

THE CONVOLUTION SUMS $\sum_{m=1}^{n-1} m^k \sigma_i(m) \sigma_j(n-m)$

ABSTRACT. In this paper we obtain some various convolution sum formulae especially, focusing on the form

$$\sum_{m=1}^{n-1} m^k \sigma_i(m) \sigma_j(n-m)$$

for $k \in \mathbb{N}$ ($1 \leq k \leq 5$) and an odd positive integer i, j . Furthermore, we obtain some identities related to the convolution sums.

1. INTRODUCTION

For $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define necessary divisor function and $\Delta(q)$ which appear in many areas of number theory:

$$(1) \quad \sigma_s(n) = \sum_{d|n} d^s, \quad \Delta(q) := \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are given by

$$(2) \quad L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$(3) \quad M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$(4) \quad N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

in [1, p. 318]. Lahiri [4, p. 149] has derived the following identities from the work of Ramanujan [6] :

$$(5) \quad L^2(q) = 1 - 288 \sum_{n=1}^{\infty} n \sigma_1(n) q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$(6) \quad L^3(q) = 1 - 1728 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 2160 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

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$$(7) \quad L^4(q) = 1 - 6912 \sum_{n=1}^{\infty} n^3 \sigma_1(n) q^n + 10368 \sum_{n=1}^{\infty} n^2 \sigma_3(n) q^n - 4032 \sum_{n=1}^{\infty} n \sigma_5(n) q^n \\ + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(8) \quad L^5(q) = 1 - 20736 \sum_{n=1}^{\infty} n^4 \sigma_1(n) q^n + 34560 \sum_{n=1}^{\infty} n^3 \sigma_3(n) q^n - 17280 \sum_{n=1}^{\infty} n^2 \sigma_5(n) q^n \\ + 3600 \sum_{n=1}^{\infty} n \sigma_7(n) q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

$$(9) \quad M^2(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(10) \quad M^3(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n + \frac{432000}{691} \sum_{n=1}^{\infty} \tau(n) q^n,$$

$$(11) \quad L(q)M(q) = 1 + 720 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$(12) \quad L^2(q)M(q) = 1 + 1728 \sum_{n=1}^{\infty} n^2 \sigma_3(n) q^n - 2016 \sum_{n=1}^{\infty} n \sigma_5(n) q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(13) \quad L(q)M^2(q) = 1 + 720 \sum_{n=1}^{\infty} n \sigma_7(n) q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

$$(14) \quad L(q)N(q) = 1 - 1008 \sum_{n=1}^{\infty} n \sigma_5(n) q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(15) \quad M(q)N(q) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

Here for $e, f, n \in \mathbb{N}$ we follow the definition of the convolution sum $S_{e,f}(n)$ given by

$$(16) \quad S_{e,f}(n) := \sum_{m=1}^{n-1} \sigma_e(m) \sigma_f(n-m),$$

which is found in [2, p. 39]. Ramanujan [6] and Lahiri [4], [5] have shown that $S_{e,f}$ can be expressed as :

$$\begin{aligned}
 S_{1,1}(n) &= \frac{5}{12} \sigma_3(n) + \frac{(1-6n)}{12} \sigma_1(n), \\
 S_{1,3}(n) &= \frac{7}{80} \sigma_5(n) + \frac{(1-3n)}{24} \sigma_3(n) - \frac{1}{240} \sigma_1(n), \\
 S_{1,5}(n) &= \frac{5}{126} \sigma_7(n) + \frac{(1-2n)}{24} \sigma_5(n) + \frac{1}{504} \sigma_1(n), \\
 S_{3,3}(n) &= \frac{1}{120} \sigma_7(n) - \frac{1}{120} \sigma_3(n), \\
 S_{1,7}(n) &= \frac{11}{480} \sigma_9(n) + \frac{(2-3n)}{48} \sigma_7(n) - \frac{1}{480} \sigma_1(n), \\
 S_{3,5}(n) &= \frac{11}{5040} \sigma_9(n) - \frac{1}{240} \sigma_5(n) + \frac{1}{504} \sigma_3(n), \\
 S_{1,9}(n) &= \frac{455}{30404} \sigma_{11}(n) + \frac{(5-6n)}{120} \sigma_9(n) + \frac{1}{264} \sigma_1(n) - \frac{36}{3455} \tau(n), \\
 S_{3,7}(n) &= \frac{91}{110560} \sigma_{11}(n) - \frac{1}{240} \sigma_7(n) - \frac{1}{480} \sigma_3(n) + \frac{15}{2764} \tau(n), \\
 S_{5,5}(n) &= \frac{65}{174132} \sigma_{11}(n) + \frac{1}{252} \sigma_5(n) - \frac{3}{691} \tau(n), \\
 S_{1,11}(n) &= \frac{691}{65520} \sigma_{13}(n) + \frac{(1-n)}{24} \sigma_{11}(n) - \frac{691}{65520} \tau(n), \\
 S_{3,9}(n) &= \frac{1}{2640} \sigma_{13}(n) - \frac{1}{240} \sigma_9(n) + \frac{1}{264} \sigma_3(n), \\
 S_{5,7}(n) &= \frac{1}{10080} \sigma_{13}(n) + \frac{1}{504} \sigma_7(n) - \frac{1}{480} \sigma_5(n).
 \end{aligned}$$

Also for $e, f, k, m, n \in \mathbb{N}$ we define

$$\begin{aligned}
 I_{m,e,f}^k(n) &:= \sum_{m=1}^{n-1} m^k \sigma_e(m) \sigma_f(n-m), \\
 (17) \quad I_{m,e,f}(n) &:= I_{m,e,f}^1(n) = \sum_{m=1}^{n-1} m \sigma_e(m) \sigma_f(n-m)
 \end{aligned}$$

and we have already obtained :

Proposition 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a) (See [7, p. 155])

$$I_{m,1,1}(n) = \frac{1}{24} n \{5\sigma_3(n) - (6n-1)\sigma_1(n)\},$$

(b) (See [7, p. 157])

$$\begin{aligned}
 I_{m,1,3}(n) &= \frac{7}{240} n \sigma_5(n) - \frac{1}{40} n^2 \sigma_3(n) - \frac{1}{240} n \sigma_1(n), \\
 I_{m,3,1}(n) &= \frac{7}{120} n \sigma_5(n) + \left(\frac{1}{24} n - \frac{1}{10} n^2 \right) \sigma_3(n),
 \end{aligned}$$

(c) (See [3, Theorem 3.3, Theorem 3.1], [7, p. 155])

$$I_{m,1,5}(n) = \frac{1}{504} n \{5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n)\},$$

$$I_{m,3,3}(n) = \frac{1}{240} n \{\sigma_7(n) - \sigma_3(n)\},$$

$$I_{m,5,1}(n) = \frac{1}{168} n \{5\sigma_7(n) - (12n - 7)\sigma_5(n)\},$$

(d) (See [3, Theorem 3.8 (a), (b), Theorem 3.7, Theorem 3.5])

$$I_{m,1,7}(n) = \frac{1}{7200} \{33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n)\},$$

$$I_{m,3,5}(n) = \frac{1}{12600} \{11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n)\},$$

$$I_{m,5,3}(n) = \frac{1}{8400} \{11n\sigma_9(n) - 35n\sigma_5(n) + 24\tau(n)\},$$

$$I_{m,7,1}(n) = \frac{1}{1800} \{33n\sigma_9(n) - 25n(4n - 3)\sigma_7(n) - 8\tau(n)\}.$$

In this paper we mainly focus on obtaining the convolution sum formulae as :

Theorem 1.2. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned} I_{m,1,5}^2(n) &= \sum_{m=1}^{n-1} m^2 \sigma_1(m) \sigma_5(n-m) \\ &= \frac{1}{3024} \{10n^2 \sigma_7(n) - 9n^3 \sigma_5(n) + 6n^2 \sigma_1(n) - 7\tau(n)\}, \end{aligned}$$

(b)

$$I_{m,3,3}^2(n) = \sum_{m=1}^{n-1} m^2 \sigma_3(m) \sigma_3(n-m) = \frac{1}{2160} \{5n^2 \sigma_7(n) - 9n^2 \sigma_3(n) + 4\tau(n)\},$$

(c)

$$\begin{aligned} I_{m,5,1}^2(n) &= \sum_{m=1}^{n-1} m^2 \sigma_5(m) \sigma_1(n-m) \\ &= \frac{1}{432} \{10n^2 \sigma_7(n) - 9n^2 (3n - 2) \sigma_5(n) - \tau(n)\}, \end{aligned}$$

(d)

$$\begin{aligned} I_{m,1,7}^2(n) &= \sum_{m=1}^{n-1} m^2 \sigma_1(m) \sigma_7(n-m) \\ &= \frac{1}{7200} n \{9n\sigma_9(n) - 10n^2\sigma_7(n) - 15n\sigma_1(n) + 16\tau(n)\}, \end{aligned}$$

(e)

$$I_{m,3,5}^2(n) = \sum_{m=1}^{n-1} m^2 \sigma_3(m) \sigma_5(n-m) = \frac{1}{2520} n \{n\sigma_9(n) + 5n\sigma_3(n) - 6\tau(n)\},$$

