

On perfect commutative *EIFA*-ring.

Abstract

We consider the class $\mathcal{F}$ of endo-Artinian modules, i.e. the modules M which satisfying the descending chain condition for endomorphic images: any descending chain $Imf_1 \supset Imf_2 \supset \cdots \supset Imf_n \supset \cdots$ is stationary, where $f_i \in End(M)$. Let $\mathcal{A}$ be the class of Artinian modules. It is clear that every Artinian R -module M is endo-Artinian, so $\mathcal{A} \subset \mathcal{F}$, but the converse is not true. Indeed, $\mathbb{Q}$ is a non-Artinian $\mathbb{Z}$ -module which is endo-Artinian. The aim of this work, is to characterize commutative rings for which $\mathcal{F}$ and $\mathcal{A}$ are identical.

Keywords : Artinian; *EIFA*-ring; endo-Artinian; co-hopfian; perfect.

Introduction

The pioneering work on perfect rings was done by Hyman Bass [1]. A ring R is left perfect (right perfect) in case each of its left (right) modules has a projective cover. The descending chain condition on ideals was introduced by Artin. Rings in which any descending chain of ideals is finite are called

Artinian. We say that a module is Artinian if any descending chain of submodules is stationary. In fact, the finiteness of the dimension of a vector space E is equivalent to the Artinian condition, then we have this following property: any injective endomorphism is surjective. This condition has inspired, the concept of co-hopfian modules. Since then, we focused on other issues relating to co-hopfian properties, including: the characterization of rings over which every co-hopfian module is Artinian. A. Kaidi introduced in 2009 in [2] the class of endo-Noetherian (resp. endo-Artinian) modules. Endo-Artinian concept is the dual property of endo-Noetherian notion. Recently in 2013, M.A.Ndiaye and C.T. Gueye characterized partially $EKFN$ -rings in [3]. The purpose of this paper is to characterize rings on which an endo-Artinian module M is Artinian. Such rings will be called $EIFA$ -rings. We prove that a perfect commutative ring is an $EIFA$ -ring if and only if it is an Artinian principal ideal ring. Our paper is structured as follows: the first section covers the properties and basic aspects of certain classes of modules and rings. In the second section, we characterize perfect commutative $EIFA$ -rings.

1 Preliminaries

The rings considered in this section are associative with unit. Unless otherwise mentioned, all the modules considered are left unitary modules. Given a ring R , let's denote the left R -module R by ${}_R R$.

Definition 1.1 *An R -module M is said co-hopfian if any injective homomorphism from M to M is surjective.*

If ${}_R R$ is co-hopfian, then we say that R is a co-hopfian ring.

Definition 1.2 *An I -ring R is a ring such that every co-hopfian R -module is Artinian.*

Proposition 1.1 ([4], P.250)

Let R be an Artinian ring having at least one non-principal ideal. Then there exists an R -module M which is not finitely generated.

Proposition 1.2 ([5], P.255-256)

Let M be a module over a left Artinian ring R . Then the following properties of M are equivalent:

1. M is finitely generated;
2. M is Artinian.

Definition 1.3 An R -module M is called endo-Artinian if any descending chain of endomorphic images $Imf_1 \supset Imf_2 \supset \cdots \supset Imf_n \supset \cdots$ stabilizes, where f_i 's are endomorphisms of M , i.e. there exists a positive integer n such that $Imf_n = Imf_{n+1}$.

Proposition 1.3 [2]

Let R be a ring. Then the following statements are equivalent:

1. R is endo-Artinian;
2. R satisfies the descending chain condition for principal left ideals;
3. R is right perfect.

Theorem 1.4 Fundamental theorem of isomorphism ([6] P.54)

Let $f : M \rightarrow N$ be an R -homomorphism. Then, f gives rise to an isomorphism of R -modules $\tilde{f} : M/Ker f \xrightarrow{\sim} Imf$.

2 The main results

Let R be an associative and commutative ring with identity $1 \neq 0$.

Definition 2.1 R is said to be an EIFA-ring¹ if every endo-Artinian R -module is Artinian.

Example 2.1 A semisimple ring and an Artinian ring with principal ideal, are EIFA-rings.

Proposition 2.1 Let R be a perfect ring. If R is an EIFA-ring then R is Artinian.

Proof: Assume that R is an EIFA-ring. Since R is perfect, then by Proposition 1.3, R is endo-Artinian, so R is Artinian.

Theorem 2.2 Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C -endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$. Then every non-zero C -endomorphism F of M commuting with σ is such that either $ImF \simeq M/rM$, or is an epimorphism.

¹EIFA is derived from the two concepts namely Endo Image Finite and Artinian.

Proof: Propositions 3.4 [3] and 1.4 imply that if $F(1)$ is not invertible, then $M/rM \simeq \text{Im}F$. Conversely, if $F(1)$ is invertible then for any element $m \in M$, we have $F(rmF(1)^{-1}) = rmF(1)^{-1}.F(1) = rm$. Hence $F(rM) = rM$ (*). Let $m \in M$, then there exists an element $m' \in M$, such that $F(rm') = rm$ by (*). Thus $r(F(m') - m) = 0$. It follows that $(F(m') - m) \in rM$. As $rM \subset \text{Im}F$, by (*), we deduce that $m \in \text{Im}F$. It follows that F is an epimorphism.

Theorem 2.3 *Let R be an Artinian ring having at least one non-principal ideal. Then there exists an endo-Artinian R -module M which is non-Artinian.*

Proof: By Proposition 1.1, there exists an R -module M which is not finitely generated. Then M is non-Artinian. Let $\text{Im}f_1 \supset \text{Im}f_2 \supset \cdots \supset \text{Im}f_n \supset \cdots$ be a descending chain for R -endomorphisms of M , where $f_i \in \text{End}_R(M)$. We know that R -endomorphisms of M are C -endomorphisms of M which commute with σ , then any descending chain for R -endomorphisms of M stabilizes. Thus M is endo-Artinian. $\square$

Theorem 2.4 Characterization theorem

Let R be a perfect ring, the following conditions are equivalent:

1. R is an EIFA-ring;
2. R is an Artinian principal ideal ring.

Proof:

- Assume that R is an EIFA-ring. Since R is perfect, then by Proposition 2.1, R is Artinian. Assume that R has at least one non-principal ideal, so there exists an endo-Artinian module which is non-Artinian by Theorem 2.3, contradiction. Thus R is an Artinian principal ideal ring.
- Assume that R is an Artinian principal ideal ring. Then, R is an EIFA-ring. $\square$

References

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