of periodic points [Fiebig and Fiebig (1992)]. Coded systems were introduced by Blanchard and Hansel [Blanchard and Hansel (1986)] who also showed that the class of the coded systems is the smallest class of subshifts which contains the synchronized systems and is closed under factors (Blanchard and Hansel, 1986, proposition 4.1). A brief introduction to coded systems can be found in (Lind and Marcus, 1995, section 13.5).

In a synchronized system X , for any synchronized word $\alpha = \alpha_1 \cdots \alpha_p$, X is generated by

$$V = V_\alpha = \{v\alpha \in \mathcal{B}(X) : \alpha v\alpha \in \mathcal{B}(X), \alpha \not\subseteq v\}. \quad (3.1)$$

Let $\mathcal{G} = (G, \mathcal{L})$ be the right Fischer cover of X . We give another right resolving cover for X , denoted by $\mathcal{G}^\alpha$ which is not necessarily follower-separated. Suppose $\mathbf{v}(\alpha)$ is the unique vertex in $\mathcal{V}(G)$ where any path labeled α terminates. Any other vertex v is shown by $\mathbf{v}(\alpha u)$ where u is some path starting at $\mathbf{v}(\alpha)$ and terminating at v . To be specific, we adopt the convention that if there is u' such that $\mathbf{v}(\alpha u') = \mathbf{v}(\alpha u)$, then $|u| \leq |u'|$. Fix $I = \mathbf{v}(\alpha u_1) \in \mathcal{V}(G)$ and assume that $\mathcal{E}_I$, the set of inner edges of I , has more than one element. This implies that by tracking the paths starting at $\mathbf{v}(\alpha)$, there is another path u_2 , $|u_2| \geq |u_1|$ such that $I = \mathbf{v}(\alpha u_1) = \mathbf{v}(\alpha u_2)$. Suppose $u_i = c_{i_1} \cdots c_{i_{k_i}} \alpha_1 \cdots \alpha_{l_i} \in \mathcal{B}(X)$, $i = 1$ or 2 . We intend to do some in-split (Lind and Marcus, 1995, §2.4) for v . If one of the following holds, then we do not do the splitting.

1. both $\alpha_1 \cdots \alpha_{l_1}$ and $\alpha_1 \cdots \alpha_{l_2}$ are empty words;
2. $\alpha_1 \cdots \alpha_{l_1}$ (resp. $\alpha_1 \cdots \alpha_{l_2}$) is not empty word and $c_{i_1} \cdots c_{i_{k_1}} \alpha_1 \cdots \alpha_{l_1} \cdots \alpha_p = c_{i_1} \cdots c_{i_{k_1}} \alpha$ (resp. $c_{i_1} \cdots c_{i_{k_2}} \alpha$) is not admissible;
3. cases (1) and (2) do not hold and $l_1 = l_2$.

(1) and (2) say that if v is not a vertex on a path π_α labeled α , then in-splitting will not be done.

Now we set up to see which vertices on π_α requires in-splitting and how this happens. Note that case (3) above excludes some cases. Set $\mathcal{G}_1 = \mathcal{G}$ and let $\mathcal{V}_{G_1}(\alpha_1) = \{I \in \mathcal{V}(G_1) : I = t(e_{\alpha_1}), e_{\alpha_1}$ is the first edge labeled α_1 on a path labeled $\alpha\}$. For $I \in \mathcal{V}_{G_1}(\alpha_1)$, partition $\mathcal{E}_I$ to $P_I^1(\alpha_1) = \{e_1 : \mathcal{L}(e_1) = \alpha_1\}$ and $P_I^2(\alpha_1)$ for the remaining edges. Do an in-split for I with respect to this partition and call the new cover $\mathcal{G}_2 = (G_2, \mathcal{L}_2)$.