(f)

$$I_{m,5,3}^2(n) = \sum_{m=1}^{n-1} m^2 \sigma_5(m) \sigma_3(n-m) = \frac{1}{1200} n \{n \sigma_9(n) - 5n \sigma_5(n) + 4\tau(n)\},$$

(g)

$$\begin{aligned} I_{m,7,1}^2(n) &= \sum_{m=1}^{n-1} m^2 \sigma_7(m) \sigma_1(n-m) \\ &= \frac{1}{600} n \{9n \sigma_9(n) - 5n(6n-5) \sigma_7(n) - 4\tau(n)\}. \end{aligned}$$

Moreover we calculate

$$\begin{aligned} I_{m,5,1}^3(n) &= \sum_{m=1}^{n-1} m^3 \sigma_5(m) \sigma_1(n-m) \\ &= \frac{1}{216} \{4n^3 \sigma_7(n) - 3n^3(4n-3) \sigma_5(n) - n\tau(n)\}, \\ I_{m,3,1}^4(n) &= \sum_{m=1}^{n-1} m^4 \sigma_3(m) \sigma_1(n-m) \\ &= \frac{1}{288} \{7n^4 \sigma_5(n) - 6n^4(3n-2) \sigma_3(n) - n\tau(n)\}, \\ I_{m,1,1}^5(n) &= \sum_{m=1}^{n-1} m^5 \sigma_1(m) \sigma_1(n-m) \\ &= \frac{1}{336} \{15n^5 \sigma_3(n) - 14n^5(2n-1) \sigma_1(n) - n\tau(n)\} \end{aligned}$$

(see Theorem 2.8 (e), Theorem 2.10 (c), and Theorem 2.12). And in Section 2, we obtain some identities related to the convolution sums, for example,

Lemma 1.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\begin{aligned} L^7(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n + \frac{458640}{691} \sum_{n=1}^{\infty} n \sigma_{11}(n) q^n - \frac{36288}{5} \sum_{n=1}^{\infty} n^2 \sigma_9(n) q^n \\ &\quad + 40320 \sum_{n=1}^{\infty} n^3 \sigma_7(n) q^n - 120960 \sum_{n=1}^{\infty} n^4 \sigma_5(n) q^n + 186624 \sum_{n=1}^{\infty} n^5 \sigma_3(n) q^n \\ &\quad - \frac{497664}{5} \sum_{n=1}^{\infty} n^6 \sigma_1(n) q^n - \frac{4608}{3455} \sum_{n=1}^{\infty} n \tau(n) q^n. \end{aligned}$$

Finally, by using the above results we can easily obtain the following theorem :

Theorem 1.4. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_1(b)\sigma_1(c) = \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k\sigma_1(k)\sigma_1(m-k)\sigma_1(n-m) \\ = \frac{1}{576} n \{7\sigma_5(n) - 10(3n-1)\sigma_3(n) + (24n^2 - 12n + 1)\sigma_1(n)\},$$

(b)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_3(b)\sigma_1(c) = \frac{1}{5760} n \{5\sigma_7(n) - (18n-7)\sigma_5(n) \\ + (12n^2 - 6n - 5)\sigma_3(n) + (6n-1)\sigma_1(n)\},$$

(c)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_3(a)\sigma_1(b)\sigma_1(c) = \frac{1}{2880} n \{5\sigma_7(n) - 2(12n-7)\sigma_5(n) \\ + (24n^2 - 24n + 5)\sigma_3(n)\},$$

(d)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_5(b)\sigma_1(c) = \frac{1}{60480} \{11n\sigma_9(n) - 25n(2n-1)\sigma_7(n) \\ + 15n^2(3n-2)\sigma_5(n) + 25n\sigma_3(n) \\ - 5n(6n-1)\sigma_1(n) - \tau(n)\},$$

(e)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_3(a)\sigma_3(b)\sigma_1(c) = \frac{1}{432000} \{33n\sigma_9(n) - 25n(4n-3)\sigma_7(n) \\ - 105n\sigma_5(n) + 15n(12n-5)\sigma_3(n) - 8\tau(n)\},$$

(f)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_5(a)\sigma_1(b)\sigma_1(c) = \frac{1}{60480} \{33n\sigma_9(n) - 50n(4n-3)\sigma_7(n) \\ + 15n(18n^2 - 24n + 7)\sigma_5(n) + 2\tau(n)\},$$

(g)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_7(b)\sigma_1(c) = \frac{1}{119404800} \{6825n\sigma_{11}(n) - 2073n(18n-11)\sigma_9(n) \\ + 6910n^2(6n-5)\sigma_7(n) - 51825n\sigma_3(n) \\ + 10365n(6n-1)\sigma_1(n) - 8(2667n - 2764)\tau(n)\},$$

(h)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a \sigma_3(a) \sigma_5(b) \sigma_1(c) = \frac{1}{208958400} \{ 2275n \sigma_{11}(n) - 691n(12n-11) \sigma_9(n) \\ + 24185n \sigma_5(n) - 3455n(12n-5) \sigma_3(n) \\ + 36(647n-691) \tau(n) \},$$

(i)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a \sigma_5(a) \sigma_3(b) \sigma_1(c) = \frac{1}{139305600} \{ 2275n \sigma_{11}(n) - 691n(12n-11) \sigma_9(n) \\ - 17275n \sigma_7(n) + 3455n(12n-7) \sigma_5(n) \\ - 24(757n-691) \tau(n) \},$$

(j)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a \sigma_7(a) \sigma_1(b) \sigma_1(c) = \frac{1}{29851200} \{ 6825n \sigma_{11}(n) - 4146n(12n-11) \sigma_9(n) \\ + 3455n(24n^2-40n+15) \sigma_7(n) \\ + 8(1479n-1382) \tau(n) \}.$$

2. THE VARIOUS CONVOLUTION SUMS MULTIPLYING m^k FOR A POSITIVE INTEGER k

We obtain the similar results as Proposition 1.1 in the following theorem and in Theorem 2.4 :

Theorem 2.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$I_{m,3,7}(n) = \frac{1}{331680} n \{ 91 \sigma_{11}(n) - 691 \sigma_3(n) + 600 \tau(n) \},$$

(b)

$$I_{m,5,5}(n) = \frac{1}{348264} n \{ 65 \sigma_{11}(n) + 691 \sigma_5(n) - 756 \tau(n) \},$$

(c)

$$I_{m,7,3}(n) = \frac{1}{165840} n \{ 91 \sigma_{11}(n) - 691 \sigma_7(n) + 600 \tau(n) \}.$$

Proof. (a) By (9) and (11) we have

(18)

$$\begin{aligned}
 L(q)M^3(q) &= M^2(q) \cdot L(q)M(q) \\
 &= \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n\right) \left(1 + 720 \sum_{m=1}^{\infty} m\sigma_3(m)q^m - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^m\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{720N\sigma_3(N) - 504\sigma_5(N) + 480\sigma_7(N) \right. \\
 &\quad \left. + 480 \cdot 720 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m\sigma_3(m) - 480 \cdot 504 \sum_{m=1}^{N-1} \sigma_7(N-m)\sigma_5(m)\right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \{720N\sigma_3(N) - 504\sigma_5(N) + 480\sigma_7(N) + 480 \cdot 720 \cdot I_{m,3,7}(N) \\
 &\quad - 480 \cdot 504 \cdot S_{5,7}(N)\} q^N.
 \end{aligned}$$

Also, by (3) and (13), we obtain

$$\begin{aligned}
 L(q)M^3(q) &= L(q)M^2(q) \cdot M(q) \\
 &= \left(1 + 720 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n\right) \left(1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m\right) \\
 &= 1 + \sum_{N=1}^{\infty} \{720N\sigma_7(N) - 264\sigma_9(N) + 240\sigma_3(N) \\
 (19) \quad &\quad + 720 \cdot 240 \sum_{m=1}^{N-1} (N-m) \sigma_7(N-m)\sigma_3(m) \\
 &\quad - 264 \cdot 240 \sum_{m=1}^{N-1} \sigma_9(N-m)\sigma_3(m)\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \{720N\sigma_7(N) - 264\sigma_9(N) + 240\sigma_3(N) + 720 \cdot 240N \cdot S_{3,7}(N) \\
 &\quad - 720 \cdot 240 \cdot I_{m,3,7}(N) - 264 \cdot 240 \cdot S_{3,9}(N)\} q^N.
 \end{aligned}$$

So equating (18) with (19) we can complete the proof.

(b) We observe that

$$I_{m,5,5}(n) = \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m) = \sum_{m=1}^{n-1} (n-m) \sigma_5(n-m)\sigma_5(m),$$

which leads that

$$I_{m,5,5}(n) = \frac{n}{2} \cdot S_{5,5}(n).$$

(c) We note that

$$\begin{aligned} I_{m,7,3}(n) &= \sum_{m=1}^{n-1} m \sigma_7(m) \sigma_3(n-m) = \sum_{m=1}^{n-1} (n-m) \sigma_7(n-m) \sigma_3(m) \\ &= n \cdot S_{3,7}(n) - I_{m,3,7}(n). \end{aligned}$$

Thus we use Theorem 2.1 (a).

