

Statistical Inference for a Simple Step Stress Model Based on Censored Data from The Kumaraswamy Weibull Distribution

ABSTRACT: In this paper, step stress tests are discussed when the samples are Type II and Type I censoring. The Kumaraswamy Weibull distribution is used. The maximum likelihood estimators are derived of the unknown parameters. The reliability and hazard function are estimated at usual conditions of stress. The log linear model is applied as accelerated model. In addition, confidence intervals for the parameters model are constructed. Optimum test plans are introduced. Such optimum test plans minimize the generalized asymptotic variance of the maximum likelihood estimators of the parameters model. The hypothesis test is carried out to illustrate the relationship between the stress and the shape parameter. The Monte Carlo simulation is carried out to investigate the precision of the maximum likelihood estimates. An application is used to indicate the properties of the maximum likelihood estimators.

Key Words: *Accelerated Life Tests; Simple Step Stress; Cumulative Exposure; Type II Censoring; Type I Censoring; Confidence Intervals; Test of Hypothesis; Kumaraswamy Weibull Distribution; Optimum Test Plans; Generalized Asymptotic Variance; Monte Carlo Simulation.*

1. INTRODUCTION

Products or materials are often tested at higher levels of stress than operational conditions to quickly obtain information on the life distribution or product performance under normal use. Such testing could save much time and money. The stress can be applied in different ways: Commonly used methods are constant stress, progressive stress and step stress see, Nelson (1990) and Bagdonavicius and Nikulin (2002).

In step stress accelerated life testing (ALT), the stress for survival units is generally changed to a higher stress level at a predetermined time. This model assumes that the remaining life of a unit depends only on the current cumulative fraction failed and current stress. The unit does not remember how the exposure was accumulated. Moreover, if held at the current stress, survivors will continue failing according to the cumulative distribution function (cdf) of that stress but starting at the age corresponding to the previous fraction failed. This model is called the cumulative exposure (CE) model. Some key references in the field of the accelerated life testing include Lydersen and Rausand (1987), Trujillo and Bonat (1989), Drop *et al.* (1996), Mousa and Jaheen (2002), Wu and Yu (2005), Wang (2006), Balakrishnan and Xie (2007), Abd-ELfattah *et al.* (2008), Balakrishnan

and Han (2008), Wu *et al.* (2008), Abdel-Hamid (2009), Abdel-Hamid and AL-Hussaini (2009), Chenhua Li (2009), Wang (2009), Han and Balakrishnan (2010), Wang (2010), Ismail (2011), Bakoban (2012), Hassan and AL-Thobety (2012), and Ramadan and Ramadan (2012).

Kumaraswamy (1980) constructed a distribution with two shape parameteres on (0,1). Kumaraswamy (Kum) distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as heights of individuals, scores obtained in a test, atmospheric temperatures, hydrological data. Also, Kum distribution could be appropriate in situations where scientists use probability distributions which have infinite lower and or upper bounds to fit data, when in reality the bounds are finite. A compound between Kum distribution and any distribution was constructed by Cordeiro *et al.* (2010).

Weibull distribution is one of the most popular models, it has been extensively used for modeling data in reliability, engineering and biological studies.

The need for forms of Weibull distribution arises in many applied areas. The cumulative distribution function and the probability density function of the Kumaraswamy Weibull (KumW) distribution are obtained as follows:-

$$F(t; \theta, \beta, \phi, \lambda) = 1 - \left[1 - [1 - \exp(-(\lambda t)^\phi)]^\theta \right]^\beta, \quad t > 0, \quad (1)$$

and

$$\begin{aligned} f(t; \theta, \beta, \phi, \lambda) = & \theta \beta \phi \lambda^\phi t^{\phi-1} \exp(-(\lambda t)^\phi) [1 - \exp(-(\lambda t)^\phi)]^{\theta-1} \\ & \left[1 - [1 - \exp(-(\lambda t)^\phi)]^\theta \right]^{\beta-1}, \quad t > 0, \beta, \theta, \lambda, \phi > 0, \end{aligned} \quad (2)$$

where θ, β and ϕ are the shape parameters, λ is a scale parameter.

It has three shape parameters. These parameters allow for a high degree of flexibility of the KumW distribution. It will attract wider applications in reliability, engineering and in other areas of research.

The reliability function of KumW and the hazard rate function (hf) corresponding to (2), can be written, respectively; as follows:-

$$R(t; \theta, \beta, \phi, \lambda) = \left[1 - [1 - \exp(-(\lambda t)^\phi)]^\theta \right]^\beta, \quad t > 0. \quad (3)$$

and

$$h(t; \theta, \beta, \phi, \lambda) = \frac{\theta \beta \phi \lambda^\phi t^{\phi-1} \exp(-(\lambda t)^\phi) [1 - \exp(-(\lambda t)^\phi)]^{\theta-1}}{1 - [1 - \exp(-(\lambda t)^\phi)]^\theta}, \quad t > 0. \quad (4)$$

The KumW is a quite flexible model in analyzing positive data. It contains special sub-models like the exponentiated Weibull, exponentiated Rayliegh, exponentiated exponential, Weibull and also the new Kumaraswamy exponential distribution. The KumW distribution has three shape parameters θ, β and ϕ . The three shape parameters allow high degree for flexibility of the KumW distribution. It is described in detailed in Rezk *et al.* (2013).

The rest of this paper is organized as follows: in Section 2, the k step stress accelerated life testing is presented. the statistical inference for simple step stress life testing based on Type II censoring is obtained in Section 3. In Section 4, the statistical inference for simple step stress life testing based on Type I censoring is discussed

2. THE K STEP STRESS ACCELERATED LIFE TESTING

In the experiment, the number of steps equal k. The model of constant stress is considered in the first step, such a model affects the lifetime of the unit by a certain level of stress x_1 , where x_1 is larger than the usual stress x_u .

The following steps, other stresses are considered as $x_2, x_3, \dots, x_k$, where $x_u < x_1 < x_2 < \dots < x_k$, then the cumulative exposure model reflects the effect of moving from one stress to another one the cumulative exposure distribution of the failure time.

2.1. Basic Assumptions

In the experiment of step stress testing, the following assumptions are taken:

1- For any stress $x_1, x_2, \dots, x_k$, the lifetime distribution is KumW($\theta, \beta, \lambda, \phi$), the pdf be written as follows:

$$f(t_{ij}; \beta, \phi, \lambda, \theta_j) = \theta_j \beta \phi \lambda^\phi t_{ij}^{\phi-1} \exp(-(\lambda t_{ij})^\phi) [1 - \exp(-(\lambda t_{ij})^\phi)]^{\theta_j-1} [1 - [1 - \exp(-(\lambda t_{ij})^\phi)]^{\theta_j}]^{\beta-1}, \quad (5)$$

where $t_{ij} > 0, \beta, \theta, \lambda, \phi > 0, j=1, 2, \dots, k$ and $i=1, 2, \dots, r_j$.

2- β, λ, ϕ are constants with respect to the stress x , and the shape parameter θ is affected by the stress $x_j, j=1, 2, \dots, k$, through the log linear model in the form

$$\theta_j = \exp(a + bx_j), \quad (6)$$

where a and b are unknown parameters

3- Suppose that, for a particular pattern of stress, units run at stress x_j starting at time τ_{j-1} and reaching to time $\tau_j, j=1, 2, \dots, k, (\tau_0 = 0)$. The behavior of such units is as follows:

At step 1, the population fraction $F_1(t)$ of units failing by time τ under constant stress x_1 is

$$F_1(t) = 1 - [1 - [1 - \exp(-(\lambda t)^{\phi})]^{\exp(a+bx_1)}]^\beta, \quad 0 < t < \tau_1, \quad a, b, \beta, \lambda > 0. \quad (7)$$

If we let $F(t)$ be the population cumulative distribution fraction of units failing under step stress, then in the first step:

$$F(t) = F_1(t), \quad 0 < t < \tau_1, \quad (8)$$

where τ_1 is the time when the stress is raised from x_1 to x_2 .