Let $\mathcal{V}_{G_2}(\alpha_1\alpha_2) = \{I \in \mathcal{V}(G_2) : I = t(e_{\alpha_1}e_{\alpha_2}), e_{\alpha_1}e_{\alpha_2}$ be the first 2 edges with label $\alpha_1\alpha_2$ of a path labeled $\alpha\}$. Partition $\mathcal{E}_I$, $I \in \mathcal{V}_{G_2}(\alpha_1\alpha_2)$ to $P_I^1(\alpha_1\alpha_2) = \{e_2 : t(e_1e_2) = I \text{ for some } e_1, \mathcal{L}_2(e_1e_2) = \alpha_1\alpha_2\}$ and $P_I^2(\alpha_1\alpha_2) = \mathcal{E}_I \setminus P_I^1(\alpha_1\alpha_2)$. By the same procedure, $\mathcal{G}_{k+1}$, $P_I^1(\alpha_1\alpha_2 \cdots \alpha_k)$, and $P_I^2(\alpha_1\alpha_2 \cdots \alpha_k)$, $1 \leq k < p$ will be constructed. Set $\mathcal{G}^\alpha = \mathcal{G}_p = (G_p, \mathcal{L}_{p-1})$. Suppose in-splitting occurs at $I \in \mathcal{V}(G_k)$ and let $\mathcal{E}^I$ be the set of outer edges of I . Then corresponding to I , there are two vertices I_1 and I_2 in $\mathcal{V}(G_{k+1})$ with $\mathcal{E}^I = \mathcal{E}^{I_1} = \mathcal{E}^{I_2}$. For $e \in \mathcal{E}$ let $e(i)$ be the corresponding edge in $\mathcal{E}^{I_i}$ with the same label as e . We collect some properties of $\mathcal{G}^\alpha$ in the following theorem.

Theorem 3.1. *Let X be a synchronized system with a synchronized word $\alpha = \alpha_1 \cdots \alpha_p$, a generator V as (3.1) and the right Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then*

1. $\mathcal{G}^\alpha$ and $\mathcal{G}$ are conjugate.
2. Let $e_{\alpha_1} \cdots e_{\alpha_k}$, $1 \leq k < p$ be a subpath of a path labeled α , $\mathcal{L}_p(e_{\alpha_1} \cdots e_{\alpha_k}) = \alpha_1 \cdots \alpha_k$ and let $I = t(e_{\alpha_1} \cdots e_{\alpha_k})$. Then all the inner edges of I have the same label α_k .
3. Let $u = u_1 \cdots u_k \in \mathcal{B}(X)$ and suppose $\pi = e_{u_1} \cdots e_{u_k} e_{\alpha_i} \cdots e_{\alpha_p}$ is a path so that $\mathcal{L}_p(\pi) = u_1 \cdots u_k \alpha_i \cdots \alpha_p$ and $e_{\alpha_i} \cdots e_{\alpha_p}$ is a subpath of a path labeled α , then either $u \alpha_i \cdots \alpha_p \subseteq \alpha$ or $u \alpha_i \cdots \alpha_p = v \alpha_1 \cdots \alpha_p$ for some $v \in \mathcal{B}(X)$.
4. If X is sofic, then $\mathcal{G}^\alpha$ is a finite labeled graph. Also, if $\mathcal{G}$ is left-closing with delay D , then $\mathcal{G}^\alpha$ will be left-closing with delay $D + p - 1$.

Proof. (2), (3) and the first part of (4) follow from the definition of $\mathcal{G}^\alpha$. The second part of (4) is satisfied by (Lind and Marcus, 1995, Proposition 5.1.8). (1) can be deduced from (Lind and Marcus, 1995, Theorem 2.4.10); nevertheless, we give a sketch of proof. Set $X_G = X_{G_1}$. First we show that X_{G_1} and X_{G_2} are conjugate; then an induction argument gives the result. Define a 1-block map $\Psi : \mathcal{B}_1(X_{G_2}) \rightarrow \mathcal{B}_1(X_{G_1})$ by $\Psi(e(j)) = e$. Note that the image under Ψ of any path in G_2 is a path in G_1 . Next we define a 2-block map $\Phi : \mathcal{B}_2(X_{G_1}) \rightarrow \mathcal{B}_1(X_{G_2})$. If $fe \in \mathcal{B}_2(X_{G_1})$, then f is in $P_I^1(\alpha_1)$, $i \in \{1, 2\}$ and we define $\Phi(fe) = e(i)$. Here also, the image under Φ of any path in G_1 is a path in G_2 . Let $\psi = \Psi_\infty$ and $\phi = \Phi_{\infty}^{[0,1]}$. Then $\psi \circ \phi$ and $\phi \circ \psi$ are identity maps and the conjugacy will be achieved. $\square$

4 Intertwined Synchronized Systems

Definition 4.1. For $1 \leq i \leq \ell$, let $X_i = X_{V_i}$ be a coded system with a synchronizing word α_i and generated by

$$V_i = V_{\alpha_i} = \{v^i \alpha_i : \alpha_i v^i \alpha_i \in \mathcal{B}(X_i), \alpha_i \not\subseteq v^i\}.$$