□

Theorem 2.2. For $q \in \mathbb{C}$ with $|q| < 1$, we have

$$L(q)M^3(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n + \frac{65520}{691} \sum_{n=1}^{\infty} n \sigma_{11}(n) q^n + \frac{432000}{691} \sum_{n=1}^{\infty} n \tau(n) q^n.$$

Proof. Insert Theorem 2.1 (a) into (18).

□

And we can see the following proposition :

Proposition 2.3. (See [3, Lemma 3.6, Lemma 3.4]) For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$\begin{aligned} L^2(q)M^2(q) &= 1 + 960 \sum_{n=1}^{\infty} n^2 \sigma_7(n) q^n - \frac{3168}{5} \sum_{n=1}^{\infty} n \sigma_9(n) q^n + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \\ &\quad + \frac{37248}{3455} \sum_{n=1}^{\infty} \tau(n) q^n, \end{aligned}$$

(b)

$$\begin{aligned} L(q)M(q)N(q) &= 1 - \frac{1584}{5} \sum_{n=1}^{\infty} n \sigma_9(n) q^n + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \\ &\quad - \frac{228096}{3455} \sum_{n=1}^{\infty} \tau(n) q^n. \end{aligned}$$

Theorem 2.4. Let $n \in \mathbb{N}$. Then we have

(a)

$$\sum_{m=1}^{n-1} \sigma_1(m) \tau(n-m) = -\frac{1}{24} (n-1) \tau(n),$$

(b)

$$I_{m,1,9}(n) = \frac{1}{912120} n \{2275 \sigma_{11}(n) - 4146 n \sigma_9(n) + 3455 \sigma_1(n) - 1584 \tau(n)\},$$

(c)

$$I_{m,9,1}(n) = \frac{1}{182424} n \{2275 \sigma_{11}(n) - 691 (12n - 11) \sigma_9(n) - 1584 \tau(n)\}.$$

Proof. (a) By (2) and (10) we have

$$\begin{aligned}
 (20) \quad & L(q)M^3(q) = M^3(q) \cdot L(q) \\
 & = \left(1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{432000}{691} \sum_{n=1}^{\infty} \tau(n)q^n\right) \left(1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)q^m\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ \frac{65520}{691} \sigma_{11}(N) + \frac{432000}{691} \tau(N) - 24\sigma_1(N) \right. \\
 & \quad \left. - \frac{65520}{691} \cdot 24 \sum_{m=1}^{N-1} \sigma_{11}(N-m)\sigma_1(m) - \frac{432000}{691} \cdot 24 \sum_{m=1}^{N-1} \tau(N-m)\sigma_1(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ \frac{65520}{691} \sigma_{11}(N) + \frac{432000}{691} \tau(N) - 24\sigma_1(N) - \frac{65520}{691} \cdot 24 \cdot S_{1,11}(N) \right. \\
 & \quad \left. - \frac{432000}{691} \cdot 24 \sum_{m=1}^{N-1} \sigma_1(m)\tau(N-m) \right\} q^N.
 \end{aligned}$$

So equating (20) with Theorem 2.2, we obtain the proof.

(b) By (5) and (15), we note that

$$\begin{aligned}
 (21) \quad & L^2(q)M(q)N(q) = M(q)N(q) \cdot L^2(q) \\
 & = \left(1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n\right) \left(1 - 288 \sum_{m=1}^{\infty} m\sigma_1(m)q^m + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m\right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -288N\sigma_1(N) + 240\sigma_3(N) - 264\sigma_9(N) \right. \\
 & \quad \left. + 264 \cdot 288 \sum_{m=1}^{N-1} \sigma_9(N-m) \cdot m\sigma_1(m) - 264 \cdot 240 \sum_{m=1}^{N-1} \sigma_9(N-m)\sigma_3(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -288N\sigma_1(N) + 240\sigma_3(N) - 264\sigma_9(N) + 264 \cdot 288 \cdot I_{m,1,9}(N) \right. \\
 & \quad \left. - 264 \cdot 240 \cdot S_{3,9}(N) \right\} q^N.
 \end{aligned}$$

Similarly, by (2) and Proposition 2.3 (b), we have

$$\begin{aligned}
 (22) \\
 L^2(q)M(q)N(q) &= L(q)M(q)N(q) \cdot L(q) \\
 &= \left(1 - \frac{1584}{5} \sum_{n=1}^{\infty} n \sigma_9(n) q^n + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n - \frac{228096}{3455} \sum_{n=1}^{\infty} \tau(n) q^n\right) \\
 &\quad \times \left(1 - 24 \sum_{m=1}^{\infty} \sigma_1(m) q^m\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -\frac{1584}{5} N \sigma_9(N) + \frac{65520}{691} \sigma_{11}(N) - \frac{228096}{3455} \tau(N) - 24 \sigma_1(N) \right. \\
 &\quad + \frac{1584}{5} \cdot 24 \sum_{m=1}^{N-1} (N-m) \sigma_9(N-m) \sigma_1(m) - \frac{65520}{691} \cdot 24 \sum_{m=1}^{N-1} \sigma_{11}(N-m) \sigma_1(m) \\
 &\quad \left. + \frac{228096}{3455} \cdot 24 \sum_{m=1}^{N-1} \tau(N-m) \sigma_1(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -\frac{1584}{5} N \sigma_9(N) + \frac{65520}{691} \sigma_{11}(N) - \frac{228096}{3455} \tau(N) - 24 \sigma_1(N) \right. \\
 &\quad + \frac{1584}{5} \cdot 24 N \cdot S_{1,9}(N) - \frac{1584}{5} \cdot 24 \cdot I_{m,1,9}(N) - \frac{65520}{691} \cdot 24 \cdot S_{1,11}(N) \\
 &\quad \left. + \frac{228096}{3455} \cdot 24 \sum_{m=1}^{N-1} \sigma_1(m) \tau(N-m) \right\} q^N.
 \end{aligned}$$

Therefore, equating (21) with (22) and using Theorem 2.4 (a), we can complete the proof.

(c) We observe that

$$\begin{aligned}
 I_{m,9,1}(n) &= \sum_{m=1}^{n-1} m \sigma_9(m) \sigma_1(n-m) = \sum_{m=1}^{n-1} (n-m) \sigma_9(n-m) \sigma_1(m) \\
 &= n \cdot S_{1,9}(n) - I_{m,1,9}(n).
 \end{aligned}$$

So we refer to Theorem 2.4 (b).

□

Theorem 2.5. For $q \in \mathbb{C}$ with $|q| < 1$, we have

$$\begin{aligned}
 L^2(q)M(q)N(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n + \frac{131040}{691} \sum_{n=1}^{\infty} n \sigma_{11}(n) q^n \\
 &\quad - \frac{1728}{5} \sum_{n=1}^{\infty} n^2 \sigma_9(n) q^n - \frac{456192}{3455} \sum_{n=1}^{\infty} n \tau(n) q^n.
 \end{aligned}$$

Proof. Insert Theorem 2.4 (b) into (21).

□

Next by the definition of (17), we can see the following proposition :

Proposition 2.6. (See [7, p. 155~156]) Let $n \in \mathbb{N}$. Then we have

(a)

$$I_{m,1,1}^2(n) = \sum_{m=1}^{n-1} m^2 \sigma_1(m) \sigma_1(n-m) = \frac{1}{8} n^2 \sigma_3(n) + \left(\frac{1}{24} n^2 - \frac{1}{6} n^3 \right) \sigma_1(n),$$

(b)

$$I_{m,1,3}^2(n) = \sum_{m=1}^{n-1} m^2 \sigma_1(m) \sigma_3(n-m) = \frac{1}{80} n^2 \sigma_5(n) - \frac{1}{120} n^3 \sigma_3(n) - \frac{1}{240} n^2 \sigma_1(n),$$

$$I_{m,3,1}^2(n) = \sum_{m=1}^{n-1} m^2 \sigma_3(m) \sigma_1(n-m) = \frac{1}{24} n^2 \sigma_5(n) + \left(\frac{1}{24} n^2 - \frac{1}{12} n^3 \right) \sigma_3(n).$$

To be continued Proposition 2.6, we deduce Theorem 1.2 :

Proof of Theorem 1.2. (a) By (4) and (6), we have

$$\begin{aligned} (23) \\ L^3(q)N(q) &= N(q) \cdot L^3(q) \\ &= \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\ &\quad \times \left(1 - 1728 \sum_{m=1}^{\infty} m^2 \sigma_1(m) q^m + 2160 \sum_{m=1}^{\infty} m \sigma_3(m) q^m - 504 \sum_{m=1}^{\infty} \sigma_5(m) q^m \right) \\ &= 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2016 N \sigma_3(N) - 504 \sigma_5(N) - 504 \sigma_5(N) \right. \\ &\quad + 504 \cdot 1728 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^2 \sigma_1(m) - 504 \cdot 2160 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m \sigma_3(m) \\ &\quad \left. + 504 \cdot 504 \sum_{m=1}^{N-1} \sigma_5(N-m) \sigma_5(m) \right\} q^N \\ &= 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2016 N \sigma_3(N) - 504 \cdot 2 \sigma_5(N) \right. \\ &\quad + 504 \cdot 1728 \sum_{m=1}^{N-1} m^2 \sigma_1(m) \sigma_5(N-m) - 504 \cdot 2160 \cdot I_{m,3,5}(N) \\ &\quad \left. + 504^2 \cdot S_{5,5}(N) \right\} q^N. \end{aligned}$$