2.2. The Cumulative Exposure Model for The Remaining Steps

When step 2, starts, units have equivalent age u_1 , which have produced the same fraction failed seen at the end of step 1. In other meaning the survivors at time τ_1 will be switched to the stress x_2 beginning at the point u_1 , which can be determined as the solution of

$$F_2(u_1) = F_1(\tau_1),$$

$$\begin{aligned} & \left[1 - \left[1 - \exp(-(\lambda u_{j-1})^\phi) \right]^{\exp(a+bx_j)} \right]^\beta \\ &= \left[1 - \left[1 - \exp(-(\lambda (\Delta_{j-2} + u_{j-2}))^\phi) \right]^{\exp(a+bx_{j-1})} \right]^\beta, \end{aligned} \quad (9)$$

where $\Delta_0 = \tau_1 - \tau_0$, $u_0 = 0$ and $\Delta_{j-2} = \tau_{j-1} - \tau_{j-2}$, $j=2, 3, \dots, k$, by taking the root β and taking the root $\exp(a+bx_j)$ to two sides, we have

$$\exp(-(\lambda u_{j-1})^\phi) = 1 - \left[1 - \exp(-(\lambda (\Delta_{j-2} + u_{j-2}))^\phi) \right]^{\exp(b(x_{j-1} - x_j))},$$

by taking the logarithm for two sides, we have

$$u_{j-1} = \frac{-1}{\lambda} \left[\ln \left[1 - \left[1 - \exp(-(\lambda (\Delta_{j-2} + u_{j-2}))^\phi) \right]^{\exp(b(x_{j-1} - x_j))} \right] \right]^{\frac{1}{\phi}}, \quad (10)$$

the cumulative exposure model for j steps is written as follows:

$$\begin{aligned} F(t) &= F_j [t - \tau_{j-1} + u_{j-1}] \quad \tau_{j-1} \leq t \leq \tau_j, \\ F(t) &= 1 - \left[1 - \left[1 - \exp(-(\lambda (t - \tau_{j-1} + u_{j-1}))^\phi) \right]^{\exp(a+bx_j)} \right]^\beta. \end{aligned} \quad (11)$$

Substituting u_{j-1} in (12), it is seen that $F(t)$, for a step stress pattern consists of segments of the cdf, $F_1, F_2, \dots, F_k$, then $F(t)$, can be written in the form:

$$F(t) = \begin{cases} 0 & t \leq \tau_0 \\ F_1(t); & \tau_0 \leq t \leq \tau_1 \\ F_j(t - \tau_{j-1} + u_{j-1}); & \tau_{j-1} \leq t \leq \tau_j, j=2, 3, \dots, k-1 \\ F_k(t - \tau_{k-1} + u_{k-1}) & \tau_{k-1} \leq t \leq \infty \end{cases}, \quad (12)$$

and the associated pdf $f(t)$, is shown as the following form:

$$f(t) = \begin{cases} f_1(t); & \tau_0 \leq t \leq \tau_1 \\ f_j(t - \tau_{j-1} + u_{j-1}); & \tau_{j-1} \leq t \leq \tau_j, j=2, 3, \dots, k-1 \\ f_k(t - \tau_{k-1} + u_{k-1}) & \tau_{k-1} \leq t \leq \infty \\ 0 & \text{elsewhere} \end{cases}. \quad (13)$$

The maximum likelihood (ML) method will be used to the step stress model. The pdf for each test is shown by (13), which is the time derivative of the cdf given by (12). The sample likelihood is the product of such pdf's evaluated at failure times if complete sampling is used or of observed pdf's of such survival function evaluated at censoring time when censoring is applied. It is shown that $F(t)$, differs for units with different step stress patterns. The likelihood function is used to obtain maximum likelihood estimators (MLE's) of the parameters a , b and β .

3. INFERENCE AND OPTIMAL SIMPLE STEP STRESS ACCELERATED LIFE TESTS BASED ON TYPE II CENSORING

The experiment based on Type II censoring step stress has the following assumptions:

1- There are k levels of stress $x_1, x_2, \dots, x_k$, where $x_1 < x_2 < \dots < x_k$, are applied such that each unit is initially put under stress x_1 .

2- The experiment was assumed to begin with N units. The stress x_1 was applied at the first step and the result was n_1 failure times t_{i1} , $i=1, 2, \dots, n_1$ of test units are noticed when stress x_2 is applied at the second step, n_2 failure times t_{i2} , $i=1, 2, \dots, n_2$ are noticed. Finally, at Step j , stress x_j is applied n_j failure times t_{ij} , are observed.

3- The test begins at stress level x_1 , if the unit doesn't fail til the predetermined n_1 failure, the stress is raised to x_2 and held until n_2 failure. If it doesn't fail, stress is raised to x_3 . In general, if the unit doesn't fail until the occurrence of n_{j-1} at stress x_{j-1} , then the stress is raised to x_j at τ_{j-1} , $j=2, 3, \dots, k$, and held n_j failure because of paying attention to the case of censored samples, then the test is continued until a prespecified number of failures $\sum_{j=1}^k n_j$. At this time, there are n_c units still survived, at the step k , the data would be the failure times of $(N-n_c)$ failed units arranged in order, and units which survived beyond $t_{c(\tau_{nk})}$.

4- Then, it is shown from the previous points that:

In failure (Type II censoring) step stress, the stress x_{j-1} is raised to x_j at τ_{j-1} , $j=2, 3, \dots, k$, when exactly n_{j-1} failures are observed. It is assumed that the test is continued until all units fail when exactly n_k failures are observed. Then, the difference between time (Type I censoring) and failure (Type II censoring) step stress, is that in failure step stress, n_{j-1} , $j=2, 3, \dots, k$ are prespecified but τ_{j-1} , $j=2, 3, \dots, k$ and t_c are random variables. On the other hand, τ_{j-1} and t_c are prespecified in time step stress.

The failure time distribution is assumed to be KumW distribution and shape parameter is shown as a function of the stress through the log linear model. The likelihood function of the experiment is assumed to have the following form:

$$\begin{aligned} & L(t_{ij}; i=1, 2, \dots, n_j, j=1, 2, \dots, k) \\ &= \left[\prod_{i=1}^{n_1} f_1(t_{i1}) \right] \left[\prod_{j=2}^k \prod_{i=1}^{n_j} f_j(t_{ij} - \tau_{j-1} + u_{j-1}) \right] \\ & \times [1 - F_k(t_c - \tau_{k-1} + u_{k-1})]^{n_c}. \end{aligned} \quad (14)$$

It is shown from (14) that likelihood function consists of three parts. The first one represents likelihood of the first step which is the same as the case of constant stress. The second part shows the likelihood function of the $(k-1)$ other stresses. The third part shows the likelihood function of the survived units by time t_c . Considering the cumulative exposure model to relate cdf under step stress to the cdf under constant stress and using the previous assumptions, it is clear that:

The failure time distribution at j -th steps

$$\begin{aligned} & f(t_{ij} - \tau_{j-1} + u_{j-1}) \\ &= \beta \phi \lambda^\phi \exp(a + bx_j) (t_{ij} - \tau_{j-1} + u_{j-1})^{\phi-1} \exp \left(- \left(\lambda(t_{ij} - \tau_{j-1} + u_{j-1}) \right)^\phi \right) \\ & \times \left[1 - \exp \left(- \left(\lambda(t_{ij} - \tau_{j-1} + u_{j-1}) \right)^\phi \right) \right]^{\exp(a + bx_j) - 1} \\ & \times \left[1 - \left[1 - \exp \left(- \left(\lambda(t_{ij} - \tau_{j-1} + u_{j-1}) \right)^\phi \right) \right]^{\exp(a + bx_j)} \right]^{\beta-1}, \end{aligned} \quad (15)$$

where $u_{j-1} = \frac{1}{\lambda} \left[-\ln \left[1 - \left[1 - \exp(-(\lambda(\Delta_{j-2} + u_{j-2}))^\phi) \right]^{\exp(b(x_{j-1} - x_j))} \right] \right]^{\frac{1}{\phi}}$.

As a special case, let $k=2$, $\tau_{j-1} = \tau_1$ and $\tau_{n2} = t_c$, it is shown that:

$$F_2(u_1) = F_1(\tau_1), \quad (16)$$

then,

$$u_1 = -\frac{1}{\lambda} \left[\ln \left[1 - [1 - \exp(-(\lambda\tau_1)^\phi)^{\exp(b(x_1-x_2))}] \right] \right]^{\frac{1}{\phi}}, \quad (17)$$

hence, the population cumulative fraction of specimens failing in Step 2, by time t is given by:

$$F(t) = F_2[(t - \tau_1) + u_1], \quad \tau_1 \leq t \leq \tau_2, \quad (18)$$

$$F(t) = 1 - \left[1 - [1 - \exp(-(\lambda((t - \tau_1) + u_1))^\phi)^{\exp(a+bx_2)}] \right]^\beta. \quad (19)$$

When $k=2$ there are two steps only with two levels of stress x_1 and x_2 . In this case, the likelihood function has the following form:

$$\begin{aligned} & L(\beta, \phi, \lambda, \underline{\theta}; \underline{t}) \\ = & \prod_{i=1}^{n_1} \{ \beta \phi \lambda^\phi \exp(a+bx_1) t_{i1}^{\phi-1} \exp(-(\lambda t_{i1})^\phi) \\ & \times (1 - \exp(-(\lambda t_{i1})^\phi)^{\exp(a+bx_1)-1} \left(1 - (1 - \exp(-(\lambda t_{i1})^\phi)^{\exp(a+bx_1)})^{\beta-1} \right) \} \\ & \times \prod_{i=1}^{n_2} \{ \beta \phi \lambda^\phi \exp(a+bx_2) (t_{i2} - \tau_1 + u_1)^{\phi-1} \exp(-(\lambda(t_{i2} - \tau_1 + u_1))^\phi) \\ & \times [1 - \exp(-(\lambda(t_{i2} - \tau_1 + u_1))^\phi)]^{\exp(a+bx_2)-1} \\ & \times \left[1 - [1 - \exp(-(\lambda(t_{i2} - \tau_1 + u_1))^\phi)]^{\exp(a+bx_2)} \right]^{\beta-1} \} \\ & \times \left[1 - [1 - \exp(-(\lambda(t_c - \tau_1 + u_1))^\phi)]^{\exp(a+bx_2)} \right]^{\beta(n_c)}. \end{aligned} \quad (20)$$