The coded system $Z = Z(V_1, \dots, V_\ell)$ generated by

$$\{v^1 \alpha_1 v^2 \alpha_2 \cdots v^\ell \alpha_\ell : v^i \alpha_i \in V_i\}$$

is called the *intertwined system* of $X_1, \dots, X_\ell$ and is denoted by

$$Z = X_1 \& X_2 \& \cdots \& X_\ell.$$

Let $\mathcal{G}_i = (G_i, \mathcal{L}_i)$ be the right Fischer cover for X_i and $\mathcal{G}_i^{\alpha_i}$ the cover provided by Theorem 3.1 for X_i , $1 \leq i \leq \ell$. Then all α_i -paths, that is paths labeled α_i in $G_i^{\alpha_i}$ terminate at $v(\alpha_i)$. Now $\mathcal{G}_Z =$

$\mathcal{G}_Z(\mathcal{G}_{X_1}, \dots, \mathcal{G}_{X_\ell})$ which is a cover obtained by choosing α_i -paths being terminated at $\mathbf{v}(\alpha_{(i+1) \bmod \ell})$ instead of $\mathbf{v}(\alpha_i)$ is a cover for Z . We call $\mathcal{G}_Z$ the *intertwined cover* for Z . So to construct $\mathcal{G}_Z$, we cut off all the α_i -paths from $\mathbf{v}(\alpha_i)$ in $G_i^{\alpha_i}$ and we glue them to $\mathbf{v}(\alpha_j)$ in $G_j^{\alpha_j}$ where $j = i + 1 \bmod \ell$.

Since the problems arising from intertwining of finite systems are basically the same as intertwining of two systems, from now on we concentrate on intertwining of two systems $X = X_V$ and $Y = Y_W$ generated by

$$V = V_\alpha = \{v\alpha : \alpha v\alpha \in \mathcal{B}(X), \alpha \not\subseteq v\} \quad \text{and} \quad W = W_\beta = \{w\beta : \beta w\beta \in \mathcal{B}(Y), \beta \not\subseteq w\} \quad (4.1)$$

respectively. So α -paths (resp. β -paths) terminate at $\mathbf{v}(\beta)$ (resp. $\mathbf{v}(\alpha)$).

Definition 4.2. Let G_{Z_X} be the subgraph of G_Z corresponding to G_X , that is, consisting of all the paths in G_Z labeled $v\alpha$, $v \in V$ and starting from $\mathbf{v}(\alpha)$ and terminating at $\mathbf{v}(\beta)$.

Note that $\mathcal{G}_{Z_X}$ is not irreducible and $\mathcal{G}_{Z_X}$ and $\mathcal{G}_{Z_Y}$ have only vertices $\mathbf{v}(\alpha)$ and $\mathbf{v}(\beta)$ in common.

Now by giving an example, we show that $X_V \& X_W$ depends on V and W and so on α and β . Let $X = X(S)$ be a S -gap shift [Lind and Marcus (1995)] for $S = \{1, 2\}$ and Y a β -shift [Thomsen (2005)] for $1_\beta = 1101$; both systems are SFT. It is obvious that 1 is a synchronizing word for X and 00, 11 are synchronizing words for Y . Let $V = \{01, 001\}$, $W_1 = \{u00 : 00u00 \in \mathcal{B}(Y), 00 \not\subseteq u\}$ and $W_2 = \{u11 : 11u11 \in \mathcal{B}(Y), 11 \not\subseteq u\}$. Then $X = X_V$ and $Y = X_{W_1} = X_{W_2}$. By a straightforward routine, that is by finding the largest positive eigenvalues of adjacency matrices, one has $h(X \& X_{W_1}) = \log 1.63$ while $h(X \& X_{W_2}) = \log 1.7$. So $X \& X_{W_1}$ and $X \& X_{W_2}$ are not conjugate. In particular, the intertwined system of conjugate systems are not necessarily conjugate.

Theorem 4.1. Let X and Y be two synchronized systems generated by $V = V_\alpha$ and $W = W_\beta$ as in 4.1. Then X and Y are sofic if and only if $Z = X \& Y = X_V \& Y_W$ is sofic.

Proof. Let $X = X_V$ and $Y = X_W$ be two sofic systems and $\mathcal{G}_X$ (resp. $\mathcal{G}_Y$) be the Fischer covers of X (resp. Y). Then $\mathcal{G}_X^\alpha$ and $\mathcal{G}_Y^\beta$ and their intertwined cover $\mathcal{G}_Z$ have finite vertices. But any symbolic system with a finite labeled graph is sofic and we are done.

