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Free divisors versus stability and coincidence thresholds

Short Research Article

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Abstract

We show that there is an unexpected relation between free divisors and stability and coincidence thresholds for projective hypersurfaces.

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1 Introduction

Let $S = \mathbb{C}[x_0, ..., x_n]$ be the graded ring of polynomials in $x_0, , ..., x_n$ with complex coefficients and denote by S_r the vector space of homogeneous polynomials in S of degree r. For any polynomial $f \in S_r$ we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives $f_0, ..., f_n$ of f with respect to $x_0, ..., x_n$.

The Hilbert-Poincaré series of a graded S-module M of finite type is defined by

$$HP(M)(t) = \sum_{k \geq 0} \dim M_k \cdot t^k \tag{1.1}$$

and it is known, to be a rational function of the form

$$HP(M)(t) = \frac{P(M)(t)}{(1-t)^{n+1}} = \frac{Q(M)(t)}{(1-t)^d}. \tag{1.2}$$

For any polynomial $f \in S_r$ we define the corresponding graded *Milnor* (or *Jacobian*) *algebra* by

$$M = M(f) = S/J_f. \tag{1.3}$$

Smooth hypersurfaces of degree d have all the same Hilbert-Poincaré series, namely

$$HP(M(f))(t) = \frac{(1-t^{d-1})^{n+1}}{(1-t)^{n+1}} = (1 + t + t^2 + \ldots + t^{d-2})^{n+1}. \tag{1.4}$$

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As soon as the hypersurface $V(f)$ acquires some singularities, the series $HP(M(f))$ is an infinite sum.

Let $\tau(V(f)) = \sum_{j=1,p} \tau(V(f), a_j)$ be the global Tjurina number of the hypersurface $V(f)$. In particular, the Hilbert polynomial $H(M(f))$ is constant and this constant is $\tau(V(f))$.

For a hypersurface $D : f = 0$ in $\mathbb{P}^n$ with isolated singularities we recall the following invariants, introduced in Dimca and Sticlaru (2012).

Definition 1.1. (i) The *coincidence threshold* $ct(D)$ defined as

$$ct(D) = \max\{q \ : \ \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of degree $d = \deg f$ such that $D_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$.

(ii) The *stability threshold* $st(D)$ defined as

$$st(D) = \min\{q \ : \ \dim M(f)_k = \tau(D) \text{ for all } k \geq q\}$$

where $\tau(D)$ is the total Tjurina number of D, i.e. the sum of all the Tjurina numbers of the singularities of D.

We refer for the definiton of free divisors and their main properties to Simis and Tohaneanu (2012) and the extensive reference list there.

2 Main result

The main result of this note is the following.

Theorem 2.1. *Let $f \in S_d$ and set as usual $T = (n+1)(d-2)$. Assume that $st(D) < ct(D)$. Then the following hold.*

1. *The degree d is odd, $ct(D) = (T+1)/2$ and $st(D) = (T-1)/2$.*
2. *If in addition $n = 2$, then the curve $D : f = 0$ is a free divisor.*

To prove the first claim is fairly easy. Recall that the coefficients c_k of the Hilbert-Poincaré series of $M(f)$ form a (strict) log-concave sequence by Sticlaru (2013), symmetric with respect to the middle point $T/2$, i.e. $c_k = c_{T-k}$. It follows that one has strict inequalities $c_{k-1} < c_k$ for all $k \leq T/2$. This implies that $st(D) \geq T/2$ when d is even and $st(D) \geq (T-1)/2$ when d is even.

Suppose we are in the case d is even. Then $st(D) > T/2$ implies $st(D) \geq ct(D)$ as beyond $T/2$ the sequence c_k is strictly decreasing. Hence one should have $st(D) = T/2$, which implies $ct(D) = st(D)$.

Suppose now that we are in the case d is odd. Then again $st(D) > (T-1)/2$ implies $st(D) \geq ct(D)$ as beyond $T/2$ the sequence c_k is strictly decreasing. Hence one should have $st(D) = (T-1)/2$, which implies $ct(D) = st(D) + 1 = (T+1)/2$.

The proof of the second claim is much more subtle.

Recall that D is a free divisor if and only if the Jacobian ideal J_f is a perfect ideal, see for instance the bottom of page 1 in Simis and Tohaneanu (2012). Next, the Corollary 1.2 in Simis and Tohaneanu (2012) implies the following fact: if the Castelnuovo-Mumford regularity $reg(M(f))$ satisfies

$$reg(M(f)) \leq (3d-7)/2, \tag{2.1}$$

then J_f is perfect and one has $reg(M(f)) \leq (3d-7)/2$ with d odd.

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Using Dimca (2013), Corollary 4, we know that

$$reg(M(f)) = \max(T - ct(D), sat(J_f) - 1),$$

where for any homogeneous ideal I, we let $\widehat{I}$ be the associated saturated ideal and we consider the graded artinian S-module

$$SD(I) = \frac{\widehat{I}}{I}, \tag{2.2}$$

called the *saturation defect module* of I and the *saturation threshold* $sat(I)$ defined as

$$sat(I) = \min\{q \ : \ \dim I_k = \dim \widehat{I}_k \text{ for all } k \geq q\}. \tag{2.3}$$

Now, it follows from (1) above that $T - ct(D) = (T-1)/2 = (3d-7)/2$, so we need to control $sat(J_f)$. Corollary 2 in Dimca (2013) tells us that

$$sat(J_f) \leq max(T - ct(D), st(D)),$$

and hence in our situation $sat(J_f) - 1 \leq (T+1)/2 - 1 = (T-1)/2 = (3d-7)/2$. Hence D is a free divisor. This completes the proof of Theorem 2.1.