Suppose λ and ϕ are known, the logarithm of the likelihood function in (20), denoted by ℓ_2 is given as follows:

$$\begin{aligned} \ell_2 &= \ln L(\beta, \phi, \lambda, \underline{\theta}; \underline{t}) \\ &= (N - n_c) \ln(\beta\phi) + \phi(N - n_c) \ln \lambda + \sum_{j=1}^2 n_j (a+bx_j) \\ &\quad + (\phi - 1) \sum_{i=1}^{n_1} \ln t_{i1} - \sum_{i=1}^{n_1} (\lambda t_{i1})^\phi + (\exp(a+bx_1) - 1) \\ &\quad \sum_{i=1}^{n_1} \ln \left[\frac{H_1(\cdot)}{1 - \exp(-(\lambda t_{i1})^\phi)} \right] + (\beta - 1) \sum_{i=1}^{n_1} \ln [1 - H_1(\cdot)] \\ &\quad + (\phi - 1) \sum_{i=1}^{n_2} \ln (H_2(\cdot)) - \sum_{i=1}^{n_2} (\lambda (H_2(\cdot)))^\phi \\ &\quad + (\exp(a+bx_2) - 1) \sum_{i=1}^{n_2} \ln \left(1 - \exp(-(\lambda (H_2(\cdot)))^\phi) \right) \\ &\quad + (\beta - 1) \sum_{i=1}^{n_2} \ln \left[1 - \left(1 - \exp(-(\lambda (H_2(\cdot)))^\phi) \right)^{\exp(a+bx_2)} \right] \\ &\quad + \beta n_c \ln \left[1 - \left(1 - \exp(-(\lambda (H_3(\cdot)))^\phi) \right)^{\exp(a+bx_2)} \right], \end{aligned} \quad (21)$$

where

$$\begin{aligned} H_1(.) &= (1 - \exp(-(\lambda t_{i1})^\phi))^{\exp(a+bx_1)}, \\ H_2(.) &= t_{i2} - \tau_1 + u_1, \\ H_3(.) &= t_c - \tau_1 + u_1, \end{aligned}$$

and

$$H_4(.) = [1 - \exp(-(\lambda \tau_1)^\phi)]^{\exp(b(x_1 - x_2))}.$$

Differentiating the logarithm of the likelihood function (21), with respect to a , b , and β , are obtained in Appendix.

Therefore, the MLE's may be found by setting the first derivatives of ℓ_2 to zero. As shown they are nonlinear equations, their solutions are numerically obtained using Newton Raphson method. They are solved numerically to obtain the MLE's $\hat{a}_2$, $\hat{b}_2$ and $\hat{\beta}_2$. Depending on the invariance property of the MLE's, the MLE of the shape parameter, θ_u of KumW distribution at usual stress x_u , can be predicted using the following equation

$$\hat{\theta}_{2u} = \exp(\hat{a}_2 + \hat{b}_2 x_{2u}), \quad (22)$$

also, the MLE of the reliability function under the same usual conditions $\hat{R}_{2u}(t_0)$, can be given by

$$\hat{R}_{2u}(t_0) = \left[1 - \left[1 - \exp(-(\lambda t_0)^\phi) \right]^{\hat{\theta}_{2u}} \right]^{\hat{\beta}_2}, \quad (23)$$

and the MLE of the hazard rate function under the same usual conditions $\hat{h}_{2u}(t_0)$, is given as follows

$$\hat{h}_{2u}(t_0) = \frac{\hat{\theta}_{2u} \hat{\beta}_2 \phi \lambda^\phi t_0^{\phi-1} \exp(-(\lambda t_0)^\phi) [1 - \exp(-(\lambda t_0)^\phi)]^{\hat{\theta}_{2u}-1}}{1 - [1 - \exp(-(\lambda t_0)^\phi)]^{\hat{\theta}_{2u}}},$$

where t_0 is a mission time.

The asymptotic Fisher information matrix can be written as follows:

$$\tilde{I}_2 = - \left[\frac{\partial^2 \ell_2}{\partial \psi_i \partial \psi_j} \right], \quad i, j = 1, 2, 3, \quad (24)$$

where $\psi_1 = a$, $\psi_2 = b$ and $\psi_3 = \beta$. where the elements of the information matrix (24) are found in Appendix.

3.1. The Confidence Intervals Based on Type II Censoring

For large sample size, the MLE's under appropriate regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the two sided approximate $100(1-\alpha)\%$ confidence intervals for the MLE say, $\hat{w}$ of a population value w can be obtained by $\mathbf{p}\left(-z \leq \frac{\hat{w}-w}{\sigma_{\hat{w}}} \leq z\right) = (1-\alpha)$ where z is the $100\left(1 - \frac{\alpha}{2}\right)$ th standard normal percentile. The two sided approximate $100(1-\alpha)\%$ confidence intervals for a , b and β will be respectively, as follows:-

$$L_w, U_w = \hat{w} \pm z_{\frac{\alpha}{2}} \sigma_{\hat{w}}, \quad (25)$$

where $\sigma_{\hat{w}}$, is the standard error, in this study $\hat{w}$ is $\hat{a}$, $\hat{b}$, and $\hat{\beta}$, respectively.

[For more details see Nelson (1982)].

3.2. Optimum Test Plans Based on Type II Censoring

The GAV of the MLE of the parameters model is the reciprocal of the determinant of the asymptotic Fisher information matrix $\tilde{I}_2$, see Wu *et al.* (2008).

That is

$$\text{GAV}(\hat{a}_2, \hat{b}_2, \hat{\beta}_2) = \tilde{I}_2^{-1}. \quad (26)$$

Thus, minimization of the determinant GAV is equivalent to maximization of the determinant of $\tilde{I}_2$. The Newton Raphson method is applied to determine numerically the best choice of the sub sample proportion allocated to each level of stress which minimizes the determinant GAV as defined previously. Accordingly, the corresponding optimal numbers of items allocated to each step of stress can be obtained by finding the first partial derivatives of $|\tilde{I}_2|$ with respect to G_1 and G_2 .

$$\frac{\partial |\tilde{I}_2|}{\partial G_j}, \quad j=1,2. \quad (27)$$

Then setting (27) equal to zero, which G_j are the sub sample proportion, $j=1, 2$ can be optimally determined by solving them simultaneously and applying the Newton Raphson method where, the determinant can be obtained as follows

$$|\tilde{I}_2| = A_{11}A_{22}A_{33} - A_{11}A_{23}^2 - A_{12}^2A_{33} + A_{12}A_{13}A_{23} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}, \quad (28)$$

Remark

When $r=n$ all the results obtained for Type II censoring will approach those of the complete sample.

3.3. Numerical Results

The main aim of this section is to make a numerical investigation to illustrate the theoretical results of both estimation and optimal design problems on basis of simulated data.

3.3.1. Simulation Algorithm

- Several data sets are generated from KumW distribution for a combination of the true parameter values of a , b , and β , and for sample sizes 100, 200, 300, 400 and 500 using 1000 replications for each sample size.
- The population N is with parameters $a=0.5$, $b=1.5$ and $\beta=1.2$, given $n_1 = 0.4N$, $n_2 = 0.5N$ and $n_c = 0.1N$.
- It is assumed that there are only two different levels of stresses ($k=2$), $x_1=1$ and $x_2=1.5$, which are higher than the stress at usual condition, $x_u=0.5$.
- Numbers of test units are allocated to each level of stress (n_j , $j=1,2$) where $G_1=0.5$, $G_2=0.5$, $r_j=80\%(n_j)$, $j=1, 2$.
- The true parameter values of a , b and β used in this simulation study to generate $(t_{ij}, j=1, 2 \text{ and } i=1, 2, \dots, r_j)$.
- Computer program is used depending on MathCad 14 using the iterative technique of Newton Raphson method to solve the derived nonlinear logarithmic likelihood equations in (37)- (39) simultaneously.
- Once the values of $\hat{a}_2$, $\hat{b}_2$ and $\hat{\beta}_2$ are obtained, the estimates are used to obtain, depending on (22) and the design stress, $x_u=0.5$, the shape parameter

under this stress, θ_u , is predicted as $\hat{\theta}_{2u} = \exp(\hat{a}_2 + \hat{b}_2 x_{2u})$. Also, the reliability function, the hazard rate function and their relative absolute bias are predicted for different values of mission times under usual conditions using (23) and (24).

- The performance of the $\hat{a}_2$, $\hat{b}_2$, and $\hat{\beta}_2$ has been evaluated through some measurements of accuracy. In order to study the precision and variation of MLE's (E_2), it is convenient to use, firstly, the relative absolute bias (RAB_2). The secondly, is the mean square error (ER_2). The thirdly, is the relative error (RE_2).
- The two sided approximate $100(1-\alpha)\%$ confidence intervals for a , b and β will be obtained through using (25). The different sample sizes of $N=100, 200, 300, 400, 500$ are considered. The results are displayed in Tables 1-5.