For the converse if Z is sofic but X is not sofic, then G_{Z_X} will be a graph with infinite vertices which is absurd. $\square$

Next example will illustrate the intertwining of two sofic systems X and Y .

Example 4.2. Consider Figure 1 and two sofic shifts X and Y with $\alpha = \alpha_1\alpha_2 = 00$ and $\beta = \beta_1\beta_2\beta_3 = 000$ as their synchronizing words respectively. The Fischer covers of X and Y are presented in that figure.

First we will construct $\mathcal{G}_X^\alpha$. We have $\mathcal{V}_{G_X}(\alpha_1) = \{F_X(\alpha_1)\}$ and only $I = F_X(\alpha_1)$ needs in-splitting. We do this and we obtain $\mathcal{G}_X^\alpha = \mathcal{G}_{X_2}$.

For $\mathcal{G}_Y^\beta$, the first in-splitting occurs in $I = F_Y(\beta_{11})$. Do this in-splitting and call the new cover $\mathcal{G}_{Y_2}$. We have $\mathcal{V}_{G_{Y_2}}(\beta_1\beta_2) = \{F_Y(\beta_1)\}$ and $F_Y(\beta_1)$ needs also in-splitting. Doing this $\mathcal{G}_Y^\beta (= \mathcal{G}_{Y_3})$ will be constructed.

Definition 4.3 (Jung (2011)). A shift space X has *specification with variable gap length* (SVGL) if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uvw \in \mathcal{B}(X)$ and $|w| \leq N$.

Note that a SVGL was called almost specified in [Jung (2011)].

Theorem 4.3. Suppose X and Y are two synchronized systems generated by $V = V_\alpha$ and $W = W_\beta$ as in 4.1. Then $Z = X \& Y = X_V \& Y_W$ has SVGL if and only if $X = X_V$ and $Y = Y_W$ have SVGL.

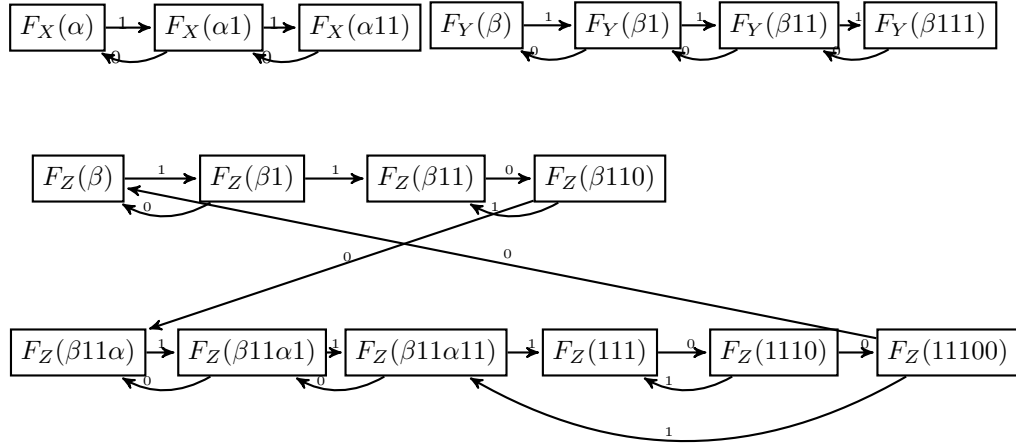


Figure 1: From above, the Fischer covers of X , Y and $Z = X \& Y$.

Proof. If $V = W$, then $Z = X$ and we are done. So suppose $W \neq V$ and pick $w_0\beta \in W \setminus V$.

First suppose Z has SVGL with the transition length M and suppose that one of X or Y , say X , does not have SVGL. Then for all n , there are $u_n, v_n \in \mathcal{B}(X)$ such that if $w \in \mathcal{B}(X)$ and $u_n w v_n \in \mathcal{B}(X)$, then $|w| \geq n$. Without loss of generality, assume that $\alpha u_n, v_n \alpha \in \mathcal{B}(X)$ for all n . Now let $z_n = \alpha w_0 \beta u_n$ and $z'_n = v_n \alpha w_0 \beta$ be the words in $\mathcal{B}(Z)$. Since Z has SVGL, there is $z''_n \in \mathcal{B}(Z)$ such that $z_n z''_n \in \mathcal{B}(Z)$ and $|z''_n| \leq M$ for all $n \in \mathbb{N}$. Note that this z''_n is a word such as $u'_n \alpha w_{i_1} \beta \cdots w_{i_k} \beta v'_n$ for some $u'_n, v'_n \in \mathcal{B}(X)$. Let $n > M$ and set $w = u'_n \alpha v'_n$. Then by the fact that α is a synchronizing word, $u_n w v_n \in X$ and $|w| \leq M$ which is absurd.