Remark 2.1. Note that in fact D is a free divisor if and only if $J_f = \widehat{J}_f$, see top of page 6 in Simis and Tohaneanu (2012), and hence if and only if $sat(J_f) = 0$. In particular, for a free divisor one has

$$reg(M(f)) = T - ct(D).$$

3 Examples of hypersurfaces with $st(D) \leq ct(D)$

Example 3.1. *(a triangle and a conic+tangent) The example $D : f = xyz = 0$ has $sat(J_f) = 0$, $2 = ct(D) > st(D) = 1$. The conic plus tangent $D : f = x(xz + y^2)$ has the same invariants and Hilbert-Poincaré series $HP(M(f))(t) = 1 + 3(t + \cdots$.*

Example 3.2. *(two free line arrangements in $\mathbb{P}^2$) Consider the line arrangement*

$$D(\mathcal{A}_3) : f = (x^2 - y^2)(y^2 - z^2)(x^2 - z^2) = 0.$$

The Hilbert-Poincaré series is:

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19(t^6 + \cdots .$$

Then $ct(D(\mathcal{A}_3)) = st(D(\mathcal{A}_3)) = 6$ and the curve $D(\mathcal{A}_3)$ is free, since one can verify by Singular the vanishing $sat(J_f) = 0$. Similarly, for the line arrangement

$$D : g = xyzf = xyz(x^2 - y^2)(y^2 - z^2)(x^2 - z^2) = 0,$$

one has $ct(D) = 10 < st(D) = 11$ and the Hilbert-Poincaré series is:

$$1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 + 36t^7 + 42t^8 + 46t^9 + 48t^{10} + 49(t^{11} + \cdots .$$

So this curve is free for the same reason as above.

Example 3.3. *(a sequence of free divisors constructed in Simis and Tohaneanu (2012), Prop. 2.2.) Consider the sequence of free divisors*

$$D_d : f = y^{d-1}z + x^d + x^2y^{d-2} = 0$$

for $d \geq 5$ Then, for $d = 5$ we have $st(D) = 4 < 5 = ct(D)$, hence we are in the setting of Theorem 2.1 (ii). For $d = 6$, one has $st(D) = 6 = ct(D)$, while for $d > 6$ one has $st(D) > ct(D)$. So Theorem 2.1 (2) has no converse.

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Example 3.4. *(Simis' free irreducible sextic, see Simis (2005)) Consider the sextic $D : f = 0$, where*

$$f = 4(x^2 + y^2 + xz)^3 - 27(x^2 + y^2)^2 z^2.$$

Then D has three E_6 singularities and one node A_1, hence $\tau(D) = 19$. Moreover one has $ct(D) = st(D) = 6$ since the Hilbert-Poincaré series of $M(f)$ can be computed using for instance Singular and we get

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19(t^6 + \cdots .$$

This is one of the very few irreducible free divisors in $\mathbb{P}^2$ known to this day.

Example 3.5. *(a sequence of curves C_d with $st(C_d) = ct(C_d)$.) Let $C_d : f_d = 0$ with*

$$f_d(x, y, z) = f = x^2y^2z^{d-4} + x^5z^{d-5} + y^5z^{d-5} + x^d + y^d$$

for $d \geq 5$. Then, for any $d \geq 5$, the curve C_d has a unique singularity at the point $(0 : 0 : 1)$ of type $T_{2,5,5}$ with $\tau = 10$. Then a computation by Singular suggests that $ct(D_d) = st(D_d) = 3d - 9$ (verified for $5 \leq d \leq 15$). Moreover all these divisors are not free since $sat(J_f) = 0$ (verified for $5 \leq d \leq 15$).

Example 3.6. *(a nodal hypersurface with one node) If D is a nodal hypersurface in $\mathbb{P}^n$ having exactly one node, it follows from Example 4.3 (i) in Dimca and Sticlaru (2013) that $st(D) = ct(D) = T$. For nodal curve of degree 5:*

$$C : f = x^4y + x^3y^2 + y^5 + xy^2z^2 + (x^2 + xy + y^2)z^3 = 0$$

our invariants are $ct(C) = st(C) = 9$ and the Hilbert-Poincaré series of $M(f)$ is

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + 10t^6 + 6t^7 + 3t^8 + (t^9 + \cdots .$$

4 Conclusion

For a hypersurface $D : f = 0$ in $\mathbb{P}^n$ with isolated singularities, we compare the Hilbert-Poincaré series of the corresponding graded *Milnor algebra* with the smooth case. These series coincides up to an index $ct(D)$ and Corollary 9 Choudary and Dimca (1994) ensures that from an index $st(D)$ we have stability.

The main result is Theorem 2.1, which establishes a relation between free divisors and stability $st(D)$ and coincidence thresholds $ct(D)$.

In the vast majority of cases one has $ct(D) \leq st(D)$, see for instance the examples given above for the case of equality. The opposite inequality $st(D) < ct(D)$ may occur but it is extremely rare.

Recall that one of the main question in this theory is the construction of irreducible free divisor of arbitrarily high degree. Simis' celebrated example in degree $d = 6$ is discussed in Example 3.4. The main conclusion of the given examples is that Theorem 2.1 (2) has no converse, even in the class of free line arrangements. Moreover, the examples show that there are a lot of free and not free divisors satisfying the equality $st(D) = ct(D)$.

All the computations for this paper were made by Singular package, see Decker et al. (2013).

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