3.3.2. Concluding Remarks

- It is clear that from Table 1, the MLE's (E_2) approximate the true values of the parameters as the sample size increases. Also, as shown in the numerical results the RAB , ER and RE are decreasing when the sample size is increasing.
- Table 2, shows that the asymptotic variances of the estimates are decreasing as sample sizes are getting large.
- Table 3, indicates that the reliability decreases when the mission time t_0 increases. The results get better in the sense that the aim of an accelerated life testing experiments is to get large number of failures (reduce the reliability) of the device with high reliability. In other words, when sample size increases, the reliability function increases. Also, the RAB_R for the reliability function decreases when the sample size increases. The hazard rate function increases when the mission time t_0 increases.
- The two-sided 95% central asymptotic confidence intervals for the parameters of KumW are described in Table 4. This table contains the standard error (SE_2), lower bound (L_2), upper bound (U_2) and the length of intervals. The interval estimate of the parameters becomes narrower as the sample size increases.
- Optimum test plans are developed numerically, it can be observed from the numerical results presented in Table 5, that the optimum test plans do not allocate the same number of the test units to each stress. Also, Table 5, includes the expected number of items that must be allocated to each level of stress represented by n_1^* , n_2^* which minimizes the GAV. As indicated from the results, the optimal GAV of the MLE of the parameters model is decreased as the sample size N is increasing.

Table 1: The E_2 , RAB_2 , ER_2 and RE_2 of the estimates for different sample sizes

N	parameter	E_2	RAB_2	ER_2	RE_2
100	a	0.6753	0.3508	0.0526	0.4587
	b	1.4602	0.0265	0.0447	0.4467
	β	1.1951	0.0041	0.0089	0.0786
200	a	0.6686	0.3372	0.0435	0.3231
	b	1.4694	0.0204	0.0438	0.4415
	β	1.1983	0.0014	0.0073	0.0712
300	a	0.6516	0.3032	0.0355	0.3768
	b	1.4798	0.0315	0.02534	0.3359
	β	1.1999	0.4001	0.0254	0.0456
400	a	0.6053	0.2106	0.0084	0.1833
	b	1.4900	0.0070	0.0168	0.1750
	β	1.2018	0.0015	0.0090	0.0251
500	a	0.5877	0.1754	0.0067	0.1637
	b	1.5086	0.0057	0.0142	0.1028
	β	1.2013	0.0011	0.0031	0.0247

Table 2: Asymptotic variances and covariances of estimates for different sample sizes

N	variance - covariance matrix		
	a	b	β
100	0.0037	0.0028	-0.0031
	0.0028	0.0268	-0.0592
	-0.0031	-0.0592	0.3776
200	0.0034	0.0026	-0.0028
	0.0026	0.0245	-0.0574
	-0.0028	-0.0574	0.3758
300	0.0031	0.0023	-0.0026
	0.0023	0.0213	-0.0541
	-0.0026	-0.0541	0.3726
400	0.0028	0.0021	-0.0023
	0.0021	0.0209	-0.0517
	-0.0023	-0.0517	0.3674
500	0.0026	0.0019	-0.0021
	0.0019	0.0188	-0.0483
	-0.0021	-0.0483	0.3446

Table 3: The estimated shape parameter and reliability function under usual condition at different sample sizes

N	$\hat{\theta}_{2u}$	t_0	$\hat{R}_{2u}(t_0)$	RAB_{R_2}	$\hat{h}_{2u}(t_0)$	RAB_{h_2}
100	4.0771	0.3	0.9909	0.0095	0.5049	0.4279
		0.5	0.8187	0.0729	5.7316	0.1612
		0.7	0.3969	0.1523	27.007	0.0515
		1	0.0435	0.2117	258.779	0.0095
200	4.0686	0.3	0.9908	0.0094	0.5104	0.4217
		0.5	0.8176	0.0714	5.7615	0.1568
		0.7	0.3952	0.1475	27.0987	0.0483
		1	0.0431	0.0072	259.4929	0.0068
300	4.0209	0.3	0.9902	0.0088	0.5350	0.3938
		0.5	0.8134	0.0659	5.8514	0.1436
		0.7	0.3907	0.1344	27.2434	0.0432
		1	0.0423	0.1783	259.9554	0.0050
400	3.8586	0.3	0.9881	0.0066	0.6255	0.2913
		0.5	0.7990	0.0449	6.1480	0.1002
		0.7	0.3762	0.0923	27.6585	0.0286
		1	.0401	0.1170	260.7626	0.0019
500	3.8267	0.3	0.9877	0.0062	0.6445	0.2698
		0.5	0.7961	0.0415	6.2033	0.0921
		0.7	0.3736	0.0848	27.7205	0.0264
		1	0.0398	0.1086	260.7319	0.0021

Table 4: Confidence intervals of the estimates at confidence level 95% for different sample sizes

N	parameter	E_2	SE_2	L_2	U_2	length
100	a	0.6753	0.2293	0.2167	1.1339	0.9172
	b	1.4602	0.6692	0.1218	2.7986	2.6760
	β	1.1951	0.0943	1.0065	1.3837	0.3772
200	a	0.6686	0.2086	0.2514	1.0858	0.8344
	b	1.4694	0.6622	0.1450	2.7938	2.6488
	β	1.1983	0.0854	1.0275	1.3691	0.3416
300	a	0.6516	0.1884	0.2748	1.0284	0.7536
	b	1.4798	0.5039	0.4720	2.4876	2.0156
	β	1.1999	0.0548	1.0902	1.3095	0.2192
400	a	0.6053	0.0916	0.4221	0.7885	0.3664
	b	1.4900	0.2625	0.9650	2.0150	1.0500
	β	1.2018	0.0302	1.1414	1.2622	0.1208
500	a	0.5877	0.0818	0.4241	0.7513	0.3272
	b	1.5086	0.1543	1.3543	1.8172	0.4629
	β	1.2013	0.0297	1.1419	1.2607	0.1188

Table 5: The results of optimal design of the life test for different sample size

N	n ₁	n ₂	G_1	G_2	n ₁ [*]	n ₂ [*]	GAV
100	40	50	0.3106	0.5106	31	51	0.007842
200	80	100	0.3227	0.5227	64	104	0.000913
300	120	150	0.3266	0.5266	98	158	0.0000652
400	160	200	0.3304	0.5304	132	212	0.00000845
500	200	250	0.3378	0.5378	169	269	0.00000572

3.3.3. Application

The main aim of this section is to demonstrate how the proposed method can be used in practice. Cordeiro *et al.* (2010) used Kolmogorov- Smirnov goodness of fit test and data points representing failure time. The data were taken from Murthy *et al.* (2004), [see p. 154]. The data were 30 items (n=30) tested with test stopped after 20th failure (r=20). It is assumed that k=2, i.e. there are only two different levels of stress $x_1=0.6$ and $x_2=1$, which are higher than the stress at usual conditions, $x_u=0.5$. The failure times in the first step are [0.0014, 0.0623, 1.3826, 2.0130, 2.5274, 2.8221, 3.1544, 4.9835, 5.5462, 5.8196, 5.8714, 7.4710] and the failure times in the second stress are [7.5080, 7.6667, 8.6122, 9.0442, 9.1153, 9.6477, 10.1547, 10.7582]

The true parameter values of a, b and β used in this application are a=0.5, b=1.5, $\beta=2$, $\phi=2$, and $\lambda=2$. Once the estimate value of a, b and β are obtained, the estimators are used to predicate θ_u , as $\hat{\theta}_{2u}=\exp(\hat{a}_1+\hat{b}_2x_{2u})$. Letting that the design stress, $x_u=0.5$. Also, the reliability function is predicted for different values of mission times under usual conditions depending on (22).

Moreover, the precision and variation of MLE's (E_2) are studied through some convenient measures such as the RAB, ER and RE are computed for each parameter in Table 6.

In Table 7, Asymptotic variances and covariances of estimates of a, b, β are obtained. The estimated shape parameter, reliability function, RAB_{R_2} , the hazard rate function, RAB_{h_2} under usual condition are shown in Table 8. Table 9 and Table 10 indicate Confidence intervals of the parameters at confidence level 95% and level 99%. These tables contains the standard error (SE_2), lower bound (L_2), upper bound (U_2) and the length of intervals. Hypothesis tests are obtained for b when $\alpha=0.5$ and df =1. It is concluded that, the decision of b=0 is reject and there is relationship between the level of stress and the shape parameter θ .