Now suppose both of X and Y have SVGL with the transition lengths M_X and M_Y . Let

$$\begin{aligned} m_1 &= \min\{|v\alpha| : v\alpha \in V\} = |v_1\alpha|, & m_2 &= \min\{|w\beta| : w\beta \in W\} = |w_1\beta|, \\ k &= \max\{n \in \mathbb{N} : n\alpha < M_X\}, & l &= \max\{n \in \mathbb{N} : n\beta < M_Y\}, \end{aligned}$$

and $M = M_X + km_2 + M_Y + lm_1$. We claim that M is a transition length for Z . Let $z_1, z_2 \in \mathcal{B}(Z)$. Different cases occur. We just prove two cases, other cases will be proved similarly. First case is when $z_1 = \gamma v_i \alpha w_j \beta z'$ and $z_2 = z'' \alpha w_p \beta \lambda$ where $\gamma, \lambda \in \mathcal{B}(Z)$ and $z', z'' \in \mathcal{B}(X)$ so that $\alpha \not\subseteq z', z''$. Since X has SVGL, there is $x = x_1 \alpha v_{i_1} \alpha \cdots v_{i_n} \alpha x_2$ such that $z' x z'' \in \mathcal{B}(X)$ and $|x| \leq M_X$. Then $z = x_1 \alpha w_1 \beta v_{i_1} \alpha w_1 \beta \cdots v_{i_n} \alpha w_1 \beta x_2 \in \mathcal{B}(Z)$ and $z_1 z z_2 \in \mathcal{B}(Z)$. Furthermore, $|z| \leq M_X + km_2 \leq M$.

The other case is when z_1 is as above and $z_2 = z'' \beta v_q \alpha \lambda$ with $\beta \not\subseteq z'' \in \mathcal{B}(Y)$. Since X and Y have SVGL, there are $x \in \mathcal{B}(X)$ and $y \in \mathcal{B}(Y)$ such that $z' x \alpha \in \mathcal{B}(X)$, $\beta y z'' \in \mathcal{B}(Y)$ and $|x| \leq M_X, |y| \leq M_Y$. We can assume that x (resp. y) does not contain α (resp. β) as a subword. Then $z_1 x \alpha y z_2 \in \mathcal{B}(Z)$. Note that $|x \alpha y| \leq M_X + m_1 + M_Y \leq M$ and we are done. $\square$

Recall that when X is a sofic shift space with non-wandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : x \text{ contains no words that are synchronizing for } R(X)\}$$

which is called the *derived shift space* of X . An irreducible sofic shift space X is *near Markov* when it is AFT and its derived shift space ∂X is a finite set [Thomsen (2004)].

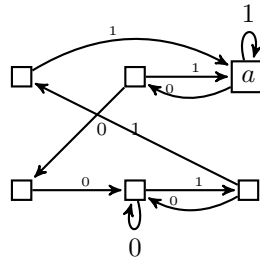


Figure 2: The Fischer cover of a non-AFT intertwined system, constructed from two AFT systems: two different paths labeled $(01)^\infty 111$ terminate at vertex a .

Theorem 4.4. *Let X and Y be two synchronized systems with $V = V_\alpha$ and $W = W_\beta$ generators for X and Y as in 4.1. If $Z = X \& Y = X_V \& Y_W$ is SFT, near Markov or AFT, then both X and Y are SFT, near Markov or AFT respectively.*

Proof. By definition of the derived shift space, if $x \in \partial X$, α is not a subword of x and then $x \in \partial Z$. That is, $\partial X \subseteq \partial Z$. By the same reasoning, $\partial Y \subseteq \partial Z$ and so

$$\partial X \cup \partial Y \subseteq \partial Z. \quad (4.2)$$

First suppose Z is SFT and one of X or Y , say X , is not SFT. Then $\partial X \neq \emptyset$ however $\partial Z = \emptyset$. So X is SFT.