And, by (5) and (14), we obtain

$$\begin{aligned}
 (24) \quad & L^3(q)N(q) = L(q)N(q) \cdot L^2(q) \\
 & = \left(1 - 1008 \sum_{n=1}^{\infty} n \sigma_5(n) q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n \right) \\
 & \quad \times \left(1 - 288 \sum_{m=1}^{\infty} m \sigma_1(m) q^m + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1008N \sigma_5(N) + 480 \sigma_7(N) - 288N \sigma_1(N) \right. \\
 & \quad + 1008 \cdot 288 \sum_{m=1}^{N-1} (N-m) \sigma_5(N-m) \cdot m \sigma_1(m) \\
 & \quad - 480 \cdot 288 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m \sigma_1(m) + 240 \sigma_3(N) \\
 & \quad - 1008 \cdot 240 \sum_{m=1}^{N-1} (N-m) \sigma_5(N-m) \sigma_3(m) \\
 & \quad \left. + 480 \cdot 240 \sum_{m=1}^{N-1} \sigma_7(N-m) \sigma_3(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1008N \sigma_5(N) + 480 \sigma_7(N) - 288N \sigma_1(N) + 1008 \cdot 288N \cdot I_{m,1,5}(N) \right. \\
 & \quad - 1008 \cdot 288 \sum_{m=1}^{N-1} m^2 \sigma_1(m) \sigma_5(N-m) - 480 \cdot 288 \cdot I_{m,1,7}(N) + 240 \sigma_3(N) \\
 & \quad \left. - 1008 \cdot 240N \cdot S_{3,5}(N) + 1008 \cdot 240 \cdot I_{m,3,5}(N) + 480 \cdot 240 \cdot S_{3,7}(N) \right\} q^N.
 \end{aligned}$$

Finally, we equate (23) with (24) and use Proposition 1.1 (c) and (d).
 (b) By (3) and (12), let us consider

$$\begin{aligned}
 (25) \quad & L^2(q)M^2(q) = M(q) \cdot L^2(q)M(q) \\
 & = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right) \\
 & \quad \times \left(1 + 1728 \sum_{m=1}^{\infty} m^2 \sigma_3(m)q^m - 2016 \sum_{m=1}^{\infty} m \sigma_5(m)q^m + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 1728 N^2 \sigma_3(N) - 2016 N \sigma_5(N) + 480 \sigma_7(N) + 240 \sigma_3(N) \right. \\
 & \quad + 240 \cdot 1728 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^2 \sigma_3(m) - 240 \cdot 2016 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m \sigma_5(m) \\
 & \quad \left. + 240 \cdot 480 \sum_{m=1}^{N-1} \sigma_3(N-m) \sigma_7(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 1728 N^2 \sigma_3(N) - 2016 N \sigma_5(N) + 480 \sigma_7(N) + 240 \sigma_3(N) \right. \\
 & \quad + 240 \cdot 1728 \sum_{m=1}^{N-1} m^2 \sigma_3(m) \sigma_3(N-m) - 240 \cdot 2016 \cdot I_{m,5,3}(N) \\
 & \quad \left. + 240 \cdot 480 \cdot S_{3,7}(N) \right\} q^N.
 \end{aligned}$$

Thus we equate (25) with Proposition 2.3 (a) and use Proposition 1.1 (d).

(c) Let us expand

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^2 \sigma_5(m) \sigma_1(n-m) \\
 & = \sum_{m=1}^{n-1} (n-m)^2 \sigma_5(n-m) \sigma_1(m) \\
 & = n^2 \cdot S_{1,5}(n) - 2n \cdot I_{m,1,5}(n) + I_{m,1,5}^2(n).
 \end{aligned}$$

So we refer to Theorem 1.2 (a).

(d) First by (6) and (9), we obtain

$$\begin{aligned}
 (26) \quad & L^3(q)M^2(q) = M^2(q) \cdot L^3(q) \\
 & = \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n \right) \\
 & \quad \times \left(1 - 1728 \sum_{m=1}^{\infty} m^2 \sigma_1(m) q^m + 2160 \sum_{m=1}^{\infty} m \sigma_3(m) q^m - 504 \sum_{m=1}^{\infty} \sigma_5(m) q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2160 N \sigma_3(N) - 504 \sigma_5(N) + 480 \sigma_7(N) \right. \\
 & \quad - 480 \cdot 1728 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m^2 \sigma_1(m) + 480 \cdot 2160 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m \sigma_3(m) \\
 & \quad \left. - 480 \cdot 504 \sum_{m=1}^{N-1} \sigma_7(N-m) \sigma_5(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2160 N \sigma_3(N) - 504 \sigma_5(N) + 480 \sigma_7(N) \right. \\
 & \quad - 480 \cdot 1728 \sum_{m=1}^{N-1} m^2 \sigma_1(m) \sigma_7(N-m) + 480 \cdot 2160 \cdot I_{m,3,7}(N) \\
 & \quad \left. - 480 \cdot 504 \cdot S_{5,7}(N) \right\} q^N.
 \end{aligned}$$

Second by (5) and (13), we have

$$\begin{aligned}
 (27) \quad & L^3(q)M^2(q) = L(q)M^2(q) \cdot L^2(q) \\
 & = \left(1 + 720 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n \right) \\
 & \quad \times \left(1 - 288 \sum_{m=1}^{\infty} m\sigma_1(m)q^m + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -288N\sigma_1(N) + 240\sigma_3(N) + 720N\sigma_7(N) \right. \\
 & \quad - 720 \cdot 288 \sum_{m=1}^{N-1} (N-m)\sigma_7(N-m) \cdot m\sigma_1(m) \\
 & \quad + 720 \cdot 240 \sum_{m=1}^{N-1} (N-m)\sigma_7(N-m)\sigma_3(m) - 264\sigma_9(N) \\
 & \quad \left. + 264 \cdot 288 \sum_{m=1}^{N-1} \sigma_9(N-m) \cdot m\sigma_1(m) - 264 \cdot 240 \sum_{m=1}^{N-1} \sigma_9(N-m)\sigma_3(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -288N\sigma_1(N) + 240\sigma_3(N) + 720N\sigma_7(N) - 720 \cdot 288N \cdot I_{m,1,7}(N) \right. \\
 & \quad + 720 \cdot 288 \sum_{m=1}^{N-1} m^2\sigma_1(m)\sigma_7(N-m) + 720 \cdot 240N \cdot S_{3,7}(N) \\
 & \quad - 720 \cdot 240 \cdot I_{m,3,7}(N) - 264\sigma_9(N) + 264 \cdot 288 \cdot I_{m,1,9}(N) \\
 & \quad \left. - 264 \cdot 240 \cdot S_{3,9}(N) \right\} q^N.
 \end{aligned}$$

Therefore, we equate (26) and (27) and use Proposition 1.1 (d), Theorem 2.1 (a), and Theorem 2.4 (b).
 (e) By (4) and (12), we note that

$$\begin{aligned}
 (28) \quad & L^2(q)M(q)N(q) = N(q) \cdot L^2(q)M(q) \\
 & = \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n \right) \\
 & \quad \times \left(1 + 1728 \sum_{m=1}^{\infty} m^2 \sigma_3(m)q^m - 2016 \sum_{m=1}^{\infty} m \sigma_5(m)q^m + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 1728 N^2 \sigma_3(N) - 2016 N \sigma_5(N) + 480 \sigma_7(N) - 504 \sigma_5(N) \right. \\
 & \quad - 504 \cdot 1728 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^2 \sigma_3(m) \\
 & \quad \left. + 504 \cdot 2016 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m \sigma_5(m) - 504 \cdot 480 \sum_{m=1}^{N-1} \sigma_5(N-m) \sigma_7(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ 1728 N^2 \sigma_3(N) - 2016 N \sigma_5(N) + 480 \sigma_7(N) - 504 \sigma_5(N) \right. \\
 & \quad - 504 \cdot 1728 \sum_{m=1}^{N-1} m^2 \sigma_3(m) \sigma_5(N-m) + 504 \cdot 2016 \cdot I_{m,5,5}(N) \\
 & \quad \left. - 504 \cdot 480 \cdot S_{5,7}(N) \right\} q^N.
 \end{aligned}$$

So we equate (28) with Theorem 2.5 and use Theorem 2.1 (b).

(f) We pay attention to

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^2 \sigma_5(m) \sigma_3(n-m) \\
 & = \sum_{m=1}^{n-1} (n-m)^2 \sigma_5(n-m) \sigma_3(m) \\
 & = n^2 \cdot S_{3,5}(n) - 2n \cdot I_{m,3,5}(n) + I_{m,3,5}^2(n).
 \end{aligned}$$

So we refer to Theorem 1.2 (e).