Table 6: The E_2 , RAB_2 , SE_2 and RE_2 of the estimates

parameter	E_2	RAB_2	ER_2	RE_2
a	0.1122	0.7766	0.1504	0.7756
b	1.6020	0.0013	0.5913	0.4806
β	1.1218	0.4391	0.7712	0.4391

Table 7: Asymptotic variances and covariances matrix

variance	covariance	matrix
a	b	β
0.7730	-1.1341	0.0807
-1.1341	1.6943	-0.1356
0.0807	-0.1356	0.0020

Table 8: The estimated shape parameter and reliability function under usual condition

$\hat{\theta}_{2u}=2.4897$				
t_0	0.3	0.5	0.7	1
$\hat{R}_{2u}(t_0)$	0.9745	0.9434	0.9065	0.8468
RAB_{R_2}	0.0214	0.0465	0.0732	0.1104
t_0	1.2	1.5	1.8	2
$\hat{h}_{2u}(t_0)$	0.8135	1.1031	1.4313	1.6750
RAB_{h_2}	2.1438	1.5808	1.2184	1.0411

Table 9: Confidence intervals of the estimates at confidence level 95%

parameter	E_2	SE_2	L_2	U_2	length
a	0.1122	0.7756	-0.6635	0.8878	1.5513
b	1.6020	0.4806	0.0621	3.1379	3.0758
β	1.1218	0.4391	0.6346	2.8782	3.5128

Table 10: Confidence intervals of the estimates at confidence level 99%

parameter	E_2	SE_2	L_2	U_2	length
a	0.1122	0.7756	-1.1575	1.3288	1.5513
b	1.6020	0.4806	-0.7068	3.9068	3.0758
β	1.1218	0.4391	-2.0298	4.0149	3.5128

4. INFERENCE AND OPTIMAL SIMPLE STEP STRESS ACCELERATED LIFE TESTS BASED ON TYPE I CENSORING

In time step stress, the stress x_{j-1} is raised to x_j at τ_{j-1} , $j=2, 3, \dots, k$. It is assumed that the test is continued until all units fail or until time L . The likelihood function has the same form as (21). Then, the difference between time and failure step stress, is that in failure step stress, n_{j-1} , $j=2, 3, \dots, k+1$, are prespecified but τ_{j-1} , $j=2, 3, \dots, k$ and L are random variables. On other hand, τ_{j-1} , and L are prespecified in time step stress and n_{j-1} , $j=2, 3, \dots, k+1$, are random variables.

4.1. The Mmaximum Likelihood Estimation Based on Type I When There are 2 Steps of Stress as a Special Case

As a special case, let $k=2$, τ_1 is the time at which the stress change from x_1 to x_2 and L is the time at which the experiment is terminated (censoring time)

The likelihood function of the experiment is considered to have the same form as (20).

Depending on the same procedure as in Section 3, the numerical result of the experiment can be summarized as follows:

4.2. Numerical Results

The main aim of this section is to make a numerical investigation to illustrate the theoretical results of both estimation and optimal design problems on basis of simulated data.

4.2.1. Simulation algorithm

- Several data sets are generated from KumW distribution for a combination of the true parameter values of a , b , and β , and for sample sizes 100, 200, 300, 400 and 500 using 1000 replications for each sample size.
- Using the same values of the parameters a , b and β considered in Section (4), with different sample sizes and the same procedure to obtain the maximum likelihood estimates by setting (37), (38) and (39) equal to zero.
- It is shown they are nonlinear equations, their solutions are numerically obtained by using Newton Raphson method depending on MathCad 14.
- The using population is with parameters $a=0.5$, $b=1.5$ and $\beta=1.2$, given $\tau_1=25$, $L=5000$ and $x_1=1$ and $x_2=1.5$.
- It is assumed that there are only two different levels of stresses ($k=2$), $x_1=1$ and $x_2=1.5$, which are higher than the stress at usual condition, $x_u=0.5$.
- The true parameter values of a , b and β used in this simulation study to generate $(t_{ij}, j = 1, 2 \text{ and } i=1, 2, \dots, n_j)$.
- Computer program is used depending on MathCad 14 using the iterative technique of Newton Raphson method to solve the derived nonlinear logarithmic likelihood equations in (37)- (39) simultaneously.
- Once the values of $\hat{a}_1$, $\hat{b}_1$ and $\hat{\beta}_1$ are obtained, the estimates are used to obtain, depending on (22) and the design stress, $x_u=0.5$, the shape parameter under this stress, θ_u , is predicted as $\hat{\theta}_{1u}=\exp(\hat{a}_1+\hat{b}_1x_{1u})$. Also, the reliability function, the hazard rate function and their relative absolute bias are predicted for different values of mission times under usual conditions using (23) and (24).
- The performance of the $\hat{a}_1$, $\hat{b}_1$, and $\hat{\beta}_1$ has been evaluated through some measurements of accuracy. In order to study the precision and variation of MLE's, it is convenient to use, firstly, the relative absolute bias (RAB₁). The secondly, is the mean square error (ER₁). The thirdly, is the relative error (RE₁).
- The two sided approximate $100(1-\alpha)\%$ confidence intervals for a , b and β will be obtained through using (25). The different sample sizes of $N=100, 200, 300, 400, 500$ are considered. The results are displayed in Tables 11-15.

4.2.2. Concluding remarks

- It is clear that from Table 11, the MLE's (E_1) approximate the true values of the parameters as the sample size increases. Also, as shown in the numerical results the RAB₁, ER₁ and RE₁ are decreasing when the sample size is increasing.
- Table 12, shows that the asymptotic variances of the estimates are decreasing as sample sizes are getting large.
- Table 13, indicates that the reliability decreases when the mission time t_0 increases. The results get better in the sense that the aim of an accelerated life

testing experiments is to get large number of failures (reduce the reliability) of the device with high reliability. In other words, when sample size increases, the reliability function increases. Also, the RAB_{R_1} for the reliability function decreases when the sample size increases. The hazard rate function increases when the mission time t_0 increases.

- The two-sided 95% central asymptotic confidence intervals for the parameters of KumW are described in Table 14. This table contains the standard error (SE_1), lower bound (L_1), upper bound (U_1) and the length of intervals. The interval estimate of the parameters becomes narrower as the sample size increases.
- As shown in Section 4, by setting the $\frac{\partial \Pi}{\partial \tau_j} = 0$, τ_1 and L can be optimally determined by solving them simultaneously and applying the Newton Raphson method.
- Then, optimum test plans are developed numerically. It is noted from the numerical results presented in Table 15, that includes the expected time, τ_1^* , at which the stress changes from x_1 to x_2 and the the expected time, L^* , at which the experiment is terminated. It is clear that the optimum test plans do not allocate the same number of the test units to each step. As indicated from the results, the optimal GAV of the MLE of the parameters model is decreasing as the sample size N is increasing.

Table 11: The E_1 , RAB_1 , ER_1 and RE_1 of the estimate for different sample sizes

N	parameter	E_1	RAB_1	ER_1	RE_1
100	a	0.6847	0.3694	0.0621	0.9984
	b	1.7480	0.1653	0.1068	0.2179
	β	1.3959	0.1630	0.0483	0.1831
200	a	0.6671	0.3342	0.0572	0.4783
	b	1.6939	0.1293	0.0735	0.1807
	β	1.3524	0.1270	0.0249	0.1096
300	a	0.6427	0.2854	0.0416	0.4079
	b	1.6255	0.0837	0.0327	0.1205
	β	1.2914	0.0761	0.0285	0.1407
400	a	0.6119	0.2238	0.0094	0.1939
	b	1.5744	0.0496	0.0183	0.0902
	β	1.2538	0.0448	0.0095	0.0812
500	a	0.5862	0.1724	0.0068	0.1649
	b	1.5237	0.0158	0.0157	0.0833
	β	1.2164	0.0137	0.0039	0.0520

Table 12: Asymptotic variances and covariances of estimates for different sample sizes

N	variance - covariance matrix		
	a	b	β
100	0.0046	0.0057	0.0063
	0.0057	0.0642	-0.0714
	0.0036	-0.0714	0.5283
200	0.0043	0.0052	0.0059
	0.0052	0.0492	-0.0688
	0.0059	-0.0688	0.4718
300	0.0038	0.0044	0.0026
	0.0049	0.0381	-0.0597
	0.0034	-0.0597	0.3982
400	0.0030	0.0038	-0.0023
	0.0038	0.0247	-0.0551
	0.0026	-0.0551	0.3768
500	0.0027	0.0024	0.0023
	0.0024	0.0198	-0.0493
	0.0023	-0.0493	0.3516

Table 13: The estimated shape parameter and reliability function under usual condition at different samples sizes

N	$\hat{\theta}_{1u}$	t_0	$\hat{R}_{1u}(t_0)$	RAB_{R_1}	$\hat{h}_{1u}(t_0)$	RAB_{h_1}
100	4.7526	0.3	0.9953	0.0109	0.3051	0.9479
		0.5	0.8458	0.1084	5.4580	0.5640
		0.7	0.3950	0.1469	29.792	0.1260
		1	0.0296	0.1755	300.288	0.1260
200	4.5451	0.3	0.9941	0.0096	0.3630	0.9380
		0.5	0.8356	0.0950	5.6350	0.5495
		0.7	0.3899	0.1321	29.382	0.1383
		1	0.0331	0.0780	291.559	0.0929
300	4.2864	0.3	0.9923	0.0078	0.4460	0.9238
		0.5	0.8356	0.0950	5.818	0.5349
		0.7	0.3859	0.1205	28.675	0.1590
		1	0.0360	0.0028	279.084	0.0461
400	4.0515	0.3	0.9902	0.0057	0.5430	0.9072
		0.5	0.8085	0.0561	6.0590	0.5156
		0.7	0.3762	0.0923	27.6585	0.0286
		1	.0370	0.0011	271.555	0.0179
500	3.8267	0.3	0.9879	0.0033	0.6380	0.8909
		0.5	0.7960	0.0431	6.2390	0.5012
		0.7	0.3710	0.0772	28.0150	0.1784
		1	0.0385	0.0724	263.952	0.0106