Now suppose X is not AFT. So there are two different infinite paths $x = \cdots e_{-1}e_0$ and $x' = \cdots e'_{-1}e'_0$ with the same label and $t(e_0) = t(e'_0)$. If $\alpha \not\subseteq \mathcal{L}_{X_\infty}(x) = \mathcal{L}_{X_\infty}(x')$, then x and x' will be two paths in G_Z where $\mathcal{L}_{Z_\infty}(x) = \mathcal{L}_{Z_\infty}(x')$ and terminating at the same vertex of $\mathcal{V}(G_Z)$. So Z is not AFT which is absurd. Otherwise, since α is a synchronizing word and so magic for $\mathcal{G}_X$, we may assume $\mathcal{L}_X(e_{-(|\alpha|-1)} \cdots e_{-1}e_0) = \alpha$ and by the proof of Theorem 4.1, both of these paths terminate at the same vertex. By technique of merging (Lind and Marcus, 1995, §3.3), one can obtain the Fischer cover of Z from $\mathcal{G}_Z$. However, two vertices of $\mathcal{G}_Z$ merge only if one in $\mathcal{V}(G_{Z_X})$ and the other is in $\mathcal{V}(G_{Z_Y})$. Hence after merging, x and x' will be yet two different paths with the same label and terminating at the same vertex. This means Z is not AFT which is absurd.

If Z is near Markov, then it is AFT and $|\partial Z| < \infty$. So X and Y are near Markov if ∂X and ∂Y are finite which is consequence of 4.2. $\square$

The converse in Theorem 4.4 does not hold necessarily. We will give an example of X and Y , both AFT, in fact SFT, such that $X_V \& Y_W$ is not AFT for some set of generators V and W .

Example 4.5. *Let $S = S' = \{0, 1, 2\}$, $X = X(S)$ and take Y to be the set of binary sequences whose runs of 1's is restricted to S' . Choose $\alpha = 00$ and $\beta = 11$ to be the synchronized words for defining the generating sets V and W respectively. The Fischer cover of $X \& Y = X_V \& Y_W$ is as in Figure 2. Observe that there are two different infinite paths terminating at the same vertex a and having the same label $(01)^\infty 111$. Therefore, $X \& Y$ is not AFT.*

Now we give sufficient conditions such that the converse of Theorem 4.4 holds. Suppose X is a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Let $G^\#$ be a new graph whose vertex set is the set $2^\mathcal{V}$ of subsets of the vertex set $\mathcal{V}$ of G . Let $\mathcal{A}$ be the alphabet of X . We draw an arrow labeled $a \in \mathcal{A}$ from a subset $F \in 2^\mathcal{V}$ to another subset $F' \in 2^\mathcal{V}$, when

$$F' = \{x \in \mathcal{V} : \text{there is an edge labeled } a \text{ from an element of } F \text{ to } x\}.$$

We denote this new labeled graph by $(G^\#, \mathcal{L}^\#)$. By (Thomsen, 2004, Proposition 6.5), $\partial X = \mathcal{L}_\infty^\#(X_{G_2^\#})$ where $G_2^\#$ denotes the subgraph of $G^\#$ obtained by erasing all vertices $F \in 2^V$ for which $\#F \neq 2$, together with all arrows to or from such a vertex.

Lemma 4.6. *Let X be a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Also let $x = p^\infty \in \partial X$ where p is primitive and let $p = \mathcal{L}(\pi_0)$ for some path π_0 in G . If there is only one cycle γ in G such that $x = \mathcal{L}_\infty(\gamma^\infty)$, then γ consists of concatenations of at least two copies of π_0 .*

Proof. Let $p = p_0 p_1 \cdots p_{n-1}$. Then there is a cycle $\lambda = e_0 e_1 \cdots e_{n-1}$ in $G_2^\#$ such that $\mathcal{L}^\#(\lambda) = p$. Also suppose the edge e_i , $0 \leq i \leq n-1$ starts from the vertex $\{I_i, J_i\}$ and terminates at $\{I_{(i+1) \bmod n}, J_{(i+1) \bmod n}\}$. Note that if $e \in \mathcal{E}(G_2^\#)$ starts from $\{K_1, L_1\}$ and terminates at $\{K_2, L_2\}$, since $K_i \neq L_i$ for $i = 1, 2$, e represents two different edges e_1 and e_2 in G such that $i(e_i) \in \{K_1, L_1\}$ and $t(e_i) \in \{K_2, L_2\}$. So there are two paths π_1 and π_2 in G such that $\mathcal{L}(\pi_i) = \mathcal{L}(\pi_0) = p$ and

$$i(\pi_i), t(\pi_i) \in \{I_0, J_0\}, \quad i = 1, 2. \quad (4.3)$$

Suppose there is only one cycle γ in G such that $x = \mathcal{L}_\infty(\gamma^\infty)$. Since $I_0 \neq J_0$, I_0 and J_0 are different states along γ and by (4.3), they are initial and terminating points for two different paths in G labeled p and we are done. $\square$