(g) We can expand

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^2 \sigma_7(m) \sigma_1(n-m) \\
 & = \sum_{m=1}^{n-1} (n-m)^2 \sigma_7(n-m) \sigma_1(m) \\
 & = n^2 \cdot S_{1,7}(n) - 2n \cdot I_{m,1,7}(n) + I_{m,1,7}^2(n).
 \end{aligned}$$

So we refer to Theorem 1.2 (d).

□

Theorem 2.7. For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$L^3(q)N(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n - \frac{4752}{5} \sum_{n=1}^{\infty} n\sigma_9(n)q^n + 2880 \sum_{n=1}^{\infty} n^2\sigma_7(n)q^n \\ - 2592 \sum_{n=1}^{\infty} n^3\sigma_5(n)q^n - \frac{29088}{3455} \sum_{n=1}^{\infty} \tau(n)q^n,$$

(b)

$$L^3(q)M^2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n)q^n + \frac{196560}{691} \sum_{n=1}^{\infty} n\sigma_{11}(n)q^n - \frac{5184}{5} \sum_{n=1}^{\infty} n^2\sigma_9(n)q^n \\ + 1152 \sum_{n=1}^{\infty} n^3\sigma_7(n)q^n + \frac{111744}{3455} \sum_{n=1}^{\infty} n\tau(n)q^n.$$

Proof. (a) Insert Theorem 1.2 (a) into (23).

(b) Insert Theorem 1.2 (d) into (26).

□

Now we can see that

$$(29) \quad I_{m,1,1}^3(n) = \sum_{m=1}^{n-1} m^3\sigma_1(m)\sigma_1(n-m) = \frac{1}{12}n^3\sigma_3(n) + \left(\frac{1}{24}n^3 - \frac{1}{8}n^4\right)\sigma_1(n)$$

in [7, p. 155], which leads us to deduce the following theorem :

Theorem 2.8. Let $n \in \mathbb{N}$. Then we have

(a)

$$I_{m,1,3}^3(n) = \sum_{m=1}^{n-1} m^3\sigma_1(m)\sigma_3(n-m) \\ = \frac{1}{3360} \{21n^3\sigma_5(n) - 12n^4\sigma_3(n) - 14n^3\sigma_1(n) + 5\tau(n)\},$$

(b)

$$I_{m,3,1}^3(n) = \sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_1(n-m) \\ = \frac{1}{671} \{21n^3\sigma_5(n) - 4n^3(12n-7)\sigma_3(n) - \tau(n)\},$$

(c)

$$I_{m,1,5}^3(n) = \sum_{m=1}^{n-1} m^3\sigma_1(m)\sigma_5(n-m) \\ = \frac{1}{3024} \{4n^3\sigma_7(n) - 3n^4\sigma_5(n) + 6n^3\sigma_1(n) - 7n\tau(n)\},$$

(d)

$$I_{m,3,3}^3(n) = \sum_{m=1}^{n-1} m^3 \sigma_3(m) \sigma_3(n-m) = \frac{1}{720} \{n^3 \sigma_7(n) - 3n^3 \sigma_3(n) + 2n\tau(n)\},$$

(e)

$$\begin{aligned} I_{m,5,1}^3(n) &= \sum_{m=1}^{n-1} m^3 \sigma_5(m) \sigma_1(n-m) \\ &= \frac{1}{216} \{4n^3 \sigma_7(n) - 3n^3 (4n-3) \sigma_5(n) - n\tau(n)\}. \end{aligned}$$

Proof. (a) First by (3) and (7), we have

$$\begin{aligned} (30) \\ L^4(q)M(q) &= M(q) \cdot L^4(q) \\ &= \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right) \left(1 - 6912 \sum_{m=1}^{\infty} m^3 \sigma_1(m) q^m + 10368 \sum_{m=1}^{\infty} m^2 \sigma_3(m) q^m \right. \\ &\quad \left. - 4032 \sum_{m=1}^{\infty} m \sigma_5(m) q^m + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^m\right) \\ &= 1 + \sum_{N=1}^{\infty} \{-6912N^3 \sigma_1(N) + 10368N^2 \sigma_3(N) - 4032N \sigma_5(N) + 480 \sigma_7(N) \\ &\quad + 240 \sigma_3(N) - 240 \cdot 6912 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^3 \sigma_1(m) \\ &\quad + 240 \cdot 10368 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^2 \sigma_3(m) - 240 \cdot 4032 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m \sigma_5(m) \\ &\quad + 240 \cdot 480 \sum_{m=1}^{N-1} \sigma_3(N-m) \sigma_7(m)\} q^N \\ &= 1 + \sum_{N=1}^{\infty} \{-6912N^3 \sigma_1(N) + 10368N^2 \sigma_3(N) - 4032N \sigma_5(N) + 480 \sigma_7(N) \\ &\quad + 240 \sigma_3(N) - 240 \cdot 6912 \sum_{m=1}^{N-1} m^3 \sigma_1(m) \sigma_3(N-m) + 240 \cdot 10368 \cdot I_{m,3,3}^2(n) \\ &\quad - 240 \cdot 4032 \cdot I_{m,5,3}(N) + 240 \cdot 480 \cdot S_{3,7}(N)\} q^N. \end{aligned}$$

Second by (6) and (11), we obtain

$$\begin{aligned}
 (31) \quad & L^4(q)M(q) = L(q)M(q) \cdot L^3(q) \\
 & = \left(1 + 720 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n \right) \\
 & \quad \times \left(1 - 1728 \sum_{m=1}^{\infty} m^2\sigma_1(m)q^m + 2160 \sum_{m=1}^{\infty} m\sigma_3(m)q^m - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) + 720N\sigma_3(N) \right. \\
 & \quad - 720 \cdot 1728 \sum_{m=1}^{N-1} (N-m)\sigma_3(N-m) \cdot m^2\sigma_1(m) \\
 & \quad + 720 \cdot 2160 \sum_{m=1}^{N-1} (N-m)\sigma_3(N-m) \cdot m\sigma_3(m) \\
 & \quad - 720 \cdot 504 \sum_{m=1}^{N-1} (N-m)\sigma_3(N-m)\sigma_5(m) - 504\sigma_5(N) \\
 & \quad + 504 \cdot 1728 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^2\sigma_1(m) - 504 \cdot 2160 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m\sigma_3(m) \\
 & \quad \left. + 504 \cdot 504 \sum_{m=1}^{N-1} \sigma_5(N-m)\sigma_5(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) + 720N\sigma_3(N) \right. \\
 & \quad - 720 \cdot 1728N \cdot I_{m,1,3}^2(n) + 720 \cdot 1728 \sum_{m=1}^{N-1} m^3\sigma_1(m)\sigma_3(N-m) \\
 & \quad + 720 \cdot 2160N \cdot I_{m,3,3}(N) - 720 \cdot 2160 \cdot I_{m,3,3}^2(n) - 720 \cdot 504N \cdot S_{3,5}(N) \\
 & \quad + 720 \cdot 504 \cdot I_{m,5,3}(N) - 504\sigma_5(N) + 504 \cdot 1728 \cdot I_{m,1,5}^2(N) \\
 & \quad \left. - 504 \cdot 2160 \cdot I_{m,3,5}(N) + 504^2 \cdot S_{5,5}(N) \right\} q^N.
 \end{aligned}$$

Consequently, we equate (30) and (31) and use Proposition 1.1 (c), (d), Proposition 2.6 (b), Theorem 1.2 (a) and (b).

(b) We can expand

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_1(n-m) \\
 & = \sum_{m=1}^{n-1} (n-m)^3\sigma_3(n-m)\sigma_1(m) \\
 & = n^3 \cdot S_{1,3}(n) - 3n^2 \cdot I_{m,1,3}(n) + 3n \cdot I_{m,1,3}^2(n) - I_{m,1,3}^3(n).
 \end{aligned}$$

So we refer to Proposition 1.1 (b), Proposition 2.6 (b), and Theorem 2.8 (a).