Table 14: Confidence bounds of the parameters at confidence level 95% for different sample sizes

N	parameters	E_1	SE_1	L_1	U_1	length
100	a	0.6847	0.2492	0.1863	1.1831	0.9968
	b	1.7480	0.3268	1.0944	2.4016	1.3072
	β	1.3959	0.2198	0.9563	1.8355	0.8792
200	a	0.6671	0.2392	0.1887	1.1455	0.9568
	b	1.6939	0.2711	1.1517	2.2361	1.0844
	β	1.3524	0.1315	1.0894	1.6154	0.5260
300	a	0.6427	0.2040	0.2347	1.0507	0.8160
	b	1.6255	0.1808	1.2633	1.9871	0.7232
	β	1.2914	0.1688	0.9538	1.6290	0.6752
400	a	0.6119	0.0970	0.4179	0.8059	0.3880
	b	1.5744	0.1353	1.3039	1.8450	0.5412
	β	1.2538	0.0974	1.0590	1.4486	0.3896
500	a	0.5862	0.0825	0.4212	0.7512	0.3300
	b	1.5237	0.1253	1.2731	1.7743	0.5012
	β	1.2164	0.0624	1.0916	1.3412	0.2496

Table 15: The results of optimal design of the life test for different sample size under Type I censoring in step stress

N	n_1	n_2	t_1^*	L^*	n_1^*	n_2^*	GAV
100	40	50	60.76	241.391	59	8	0.00187
200	78	101	62.83	224.805	113	17	0.00039
300	118	152	60.76	207.69	170	19	0.0000472
400	157	202	63.19	195.28	225	28	0.0000066
500	197	252	64.08	193.35	282	33	0.00000243

Remarks:

The results obtained in this paper can be modified to obtain results for sub-models of KumW distribution under Type I and Type II censored samples as follows:

- The Kum exponential distribution, which has pdf given by

$$f(t; \theta, \beta, \lambda) = \theta \beta \lambda \exp(-(\lambda t)) [1 - \exp(-(\lambda t))]^{\theta-1} \left[1 - [1 - \exp(-(\lambda t))]^{\theta}\right]^{\beta-1}, t > 0, \beta, \theta, \lambda > 0. \quad (29)$$

where θ and β are the shape parameters, λ is a scale parameter, can be obtained as special case by putting $\phi = 1$.

- The Kum Rayleigh distribution, which has pdf given by

$$f(t; \theta, \beta, \lambda) = 2\theta \beta \lambda^2 t \exp(-(\lambda t)^2) [1 - \exp(-(\lambda t)^2)]^{\theta-1} \left[1 - [1 - \exp(-(\lambda t)^2)]^{\theta}\right]^{\beta-1}, t > 0, \beta, \theta, \lambda > 0. \quad (30)$$

where θ and β are the shape parameters, λ is a scale parameter, can be obtained as special case by putting $\phi = 2$.

- The exponentiated Weibull distribution, which has pdf given by

$$f(t; \theta, \phi, \lambda) = \theta \phi \lambda^\phi t^{\phi-1} \exp(-(\lambda t)^\phi) [1 - \exp(-(\lambda t)^\phi)]^{\theta-1}, t > 0, \phi, \theta, \lambda > 0. \quad (31)$$

where θ and ϕ are the shape parameters, λ is a scale parameter, can be obtained as special case by putting $\beta = 1$.

- The exponentiated Rayleigh distribution, which has pdf given by

$$f(t; \theta, \lambda) = 2\theta \lambda^2 t \exp(-(\lambda t)^2) [1 - \exp(-(\lambda t)^2)]^{\theta-1}, t > 0, \theta, \lambda > 0. \quad (32)$$

where θ is the shape parameters, λ is a scale parameter, can be obtained as special case by putting $\beta = 1, \phi = 2$.

- The exponentiated exponential distribution, which has pdf given by

$$f(t; \theta, \lambda) = \theta \lambda \exp(-(\lambda t)) [1 - \exp(-(\lambda t))]^{\theta-1}, t > 0, \theta, \lambda > 0. \quad (33)$$

where θ is the shape parameters, λ is a scale parameter, can be obtained as special case by putting $\beta = \phi = 1$.

- The Weibull distribution, which has pdf given by

$$f(t; \phi, \lambda) = \phi \lambda^\phi t^{\phi-1} \exp(-(\lambda t)^\phi), t > 0, \phi, \lambda > 0. \quad (34)$$

where ϕ is the shape parameters, λ is a scale parameter, can be obtained as special case by putting $\beta = \theta = 1$, see Khamis (1997).

- The Rayleigh distribution, which has pdf given by

$$f(t; \lambda) = 2\lambda^2 t \exp(-(\lambda t)^2), t > 0, \lambda > 0. \quad (35)$$

where λ is a scale parameter, can be obtained as special case by putting $\phi = 2, \beta = \theta = 1$.

- The exponential distribution, which has pdf given by

$$f(t; \lambda) = \lambda \exp(-(\lambda t)), t > 0, \lambda > 0. \quad (36)$$

where λ is a scale parameter, can be obtained as special case by putting $\phi = \beta = \theta = 1$.

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Appendix

The first derivatives of log likelihood function defined in (20), with respect to the unknown parameters can be written as follows:

$$\begin{aligned}
 \frac{\partial \ell_2}{\partial \mathbf{a}} = & \sum_{j=1}^2 n_j + \sum_{i=1}^{n_1} \theta_1 \ln \left(\frac{H_1(.)}{1 - \exp(-(\lambda t_{i1})^\phi)} \right) + \sum_{i=1}^{n_1} \theta_1^2 \ln (1 - \exp(-(\lambda t_{i1})^\phi)) \\
 & - (\beta - 1) \sum_{i=1}^{n_1} \frac{\theta_1 \ln [1 - \exp(-(\lambda t_{i1})^\phi)] (H_1(.))}{(1 - H_1(.))} \\
 & + \sum_{i=1}^{n_2} \theta_2 \ln \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right) \\
 & - (\beta - 1) \sum_{i=1}^{n_2} \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right)^{\theta_2} \\
 & \frac{\theta_2 \ln \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right)}{\left(1 - \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right)^{\theta_2} \right)} \\
 & - \beta n_c \left(1 - \exp \left(-(\lambda (H_3(.)))^\phi \right) \right)^{\theta_2} \\
 & \frac{\theta_2 \ln \left(1 - \exp \left(-(\lambda (H_3(.)))^\phi \right) \right)}{\left(1 - \left(1 - \exp \left(-(\lambda (H_3(.)))^\phi \right) \right)^{\theta_2} \right)}, \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \ell_2}{\partial \mathbf{b}} = & \sum_{j=1}^2 n_j x_j + \sum_{i=1}^{n_1} \theta_1 x_1 \ln (H_1(.))^{-1} + \sum_{i=1}^{n_1} \ln (1 - \exp(-(\lambda t_{i1})^\phi)) \\
 & x_1 \theta_1^2 - (\beta - 1) \sum_{i=1}^{n_1} \frac{\theta_1 x_1 \ln (1 - \exp(-(\lambda t_{i1})^\phi)) H_1(.))}{(1 - H_1(.))} \\
 & + (\phi - 1) \sum_{i=1}^{n_2} (x_1 - x_2) \exp (b(x_1 - x_2)) H_4(.) \\
 & \frac{\ln (1 - H_4(.))^{\frac{1}{\phi}-1} \ln (1 - \exp(-(\lambda t_{i1})^\phi))}{\lambda \phi (H_2(.)) [1 - H_4(.)]} \\
 & + \sum_{i=1}^{n_2} (x_1 - x_2) \exp (b(x_1 - x_2)) H_4(.) \\
 & \frac{\ln (1 - H_4(.))^{\frac{1}{\phi}-1}}{\phi [1 - H_4(.)]} + \sum_{i=1}^{n_2} x_2 \exp (a + b x_2 - 1) \\
 & \ln \left(1 - \exp(-(\lambda H_2(.))^\phi) \right) + \sum_{i=1}^{n_2} (x_1 - x_2) (\lambda H_2(.))^\phi - 1 \\
 & \exp (a + b x_1 - 1) \exp(-(\lambda H_2(.))^\phi) H_4(.) \\
 & \frac{\ln \left(1 - \exp(-(\lambda H_2(.))^\phi) \right) \ln (1 - H_4(.))^{\frac{1}{\phi}-1}}{\left(1 - \exp(-(\lambda H_2(.))^\phi) \right) [1 - H_4(.)]}
 \end{aligned}$$