An immediate consequence of the above lemma is that if $x = p^\infty \in \partial X$, then there are two different paths π_1 and π_2 with $\mathcal{L}(\pi_i) = p$ for $i = 1, 2$ and either both are in a cycle γ or in the different cycles γ and γ' such that

$$p^\infty = \mathcal{L}_\infty(\gamma^\infty) = \mathcal{L}_\infty(\gamma'^\infty). \quad (4.4)$$

Lemma 4.7. *Suppose $\mathcal{G}$ is a finite right-resolving labeled graph with two different paths $\xi = \cdots e_{-1} e_0$, $\xi' = \cdots e'_{-1} e'_0$ and $\mathcal{L}(e_i) = \mathcal{L}(e'_i)$. Then there are two different cycles $e_{-m} \cdots e_{-n}$ and $e'_{-m} \cdots e'_{-n}$ in $\mathcal{G}$.*

Proof. There is at least one state v in G such that ξ meets it infinitely many often. Let $v = t(e_{-i_j})$ for $j \in \mathbb{N}$ and choose $j_m > |\mathcal{V}_G|$. Also let v'_j be the terminating state for e'_{-i_j} . We follow ξ and ξ' (backward) and simultaneously. Thus by pigeon principle, at least two states v'_{j_1} and v'_{j_2} amongst the j_m states $v'_1, \dots, v'_{j_m}$ are equal and let $v' = v'_{j_1} = v'_{j_2}$. This means that when v' returns to itself along ξ' , v returns to itself along ξ and so ξ and ξ' have met at least a cycle simultaneously on their ways. Call the cycles C_ξ and $C_{\xi'}$ respectively. Note that $C_\xi \neq C_{\xi'}$. Otherwise, there is a vertex w on the way of ξ and ξ' with two different outer edges labeling the same which violates the fact that $\mathcal{G}$ is right-resolving. $\square$

Theorem 4.8. *Let X and Y be two synchronized systems generated by $V = V_\alpha$ and $W = W_\beta$ as in 4.1 and $P_n(X) \cap P_n(Y) = \emptyset$ for all $n \in \mathbb{N}$. If $X = X_V$ and $Y = Y_W$ are SFT, AFT or near Markov, then $Z = X \& Y = X_V \& Y_W$ is SFT, AFT or near Markov, respectively.*

Proof. Suppose X and Y are SFT but Z is not so. Then $\partial X = \partial Y = \emptyset$ while $\partial Z \neq \emptyset$. Since ∂Z is a sofic subsystem of Z , there is a periodic point $p^\infty \in \partial Z$.

First suppose $\beta v \alpha \not\subseteq p$, for any $v \alpha \in V$. By the hypothesis, this means that either $p^\infty \in \mathcal{G}_{Z_X}$ or $p^\infty \in \mathcal{G}_{Z_Y}$. Suppose the former happens. Thus $\alpha \not\subseteq p^\infty$. Now choose m sufficiently large so that p^m is a synchronized word in X and $p^m \notin Y$. The existence of such m is guaranteed by the fact that X is SFT and $p^\infty \notin Y$. To have a contradiction, we show that p^m is a synchronized word for Z . So let $u p^m$ and $p^m w$ be arbitrary words for Z . Since $p^m \notin Y$, $u = u_1 u'$ and $w = w' w_1$ where $u', w' \in \mathcal{B}(X)$ and they do not have α as a subword. We are trivially done if u_1 or w_1 is an empty word. Otherwise, without loss of generality assume $u_1 = \beta$ and $w_1 = \alpha$. Therefore, $\alpha u' p^m$ and $p^m w' \alpha$ are in X and this implies $\alpha u' p^m w' \alpha \in \mathcal{B}(X)$ and we are done.

Now suppose $\beta v \alpha \subseteq p$. Then $p = v_{i_1} \alpha w_{j_1} \beta \cdots v_{i_k} \alpha w_{j_k} \beta$ where $v_{i_r} \alpha \in V$ and $w_{i_r} \beta \in W$, $1 \leq r \leq k$. Without loss of generality assume that $p = v \alpha w \beta$ and let $V' = \{v : v \alpha \in V\}$, $W' = \{w :$

$w\beta \in W\}$. If $v \notin W'$ (resp. $w \notin V'$), then $\beta v\alpha$ (resp. $\alpha w\beta$) is a synchronized word for Z and $p^\infty \notin \partial Z$. So $v, w \in V' \cap W'$ and by the definition of our generators

$$\alpha, \beta \not\subseteq v, \quad \alpha, \beta \not\subseteq w. \quad (4.5)$$

By Lemma 4.6, there are two different paths π_1 and π_2 in G_Z with $\mathcal{L}_Z(\pi_i) = p$ for $i = 1, 2$ and either both are in a cycle γ or in different cycles γ and γ' such that 4.4 holds. Consider the following cases.