(c) By (4) and (7), we obtain

$$\begin{aligned}
 (32) \\
 L^4(q)N(q) &= N(q) \cdot L^4(q) \\
 &= \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n\right) \left(1 - 6912 \sum_{m=1}^{\infty} m^3 \sigma_1(m)q^m + 10368 \sum_{m=1}^{\infty} m^2 \sigma_3(m)q^m \right. \\
 &\quad \left. - 4032 \sum_{m=1}^{\infty} m \sigma_5(m)q^m + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^m\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -6912N^3 \sigma_1(N) + 10368N^2 \sigma_3(N) - 4032N \sigma_5(N) + 480 \sigma_7(N) \right. \\
 &\quad - 504 \sigma_5(N) + 504 \cdot 6912 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^3 \sigma_1(m) \\
 &\quad - 504 \cdot 10368 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^2 \sigma_3(m) + 504 \cdot 4032 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m \sigma_5(m) \\
 &\quad \left. - 504 \cdot 480 \sum_{m=1}^{N-1} \sigma_5(N-m) \sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -6912N^3 \sigma_1(N) + 10368N^2 \sigma_3(N) - 4032N \sigma_5(N) + 480 \sigma_7(N) \right. \\
 &\quad - 504 \sigma_5(N) + 504 \cdot 6912 \sum_{m=1}^{N-1} m^3 \sigma_1(m) \sigma_5(N-m) - 504 \cdot 10368 \cdot I_{m,3,5}^2(N) \\
 &\quad \left. + 504 \cdot 4032 \cdot I_{m,5,5}(N) - 504 \cdot 480 \cdot S_{5,7}(N) \right\} q^N.
 \end{aligned}$$

Also by (6) and (14), we have

$$\begin{aligned}
 & (33) \\
 & L^4(q)N(q) = L(q)N(q) \cdot L^3(q) \\
 & = \left(1 - 1008 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n \right) \\
 & \quad \times \left(1 - 1728 \sum_{m=1}^{\infty} m^2\sigma_1(m)q^m + 2160 \sum_{m=1}^{\infty} m\sigma_3(m)q^m - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) - 1008N\sigma_5(N) \right. \\
 & \quad + 1008 \cdot 1728 \sum_{m=1}^{N-1} (N-m)\sigma_5(N-m) \cdot m^2\sigma_1(m) \\
 & \quad - 1008 \cdot 2160 \sum_{m=1}^{N-1} (N-m)\sigma_5(N-m) \cdot m\sigma_3(m) \\
 & \quad + 1008 \cdot 504 \sum_{m=1}^{N-1} (N-m)\sigma_5(N-m)\sigma_5(m) + 480\sigma_7(N) \\
 & \quad - 480 \cdot 1728 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m^2\sigma_1(m) + 480 \cdot 2160 \sum_{m=1}^{N-1} \sigma_7(N-m) \cdot m\sigma_3(m) \\
 & \quad \left. - 480 \cdot 504 \sum_{m=1}^{N-1} \sigma_7(N-m)\sigma_5(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) - 1008N\sigma_5(N) \right. \\
 & \quad + 1008 \cdot 1728N \cdot I_{m,1,5}^2(N) - 1008 \cdot 1728 \sum_{m=1}^{N-1} m^3\sigma_1(m)\sigma_5(N-m) \\
 & \quad - 1008 \cdot 2160N \cdot I_{m,3,5}(N) + 1008 \cdot 2160 \cdot I_{m,3,5}^2(N) + 1008 \cdot 504N \cdot S_{5,5}(N) \\
 & \quad - 1008 \cdot 504 \cdot I_{m,5,5}(N) + 480\sigma_7(N) - 480 \cdot 1728 \cdot I_{m,1,7}^2(N) \\
 & \quad \left. + 480 \cdot 2160 \cdot I_{m,3,7}(N) - 480 \cdot 504 \cdot S_{5,7}(N) \right\} q^N.
 \end{aligned}$$

So we equate (32) with (33) and refer to Proposition 1.1 (d), Theorem 2.1 (a), (b), Theorem 1.2 (a), (d) and (e).
 (d) We can consider

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^3 \sigma_3(m) \sigma_3(n-m) \\
 &= \sum_{m=1}^{n-1} (n-m)^3 \sigma_3(n-m) \sigma_3(m) \\
 &= n^3 \cdot S_{3,3}(n) - 3n^2 \cdot I_{m,3,3}(n) + 3n \cdot I_{m,3,3}^2(n) - \sum_{m=1}^{n-1} m^3 \sigma_3(m) \sigma_3(n-m),
 \end{aligned}$$

which shows that

$$\sum_{m=1}^{n-1} m^3 \sigma_3(m) \sigma_3(n-m) = \frac{1}{2} \{n^3 \cdot S_{3,3}(n) - 3n^2 \cdot I_{m,3,3}(n) + 3n \cdot I_{m,3,3}^2(n)\}.$$

So we use Proposition 1.1 (c) and Theorem 1.2 (b).

(e) It is similar to proof of Theorem 2.8 (b), that is, we can expand

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^3 \sigma_5(m) \sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} (n-m)^3 \sigma_5(n-m) \sigma_1(m) \\
 &= n^3 \cdot S_{1,5}(n) - 3n^2 \cdot I_{m,1,5}(n) + 3n \cdot I_{m,1,5}^2(n) - I_{m,1,5}^3(n).
 \end{aligned}$$

So we refer to Proposition 1.1 (c), Theorem 1.2 (a), and Theorem 2.8 (c). □

Theorem 2.9. For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$\begin{aligned}
 L^4(q)M(q) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n - \frac{6336}{5} \sum_{n=1}^{\infty} n \sigma_9(n) q^n + 5760 \sum_{n=1}^{\infty} n^2 \sigma_7(n) q^n \\
 &\quad - 10368 \sum_{n=1}^{\infty} n^3 \sigma_5(n) q^n + \frac{41472}{7} \sum_{n=1}^{\infty} n^4 \sigma_3(n) q^n - \frac{4608}{24185} \sum_{n=1}^{\infty} \tau(n) q^n,
 \end{aligned}$$

(b)

$$\begin{aligned}
 L^4(q)N(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n + \frac{262080}{691} \sum_{n=1}^{\infty} n \sigma_{11}(n) q^n - \frac{10368}{5} \sum_{n=1}^{\infty} n^2 \sigma_9(n) q^n \\
 &\quad + 4608 \sum_{n=1}^{\infty} n^3 \sigma_7(n) q^n - 3456 \sum_{n=1}^{\infty} n^4 \sigma_5(n) q^n - \frac{116352}{3455} \sum_{n=1}^{\infty} n \tau(n) q^n.
 \end{aligned}$$

Proof. (a) Insert Theorem 2.8 (a) into (30).

(b) Insert Theorem 2.8 (c) into (32). □

Theorem 2.10. Let $n \in \mathbb{N}$. Then we have

(a)

$$\begin{aligned} I_{m,1,1}^4(n) &= \sum_{m=1}^{n-1} m^4 \sigma_1(m) \sigma_1(n-m) \\ &= \frac{1}{840} \{50n^4 \sigma_3(n) - 7n^4 (12n-5) \sigma_1(n) - \tau(n)\}, \end{aligned}$$

(b)

$$\begin{aligned} I_{m,1,3}^4(n) &= \sum_{m=1}^{n-1} m^4 \sigma_1(m) \sigma_3(n-m) \\ &= \frac{1}{10080} \{35n^4 \sigma_5(n) - 18n^5 \sigma_3(n) - 42n^4 \sigma_1(n) + 25n\tau(n)\}, \end{aligned}$$

(c)

$$\begin{aligned} I_{m,3,1}^4(n) &= \sum_{m=1}^{n-1} m^4 \sigma_3(m) \sigma_1(n-m) \\ &= \frac{1}{288} \{7n^4 \sigma_5(n) - 6n^4 (3n-2) \sigma_3(n) - n\tau(n)\}. \end{aligned}$$

Proof. (a) Now by (2) and (8), we have

(34)

$$\begin{aligned} L^6(q) &= L(q) \cdot L^5(q) \\ &= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n\right) \left(1 - 20736 \sum_{m=1}^{\infty} m^4 \sigma_1(m) q^m + 34560 \sum_{m=1}^{\infty} m^3 \sigma_3(m) q^m \right. \\ &\quad \left. - 17280 \sum_{m=1}^{\infty} m^2 \sigma_5(m) q^m + 3600 \sum_{m=1}^{\infty} m \sigma_7(m) q^m - 264 \sum_{m=1}^{\infty} \sigma_9(m) q^m\right) \\ &= 1 + \sum_{N=1}^{\infty} \left\{ -20736 N^4 \sigma_1(N) + 34560 N^3 \sigma_3(N) - 17280 N^2 \sigma_5(N) + 3600 N \sigma_7(N) \right. \\ &\quad \left. - 264 \sigma_9(N) - 24 \sigma_1(N) + 24 \cdot 20736 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^4 \sigma_1(m) \right. \\ &\quad \left. - 24 \cdot 34560 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^3 \sigma_3(m) + 24 \cdot 17280 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^2 \sigma_5(m) \right. \\ &\quad \left. - 24 \cdot 3600 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m \sigma_7(m) + 24 \cdot 264 \sum_{m=1}^{N-1} \sigma_1(N-m) \sigma_9(m) \right\} q^N \\ &= 1 + \sum_{N=1}^{\infty} \left\{ -20736 N^4 \sigma_1(N) + 34560 N^3 \sigma_3(N) - 17280 N^2 \sigma_5(N) + 3600 N \sigma_7(N) \right. \\ &\quad \left. - 264 \sigma_9(N) - 24 \sigma_1(N) + 24 \cdot 20736 \sum_{m=1}^{N-1} m^4 \sigma_1(m) \sigma_1(N-m) \right. \\ &\quad \left. - 24 \cdot 34560 \cdot I_{m,3,1}^3(N) + 24 \cdot 17280 \cdot I_{m,5,1}^2(N) - 24 \cdot 3600 \cdot I_{m,7,1}(N) \right. \\ &\quad \left. + 24 \cdot 264 \cdot S_{1,9}(N) \right\} q^N. \end{aligned}$$