$$\begin{aligned}
& -(\beta - 1) \sum_{i=1}^{n_2} x_2 \{ (1 - \exp(-(\lambda H_2(.))^\phi))^{\theta_2} \\
& \quad \frac{\theta_2 \ln(1 - \exp(-(\lambda H_2(.))^\phi))}{\left(1 - \left(1 - \exp(-(\lambda H_2(.))^\phi)\right)^{\theta_2}\right)} - (x_1 - x_2) H_4(.) \\
& \quad \theta_1 \exp\left(-(\lambda H_2(.))^\phi\right) \ln(1 - H_4(.))^{\frac{1}{\phi}-1} \ln(1 - \exp(-(\lambda \tau_1)^\phi)) \\
& \quad \frac{\left(1 - \exp(-(\lambda H_2(.))^\phi)\right)^{\theta_2-1} (\lambda H_2(.))^{\phi-1}}{(1 - H_4(.)) \left(1 - \left(1 - \exp(-(\lambda H_2(.))^\phi)\right)^{\theta_2}\right)} \} \\
& -\beta n_c x_2 \{ (1 - \exp(-(\lambda H_3(.))^\phi))^{\theta_2} \\
& \quad \frac{\theta_2 \ln(1 - \exp(-(\lambda H_3(.))^\phi))}{\left(1 - \left(1 - \exp(-(\lambda H_3(.))^\phi)\right)^{\theta_2}\right)} + (x_1 - x_2) \theta_1 \\
& \quad H_4(.) \exp\left(-(\lambda H_3(.))^\phi\right) \ln(1 - H_4(.))^{\frac{1}{\phi}-1} (\lambda H_3(.))^{\phi-1} \\
& \quad \frac{\ln(1 - \exp(-(\lambda \tau_1)^\phi)) \left(1 - \exp(-(\lambda H_3(.))^\phi)\right)^{\theta_2-1}}{(1 - H_4(.)) \left(1 - \left(1 - \exp(-(\lambda H_3(.))^\phi)\right)^{\theta_2}\right)} \}, \tag{38}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell_2}{\partial \beta} &= \frac{(N - n_c)}{\beta} + \sum_{i=1}^{n_1} \ln[1 - H_1(.)] \\
&+ \sum_{i=1}^{n_2} \ln \left[1 - \left(1 - \exp \left(-(\lambda (H_2(.))^\phi \right) \right)^{\theta_2} \right] \\
&+ n_c \ln \left[1 - \left(1 - \exp \left(-(\lambda (H_3(.))^\phi \right) \right)^{\theta_2} \right]. \tag{39}
\end{aligned}$$

The second derivatives of the log likelihood function defined in (20) with respect to the unknown parameters can be written as follows

$$\begin{aligned}
\frac{\partial^2 \ell_2}{\partial a^2} = & 3\theta_1^2 \sum_{i=1}^{n_1} \ln \left(1 - \exp \left(-(\lambda t_{i1})^\phi \right) \right) + \theta_1 \sum_{i=1}^{n_1} \ln \left(\frac{H_1(.)}{1 - \exp \left(-(\lambda t_{i1})^\phi \right)} \right) \\
& - (\beta - 1) \sum_{i=1}^{n_1} \frac{\theta_1 \ln \left(1 - \exp \left(-(\lambda t_{i1})^\phi \right) \right) H_1(.)}{[1 - H_1(.)]} \{1 + \theta_1 \\
& \ln \left(1 - \exp \left(-(\lambda t_{i1})^\phi \right) \right) - \frac{\theta_1 \ln \left(1 - \exp \left(-(\lambda t_{i1})^\phi \right) \right) H_1(.)}{[1 - H_1(.)]} \} \\
& + \theta_2 \sum_{i=1}^{n_2} \ln \left(1 - \exp \left(-(\lambda H_2(.))^\phi \right) \right)^2 - (\beta - 1) \sum_{i=1}^{n_2} \theta_2 \\
& \frac{(1 - \exp - (\lambda H_2(.)))^{\theta_2} (\ln (1 - \exp - (\lambda H_2(.))))^2}{[1 - \exp - (\lambda H_2(.))^{\theta_2}]} \{ \ln \left(1 - \exp - (\lambda H_2(.))^\phi \right) \\
& \theta_2 + 1 - \frac{\theta_2 (1 - \exp - (\lambda H_2(.)))^{\theta_2} (\ln (1 - \exp - (\lambda H_2(.))))^2}{[1 - (\exp - (\lambda H_2(.))^{\theta_2}]} \} \\
& - \frac{\beta n_c \theta_2 (\exp - (\lambda H_3(.)))^{\theta_2} (\ln (\exp - (\lambda H_3(.))))^2}{[1 - \exp - (\lambda H_3(.))^{\theta_2}]} \\
& \{1 + \theta_2 (\ln (\exp - (\lambda H_3(.))))\} \\
& - \frac{\theta_2 (1 - \exp - (\lambda H_2(.)))^{\theta_2} (\ln (1 - \exp - (\lambda H_2(.))))^2}{[1 - (\exp - (\lambda H_2(.))^{\theta_2}]} \}, \tag{40}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial b^2} = & 3\theta_1^2 x_1^2 \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) + \theta_1 x_1^2 \ln \left(\frac{H_1(.)}{1 - \exp(-(\lambda t_{i1})^\phi)} \right) \\
& - (\beta - 1) \sum_{i=1}^{n_1} \frac{x_1^2 \theta_1 \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) \left(1 - \exp(-(\lambda t_{i1})^\phi) \right)^{\theta_1}}{(1 - H_1(.))} \{1 + \theta_1 \\
& \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) - \frac{\theta_1 \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) \left(1 - \exp(-(\lambda t_{i1})^\phi) \right)^{\theta_1}}{[1 - H_1(.)]} \} \\
& + (\phi - 1) \sum_{i=1}^{n_2} \exp(-b(x_1 - x_2)) \ln(1 - H_4(.))^{\frac{1}{\phi}-1} \left(\ln \left(1 - \exp(-(\lambda \tau)^\phi) \right) \right)^2 \\
& \frac{(x_2 - x_1)^2 H_4(.)}{\lambda \phi (H_2(.)) (1 - H_4(.))} \left\{ 1 + \frac{\exp(-b(x_1 - x_2))}{\ln \left(1 - \exp(-(\lambda \tau)^\phi) \right)} - \frac{H_4(.)}{(1 - H_4(.))} \right. \\
& \left. - \frac{\ln(1 - H_4(.))^{\frac{1}{\phi}-1} H_4(.)}{\lambda \phi (H_2(.)) (1 - H_4(.))} + \frac{H_4(.) \left(\frac{1}{\phi} - 1 \right)}{(1 - H_4(.)) \ln(1 - H_4(.))} \right\} \\
& - \sum_{i=1}^{n_2} \exp(2b(x_1 - x_2)) \ln(1 - H_4(.))^{\frac{1}{\phi}-1} (H_4(.)) (x_1 - x_2)^2 \\
& \frac{\left(\ln \left(1 - \exp(-(\lambda \tau)^\phi) \right) \right)^2 (\lambda H_2(.))^{\phi-1}}{(1 - H_4(.))} \left\{ 1 - \frac{\exp(-b(x_1 - x_2))}{\ln \left(1 - \exp(-(\lambda \tau)^\phi) \right)} \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{H_4(.)}{(1-H_4(.))} + \frac{\ln(1-H_4(.))^{\frac{1}{\phi}-1} H_4(.)}{\phi(\lambda H_2(.))^2(1-H_4(.))} + \frac{H_4(.)\left(\frac{1}{\phi}-1\right)}{(1-H_4(.))\ln(1-H_4(.))}\} \\
& + \sum_{i=1}^{n_2} \left\{ \exp(a + bx_2 - 1) \ln\left(1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right) x_2^2 - H_4(.) \right. \\
& \exp(a + 2bx_1 - bx_2 - 1) \exp\left(-(\lambda H_2(.))^{\phi}\right) \ln(1-H_4(.))^{\frac{1}{\phi}-1} \\
& \frac{\left(\ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right)\right)^2 (x_1 - x_2)^2 (\lambda H_2(.))^{\phi-1}}{\left(1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right) [1 - H_4(.)]} \left\{ \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right) \right. \\
& \exp b(x_2 - x_1) + \frac{(\lambda H_2(.))^{\phi-1} \ln(1-H_4(.))^{\frac{1}{\phi}-1} \exp\left(-(\lambda H_2(.))^{\phi}\right) H_4(.)}{\left(1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right) [1 - H_4(.)]} \\
& + 1 + \frac{\ln(1-H_4(.))^{\frac{1}{\phi}-1} H_4(.) (\lambda H_2(.))^{\phi-1} - H_4(.) + \left(\frac{1}{\phi}-1\right) \ln(1-H_4(.))^{-1}}{[1 - H_4(.)]} \\
& + \frac{\ln(1-H_4(.))^{\frac{1}{\phi}-1} (\phi-1) H_4(.) (\lambda H_2(.))^{-1}}{\phi [1 - H_4(.)]} + 2x_2(x_2 - x_1) \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right)\} \\
& + (\beta-1) \sum_{i=1}^{n_2} x_2^2 \exp(a + bx_2 - 1) \ln\left[1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right] \\
& - \exp\left(-(\lambda H_2(.))^{\phi}\right) \exp(a + 2bx_2 - bx_1 - 1) (\lambda H_2(.))^{\phi-1} \\
& \frac{\ln[1-H_4(.)]^{\frac{1}{\phi}-1} \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right)^2 (x_1 - x_2)^2 H_4(.)}{[1 - H_4(.)] \left[1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right]} \\
& \{1 + \exp b(x_2 - x_1) \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right) - \exp\left(-(\lambda H_2(.))^{\phi}\right) H_4(.) \\
& \frac{\ln[1-H_4(.)]^{\frac{1}{\phi}} (\lambda H_2(.))^{\phi-1}}{[1 - H_4(.)] \left[1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right]} + x_2(x_1 - x_2)^{-1} \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right) \\
& + \frac{\ln[1-H_4(.)]^{\frac{1}{\phi}-1} (\lambda H_2(.))^{\phi-1} - H_4(.) (1 - (\phi-1) \ln[1-H_4(.)]^{-1})}{[1 - H_4(.)]} \\
& \frac{(\phi-1) H_4(.) (\lambda H_2(.))^{-1} \ln[1-H_4(.)]^{\frac{1}{\phi}-1}}{\phi [1 - H_4(.)]}\} + \beta n_c x_2^2 \exp(a + bx_2 - 1) \\
& \ln\left[1 - \exp\left(-(\lambda H_3(.))^{\phi}\right)\right] - \exp\left(-(\lambda H_3(.))^{\phi}\right) (\lambda H_3(.))^{\phi-1} H_4(.) \\
& \frac{\ln[1-H_4(.)]^{\frac{1}{\phi}-1} \left[\ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right)\right]^2 (x_1 - x_2)^2 \exp(a + 2bx_2 - bx_1 - 1)}{[1 - H_4(.)] \left[1 - \exp\left(-(\lambda H_3(.))^{\phi}\right)\right]} \\
& \{1 + \exp b(x_2 - x_1) \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right) - \exp\left(-(\lambda H_2(.))^{\phi}\right) H_4(.) \\
& \frac{\ln[1-H_4(.)]^{\frac{1}{\phi}} (\lambda H_2(.))^{\phi-1}}{[1 - H_4(.)] \left[1 - \exp\left(-(\lambda H_2(.))^{\phi}\right)\right]} + x_2(x_1 - x_2)^{-1} \ln\left(1 - \exp\left(-(\lambda \tau)^{\phi}\right)\right)
\end{aligned}$$