1. There are more than one cycle. Then (4.4) implies that $(v\alpha w\beta)^\infty = (v\beta w\alpha)^\infty$. By (4.5), either $v = w$ or $\alpha = \beta$. Considering the fact that any path labeled $v\alpha \in V$ (resp. $w\beta \in W$) terminates to the same state, the former will not allow $\mathcal{G}_Z$ being right-resolving and the latter contradicts our hypothesis $P_n(X) \cap P_n(Y) = \emptyset$ for all n .
2. There is only one cycle γ with $p^\infty = \mathcal{L}_{Z_\infty}(\gamma^\infty)$. Then the label of this unique cycle γ must be $v\alpha w\beta$. But by Lemma 4.6, this cycle must be formed from the concatenation of at least two paths with the same label and (4.5) implies that in our situation $v\alpha = w\beta$ and this in turn implies $P_n(X) \cap P_n(Y) \neq \emptyset$ for some n .

As a result, $\partial Z = \emptyset$ and Z is SFT.

Suppose X and Y are AFT but Z is not AFT. So there are two different paths $\xi = \dots e_{-1}e_0$ and $\xi' = \dots e'_{-1}e'_0$ in G_Z with the same label and terminating at the same vertex. Also we may assume $e_0 \neq e'_0$ and let $C_\xi = e_{-m} \dots e_{-n}$ and $C_{\xi'} = e'_{-m} \dots e'_{-n}$ be two different cycles provided by Lemma 4.7.

1. If C_ξ (resp. $C_{\xi'}$) is a cycle in G_{Z_X} (resp. G_{Z_Y}), then

$$\mathcal{L}(e_{-m} \dots e_{-n}) = \mathcal{L}(e'_{-m} \dots e'_{-n}) \in P_{n+m}(X) \cap P_{n+m}(X)$$

violating our hypothesis.

2. If C_ξ and $C_{\xi'}$ are both cycles in G_{Z_X} , then we may assume that $t(e_0) = t(e'_0) = \mathbf{v}(\beta)$; otherwise, we may continue ξ and ξ' on a common path to get to $\mathbf{v}(\beta)$. But then we will have two different infinite paths labeled the same and terminating at the same vertex in $\mathcal{G}_X$ violating the fact that $\mathcal{G}_X^\alpha$ is left closing by Theorem 3.1.
3. Note that in (1) and (2), $\mathcal{L}(C_\xi) = \mathcal{L}(C_{\xi'})$ does not have α or β as its subword. So the remaining case is that when $\alpha, \beta \subseteq \mathcal{L}(C_\xi)$. This implies $\mathcal{L}(C_\xi) = \mathcal{L}(C_{\xi'}) = w_{i_1}\beta v_{j_1}\alpha \dots \alpha$. Let π_{i_1} and π'_{i_1} be the subpaths of C_ξ and $C_{\xi'}$ such that $\mathcal{L}(\pi_{i_1}) = \mathcal{L}(\pi'_{i_1}) = w_{i_1}$. The fact that $\mathcal{G}_Z$ is right-resolving and paths labeled α terminate at the same vertex, implies that $\pi_{i_1} = \pi'_{i_1}$. By the same reasoning, paths in C_ξ and $C_{\xi'}$ labeled v_{j_1} are identical and carrying out this reasoning for all the subpaths of C_ξ and $C_{\xi'}$ we will have $C_\xi = C_{\xi'}$ which is absurd.

Now let X and Y be near Markov. So they are AFT and $|\partial X|, |\partial Y| < \infty$. Moreover, Z is AFT. If $|\partial Z| = \infty$, then since ∂Z is sofic there will be infinitely many periodic points in ∂Z . Apply the same reasoning as in the SFT – for the second part where $\beta v\alpha \subseteq p$ – to see that for any $p^\infty \in \partial Z \setminus (\partial X \cup \partial Y)$, we will have a contradiction. Thus $\partial Z \subseteq (\partial X \cup \partial Y)$ and we are done. $\square$

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