And by (5) and (7), we obtain

$$\begin{aligned}
 (35) \\
 L^6(q) &= L^2(q) \cdot L^4(q) \\
 &= \left(1 - 288 \sum_{n=1}^{\infty} n \sigma_1(n) q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \left(1 - 6912 \sum_{m=1}^{\infty} m^3 \sigma_1(m) q^m \right. \\
 &\quad \left. + 10368 \sum_{m=1}^{\infty} m^2 \sigma_3(m) q^m - 4032 \sum_{m=1}^{\infty} m \sigma_5(m) q^m + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^m \right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -6912 N^3 \sigma_1(N) + 10368 N^2 \sigma_3(N) - 4032 N \sigma_5(N) + 480 \sigma_7(N) \right. \\
 &\quad - 288 N \sigma_1(N) + 288 \cdot 6912 \sum_{m=1}^{N-1} (N-m) \sigma_1(N-m) \cdot m^3 \sigma_1(m) \\
 &\quad - 288 \cdot 10368 \sum_{m=1}^{N-1} (N-m) \sigma_1(N-m) \cdot m^2 \sigma_3(m) \\
 &\quad + 288 \cdot 4032 \sum_{m=1}^{N-1} (N-m) \sigma_1(N-m) \cdot m \sigma_5(m) \\
 &\quad - 288 \cdot 480 \sum_{m=1}^{N-1} (N-m) \sigma_1(N-m) \sigma_7(m) + 240 \sigma_3(N) \\
 &\quad - 240 \cdot 6912 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^3 \sigma_1(m) + 240 \cdot 10368 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^2 \sigma_3(m) \\
 &\quad \left. - 240 \cdot 4032 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m \sigma_5(m) + 240 \cdot 480 \sum_{m=1}^{N-1} \sigma_3(N-m) \sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -6912 N^3 \sigma_1(N) + 10368 N^2 \sigma_3(N) - 4032 N \sigma_5(N) + 480 \sigma_7(N) \right. \\
 &\quad - 288 N \sigma_1(N) + 288 \cdot 6912 N \cdot I_{m,1,1}^3(N) - 288 \cdot 6912 \sum_{m=1}^{N-1} m^4 \sigma_1(m) \sigma_1(N-m) \\
 &\quad - 288 \cdot 10368 N \cdot I_{m,3,1}^2(N) + 288 \cdot 10368 \cdot I_{m,3,1}^3(N) + 288 \cdot 4032 N \cdot I_{m,5,1}(N) \\
 &\quad - 288 \cdot 4032 \cdot I_{m,5,1}^2(N) - 288 \cdot 480 N \cdot S_{1,7}(N) + 288 \cdot 480 \cdot I_{m,7,1}(N) \\
 &\quad + 240 \sigma_3(N) - 240 \cdot 6912 \cdot I_{m,1,3}^3(N) + 240 \cdot 10368 \cdot I_{m,3,3}^2(N) \\
 &\quad \left. - 240 \cdot 4032 \cdot I_{m,5,3}(N) + 240 \cdot 480 \cdot S_{3,7}(N) \right\} q^N.
 \end{aligned}$$

So we equate (34) and (35) and use Proposition 1.1 (c), (d), Proposition 2.6 (b), Theorem 1.2 (b), (c), (29), Theorem 2.8 (a), (b).

(b) First by (3) and (8), we have

$$\begin{aligned}
 (36) \quad & L^5(q)M(q) = M(q) \cdot L^5(q) \\
 & = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right) \left(1 - 20736 \sum_{m=1}^{\infty} m^4 \sigma_1(m)q^m + 34560 \sum_{m=1}^{\infty} m^3 \sigma_3(m)q^m \right. \\
 & \quad \left. - 17280 \sum_{m=1}^{\infty} m^2 \sigma_5(m)q^m + 3600 \sum_{m=1}^{\infty} m \sigma_7(m)q^m - 264 \sum_{m=1}^{\infty} \sigma_9(m)q^m \right) \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -20736N^4 \sigma_1(N) + 34560N^3 \sigma_3(N) - 17280N^2 \sigma_5(N) + 3600N \sigma_7(N) \right. \\
 & \quad - 264 \sigma_9(N) + 240 \sigma_3(N) - 240 \cdot 20736 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^4 \sigma_1(m) \\
 & \quad + 240 \cdot 34560 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^3 \sigma_3(m) \\
 & \quad - 240 \cdot 17280 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m^2 \sigma_5(m) \\
 & \quad \left. + 240 \cdot 3600 \sum_{m=1}^{N-1} \sigma_3(N-m) \cdot m \sigma_7(m) - 240 \cdot 264 \sum_{m=1}^{N-1} \sigma_3(N-m) \sigma_9(m) \right\} q^N \\
 & = 1 + \sum_{N=1}^{\infty} \left\{ -20736N^4 \sigma_1(N) + 34560N^3 \sigma_3(N) - 17280N^2 \sigma_5(N) + 3600N \sigma_7(N) \right. \\
 & \quad - 264 \sigma_9(N) + 240 \sigma_3(N) - 240 \cdot 20736 \sum_{m=1}^{N-1} m^4 \sigma_1(m) \sigma_3(N-m) \\
 & \quad + 240 \cdot 34560 \cdot I_{m,3,3}^3(N) - 240 \cdot 17280 \cdot I_{m,5,3}^2(N) + 240 \cdot 3600 \cdot I_{m,7,3}(N) \\
 & \quad \left. - 240 \cdot 264 \cdot S_{3,9}(N) \right\} q^N.
 \end{aligned}$$

Second by (7) and (11) we obtain

$$\begin{aligned}
 (37) \\
 L^5(q)M(q) &= L(q)M(q) \cdot L^4(q) \\
 &= \left(1 + 720 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \left(1 - 6912 \sum_{m=1}^{\infty} m^3 \sigma_1(m) q^m \right. \\
 &\quad \left. + 10368 \sum_{m=1}^{\infty} m^2 \sigma_3(m) q^m - 4032 \sum_{m=1}^{\infty} m \sigma_5(m) q^m + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^m \right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -6912 N^3 \sigma_1(N) + 10368 N^2 \sigma_3(N) - 4032 N \sigma_5(N) + 480 \sigma_7(N) \right. \\
 &\quad + 720 N \sigma_3(N) - 720 \cdot 6912 \sum_{m=1}^{N-1} (N-m) \sigma_3(N-m) \cdot m^3 \sigma_1(m) \\
 &\quad + 720 \cdot 10368 \sum_{m=1}^{N-1} (N-m) \sigma_3(N-m) \cdot m^2 \sigma_3(m) \\
 &\quad - 720 \cdot 4032 \sum_{m=1}^{N-1} (N-m) \sigma_3(N-m) \cdot m \sigma_5(m) \\
 &\quad + 720 \cdot 480 \sum_{m=1}^{N-1} (N-m) \sigma_3(N-m) \sigma_7(m) - 504 \sigma_5(N) \\
 &\quad + 504 \cdot 6912 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^3 \sigma_1(m) \\
 &\quad - 504 \cdot 10368 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^2 \sigma_3(m) \\
 &\quad \left. + 504 \cdot 4032 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m \sigma_5(m) - 504 \cdot 480 \sum_{m=1}^{N-1} \sigma_5(N-m) \sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -6912 N^3 \sigma_1(N) + 10368 N^2 \sigma_3(N) - 4032 N \sigma_5(N) + 480 \sigma_7(N) \right. \\
 &\quad + 720 N \sigma_3(N) - 720 \cdot 6912 N \cdot I_{m,1,3}^3(N) + 720 \cdot 6912 \sum_{m=1}^{N-1} m^4 \sigma_1(m) \sigma_3(N-m) \\
 &\quad + 720 \cdot 10368 N \cdot I_{m,3,3}^2(N) - 720 \cdot 10368 \cdot I_{m,3,3}^3(N) - 720 \cdot 4032 N \cdot I_{m,5,3}(N) \\
 &\quad + 720 \cdot 4032 \cdot I_{m,5,3}^2(N) + 720 \cdot 480 N \cdot S_{3,7}(N) - 720 \cdot 480 \cdot I_{m,7,3}(N) \\
 &\quad - 504 \sigma_5(N) + 504 \cdot 6912 \cdot I_{m,1,5}^3(N) - 504 \cdot 10368 \cdot I_{m,3,5}^2(N) \\
 &\quad \left. + 504 \cdot 4032 \cdot I_{m,5,5}(N) - 504 \cdot 480 \cdot S_{5,7}(N) \right\} q^N.
 \end{aligned}$$

Thus we equate (36) and (37) and use Proposition 1.1 (d), Theorem 2.1 (b), (c), Theorem 1.2 (b), (e), (f), Theorem 2.8 (a), (c), and (d).
(c) We can consider

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m^4 \sigma_3(m) \sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} (n-m)^4 \sigma_3(n-m) \sigma_1(m) \\
 &= n^4 \cdot S_{1,3}(n) - 4n^3 \cdot I_{m,1,3}(n) + 6n^2 \cdot I_{m,1,3}^2(n) - 4n \cdot I_{m,1,3}^3(n) + I_{m,1,3}^4(n),
 \end{aligned}$$

which needs Proposition 1.1 (b), Proposition 2.6 (b), Theorem 2.8 (a), and Theorem 2.10 (b). $\square$