$$+ \frac{\ln [1 - H_4 (.)]^{\frac{1}{\phi}-1} (\lambda H_2 (.))^{\phi-1} - (1 - (\phi - 1) \ln [1 - H_4 (.)]^{-1})}{[1 - H_4 (.)]} \\ \frac{(\phi - 1) H_4 (.) (\lambda H_2 (.))^{-1} \ln [1 - H_4 (.)]^{\frac{1}{\phi}-1} H_4 (.)}{\phi [1 - H_4 (.)]}, \quad (41)$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = - \frac{(N - n_c)}{\beta^2}, \quad (42)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial a \partial b} = & 3 \sum_{i=1}^{n_1} x_1 \theta_1 \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) + \sum_{i=1}^{n_1} x_1 \theta_1 \ln (H_1 (.))^{-1} \\ & + 2 \sum_{i=1}^{n_1} x_1 \theta_1^2 \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) + (\beta - 1) \sum_{i=1}^{n_1} \theta_1 x_1 \\ & \frac{\left[\ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) \right]^2 H_1 (.)}{(1 - (H_1 (.)))} \left\{ \frac{1}{\ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right)} \right. \\ & \left. - \theta_1 (1 - (H_1 (.)))^{-1} H_1 (.) + \theta_1 \right\} + \sum_{i=1}^{n_2} x_2 \exp(a + bx_2 - 1) \\ & \ln \left(1 - \exp \left(-(\lambda H_2 (.))^\phi \right) \right) + \sum_{i=1}^{n_2} x_2 \theta_1 (x_1 - x_2) \\ & H_4 (.) \frac{(1 - H_4 (.))^{\frac{1}{\phi}-1} \ln \left(1 - \exp(-(\lambda \tau)^\phi) \right)}{\lambda \phi H_2 (.) (1 - (H_4 (.)))} + (\beta - 1) \sum_{i=1}^{n_2} \{ \theta_2 \\ & \ln \left[1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right] \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2} \\ & \{ x_2 \ln \left[1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right] \theta_2 \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2} \\ & + \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2-1} (\lambda (H_2 (.)))^\phi (H_4 (.)) (x_1 - x_2) \\ & \theta_1 \exp \left(-(\lambda (H_2 (.)))^\phi \right) \ln (1 - (H_4 (.)))^{\frac{1}{\phi}-1} \ln (1 - (H_4 (.)))^{\frac{1}{\phi}-1} \\ & \left. \frac{\theta_2}{(1 - (H_4 (.)))} \right\} \\ & - \frac{\ln \left[1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right] \theta_2 x_2 \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2}}{\left(1 - \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2} \right)} \\ & - \frac{\ln \left[1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right] x_2 \theta_2 \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2}}{\left(1 - \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2} \right)} \\ & + \left(1 - \exp \left(-(\lambda (H_2 (.)))^\phi \right) \right)^{\theta_2-1} \ln \left[1 - \exp \left(-(\lambda \tau)^\phi \right) \right] (\lambda (H_2 (.)))^{\phi-1} \end{aligned}$$

$$\begin{aligned}
& \frac{H_4(.) \theta_1 \exp\left(-(\lambda(H_2(.)))^\phi\right) \ln(1 - (H_4(.)))^{\frac{1}{\phi}-1}}{\left(1 - \left(1 - \exp\left(-(\lambda(H_2(.)))^\phi\right)\right)^{\theta_2}\right) (1 - (H_4(.)))} + \theta_1 \\
& \ln(1 - (H_4(.)))^{\frac{1}{\phi}-1} \ln\left[1 - \exp\left(-(\lambda\tau)^\phi\right)\right] (x_1 - x_2) (H_4(.)) (\lambda(H_2(.)))^{\phi-1} \\
& \frac{\left(1 - \exp\left(-(\lambda(H_2(.)))^\phi\right)\right)^{\theta_2} \exp\left(-(\lambda(H_2(.)))^\phi\right)}{\left(1 - \exp\left(-(\lambda(H_2(.)))^\phi\right)\right) (1 - (H_4(.))) \left(1 - \exp\left(-(\lambda(H_2(.)))^\phi\right)\right)^{\theta_2}} \Big\} \\
& + \beta n_c \{ \theta_2 \ln\left[1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right] \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2} \\
& \{x_2 \ln\left[1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right] \theta_2 \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2} \\
& + \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2-1} (\lambda(H_3(.)))^\phi (H_4(.)) (x_1 - x_2) \\
& \theta_1 \exp\left(-(\lambda(H_3(.)))^\phi\right) \ln(1 - (H_4(.)))^{\frac{1}{\phi}-1} \ln(1 - (H_4(.)))^{\frac{1}{\phi}-1} \\
& \frac{\Big\}}{(1 - (H_4(.)))} \\
& - \frac{\ln\left[1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right] \theta_2 x_2 \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2}}{\left(1 - \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2}\right)} \\
& - \frac{\ln\left[1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right] x_2 \theta_2^2 \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2}}{\left(1 - \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2}\right)} \\
& + \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2-1} \ln\left(1 - \exp(-(\lambda\tau)^\phi)\right) H_4(.) (\lambda(H_3(.)))^{\phi-1} \\
& \frac{\theta_1 \exp\left(-(\lambda(H_3(.)))^\phi\right) \ln(1 - (H_4(.)))^{\frac{1}{\phi}-1}}{\left(1 - \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2}\right) (1 - (H_4(.)))} + \theta_1 \\
& \ln(1 - (H_4(.)))^{\frac{1}{\phi}-1} \ln\left(1 - \exp(-(\lambda\tau)^\phi)\right) (x_1 - x_2) (H_4(.)) (\lambda(H_3(.)))^{\phi-1} \\
& \frac{\left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2} \exp\left(-(\lambda(H_3(.)))^\phi\right)}{\left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right) (1 - (H_4(.))) \left(1 - \exp\left(-(\lambda(H_3(.)))^\phi\right)\right)^{\theta_2}} \Big\}, (43)
\end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial^2 \ln L}{\partial a \partial \beta} = & - \sum_{i=1}^{n_1} \frac{\exp(a+bx_1) \ln \left(1 - \exp(-(\lambda t_{i1})^\phi) \right) (H_1(.))}{(1 - (H_1(.)))} \\
 & - \sum_{i=1}^{n_2} \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right)^{\theta_2} \\
 & \frac{\theta_2 \ln \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right)}{\left(1 - \left(1 - \exp \left(-(\lambda (H_2(.)))^\phi \right) \right)^{\theta_2} \right)} \\
 & - n_c \frac{\theta_2 \ln \left(1 - \exp \left(-(\lambda (H_3(.)))^\phi \right) \right)}{\left(1 - \left(1 - \exp \left(-(\lambda (H_3(.)))^\phi \right) \right)^{\theta_2} \right)} \\
 & \left(1 - \exp \left(-(\lambda (H_3(.)))^\phi \right) \right)^{\theta_2}.
 \end{aligned} \tag{44}$$

The asymptotic Fisher information matrix can be written in as follows:

$$\tilde{I}_2 = - \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \downarrow \left(\hat{a}_2, \hat{b}_2, \hat{\beta}_2 \right), \tag{45}$$