Theorem 2.11. For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$\begin{aligned}
 L^6(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n - \frac{9504}{5} \sum_{n=1}^{\infty} n \sigma_9(n) q^n + 14400 \sum_{n=1}^{\infty} n^2 \sigma_7(n) q^n \\
 - 51840 \sum_{n=1}^{\infty} n^3 \sigma_5(n) q^n + \frac{622080}{7} \sum_{n=1}^{\infty} n^4 \sigma_3(n) q^n - \frac{248832}{5} \sum_{n=1}^{\infty} n^5 \sigma_1(n) q^n \\
 - \frac{4608}{24185} \sum_{n=1}^{\infty} \tau(n) q^n,
 \end{aligned}$$

(b)

$$\begin{aligned}
 L^5(q)M(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n + \frac{327600}{691} \sum_{n=1}^{\infty} n \sigma_{11}(n) q^n - 3456 \sum_{n=1}^{\infty} n^2 \sigma_9(n) q^n \\
 + 11520 \sum_{n=1}^{\infty} n^3 \sigma_7(n) q^n - 17280 \sum_{n=1}^{\infty} n^4 \sigma_5(n) q^n \\
 + \frac{62208}{7} \sum_{n=1}^{\infty} n^5 \sigma_3(n) q^n - \frac{4608}{4837} \sum_{n=1}^{\infty} n \tau(n) q^n.
 \end{aligned}$$

Proof. (a) Insert Theorem 2.10 (a) into (34).

(b) Insert Theorem 2.10 (b) into (36). $\square$

Theorem 2.12. Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
 I_{m,1,1}^5(n) &= \sum_{m=1}^{n-1} m^5 \sigma_1(m) \sigma_1(n-m) \\
 &= \frac{1}{336} \{15n^5 \sigma_3(n) - 14n^5 (2n-1) \sigma_1(n) - n \tau(n)\}.
 \end{aligned}$$

Proof. We can show that

$$\begin{aligned} \sum_{m=1}^{n-1} m^5 \sigma_1(m) \sigma_1(n-m) &= \sum_{m=1}^{n-1} (n-m)^5 \sigma_1(n-m) \sigma_1(m) \\ &= n^5 \cdot S_{1,1}(n) - 5n^4 \cdot I_{m,1,1}(n) + 10n^3 \cdot I_{m,1,1}^2(n) - 10n^2 \cdot I_{m,1,1}^3(n) + 5n \cdot I_{m,1,1}^4(n) \\ &\quad - \sum_{m=1}^{n-1} m^5 \sigma_1(m) \sigma_1(n-m), \end{aligned}$$

which leads that

$$\begin{aligned} \sum_{m=1}^{n-1} m^5 \sigma_1(m) \sigma_1(n-m) &= \frac{1}{2} \left\{ n^5 \cdot S_{1,1}(n) - 5n^4 \cdot I_{m,1,1}(n) + 10n^3 \cdot I_{m,1,1}^2(n) \right. \\ &\quad \left. - 10n^2 \cdot I_{m,1,1}^3(n) + 5n \cdot I_{m,1,1}^4(n) \right\}. \end{aligned}$$

So we refer to Proposition 1.1 (a), Proposition 2.6 (a), (29), and Theorem 2.10 (a).

□

Proof of Lemma 1.3. We multiply Theorem 2.11 (a) by (2) :

$$\begin{aligned}
L^7(q) &= L(q) \cdot L^6(q) \\
&= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n\right) \left(1 + \frac{65520}{691} \sum_{m=1}^{\infty} \sigma_{11}(m) q^m - \frac{9504}{5} \sum_{m=1}^{\infty} m \sigma_9(m) q^m \right. \\
&\quad + 14400 \sum_{m=1}^{\infty} m^2 \sigma_7(m) q^m - 51840 \sum_{m=1}^{\infty} m^3 \sigma_5(m) q^m + \frac{622080}{7} \sum_{m=1}^{\infty} m^4 \sigma_3(m) q^m \\
&\quad \left. - \frac{248832}{5} \sum_{m=1}^{\infty} m^5 \sigma_1(m) q^m - \frac{4608}{24185} \sum_{m=1}^{\infty} \tau(m) q^m\right) \\
&= 1 + \sum_{N=1}^{\infty} \left\{ \frac{65520}{691} \sigma_{11}(N) - \frac{9504}{5} N \sigma_9(N) + 14400 N^2 \sigma_7(N) - 51840 N^3 \sigma_5(N) \right. \\
&\quad + \frac{622080}{7} N^4 \sigma_3(N) - \frac{248832}{5} N^5 \sigma_1(N) - \frac{4608}{24185} \tau(N) - 24 \sigma_1(N) \\
&\quad - 24 \cdot \frac{65520}{691} \sum_{m=1}^{N-1} \sigma_1(N-m) \sigma_{11}(m) + 24 \cdot \frac{9504}{5} \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m \sigma_9(m) \\
&\quad - 24 \cdot 14400 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^2 \sigma_7(m) + 24 \cdot 51840 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^3 \sigma_5(m) \\
&\quad - 24 \cdot \frac{622080}{7} \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^4 \sigma_3(m) \\
&\quad \left. + 24 \cdot \frac{248832}{5} \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^5 \sigma_1(m) + 24 \cdot \frac{4608}{24185} \sum_{m=1}^{N-1} \sigma_1(N-m) \tau(m) \right\} q^N \\
&= 1 + \sum_{N=1}^{\infty} \left\{ \frac{65520}{691} \sigma_{11}(N) - \frac{9504}{5} N \sigma_9(N) + 14400 N^2 \sigma_7(N) - 51840 N^3 \sigma_5(N) \right. \\
&\quad + \frac{622080}{7} N^4 \sigma_3(N) - \frac{248832}{5} N^5 \sigma_1(N) - \frac{4608}{24185} \tau(N) - 24 \sigma_1(N) \\
&\quad - 24 \cdot \frac{65520}{691} \cdot S_{1,11}(N) + 24 \cdot \frac{9504}{5} \cdot I_{m,9,1}(N) - 24 \cdot 14400 \cdot I_{m,7,1}^2(N) \\
&\quad + 24 \cdot 51840 \cdot I_{m,5,1}^3(N) - 24 \cdot \frac{622080}{7} \cdot I_{m,3,1}^4(N) + 24 \cdot \frac{248832}{5} \cdot I_{m,1,1}^5(N) \\
&\quad \left. + 24 \cdot \frac{4608}{24185} \sum_{m=1}^{N-1} \sigma_1(N-m) \tau(m) \right\} q^N.
\end{aligned}$$

Therefore we use Theorem 2.4 (a), (c), Theorem 1.2 (g), Theorem 2.8 (e), Theorem 2.10 (c), and Theorem 2.12.

□

3. PROOF OF THEOREM 1.4

In this section we construct the new convolution sums by multiplying $\sigma_1(n)$ for an positive integer n into the given convolution sums in Proposition 1.1 and so we can obtain Theorem 1.4.

Proof of Theorem 1.4. Since proofs are similar, so I only prove Theorem 1.4 (j). Now by Proposition 1.1 (d), we can expand

$$\begin{aligned}
 \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a \sigma_7(a) \sigma_1(b) \sigma_1(c) &= \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k \sigma_1(k) \sigma_1(m-k) \sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} \left[\frac{1}{1800} \{33m \sigma_9(m) - 25m(4m-3) \sigma_7(m) - 8\tau(m)\} \right] \sigma_1(n-m) \\
 &= \frac{1}{1800} \left\{ 33 \sum_{m=1}^{n-1} m \sigma_9(m) \sigma_1(n-m) - 25 \cdot 4 \sum_{m=1}^{n-1} m^2 \sigma_7(m) \sigma_1(n-m) \right. \\
 &\quad \left. + 25 \cdot 3 \sum_{m=1}^{n-1} m \sigma_7(m) \sigma_1(n-m) - 8 \sum_{m=1}^{n-1} \tau(m) \sigma_1(n-m) \right\} \\
 &= \frac{1}{1800} \left\{ 33 \cdot I_{m,9,1}(n) - 25 \cdot 4 \cdot I_{m,7,1}^2(n) + 25 \cdot 3 \cdot I_{m,7,1}(n) \right. \\
 &\quad \left. - 8 \sum_{m=1}^{n-1} \tau(m) \sigma_1(n-m) \right\}.
 \end{aligned}$$

So we use Proposition 1.1 (d), Theorem 2.4 (a), (c), and Theorem 1.2 (g).

□

4. APPENDIX

The first twenty values of $\tau(n)$ are given in the Table 1.

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	-6048	11	534612	16	987136
2	-24	7	-16744	12	-370944	17	-6905934
3	252	8	84480	13	-577738	18	2727432
4	-1472	9	-113643	14	401856	19	10661420
5	4830	10	-115920	15	1217160	20	-7109760

TABLE 1. $\tau(n)$ for n ($1 \leq n \leq 20$)

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