

Generalized Sturm-Liouville Problems and Chebychev Collocation Method

Abstract

In this paper, we present an algorithm for approximating the eigenvalues of Sturm-Liouville problems with parameter-dependent boundary conditions. The algorithm is based on the Chebychev method. A few examples shall be presented to illustrate the power of the method and a comparison made with the regularized sampling method.

Keywords: Eigenvalues, Sturm-Liouville, Chebychev, polynomial.

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1 Introduction

The study of many physical phenomena, such as the vibration of strings, the interaction of atomic particles, or the earth's free oscillations yields Sturm-Liouville (SL) eigenvalue problems. The general form of Sturm-Liouville problems that concerns this paper is

$$\sum_{j=0}^2 P_j(x) u^{(j)}(x) = \lambda^2 q(x) u(x), \quad x \in J = [0, 1] \quad (1.1)$$

subject to the boundary conditions

$$\begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) & b_{11}(\lambda) & b_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & b_{21}(\lambda) & b_{22}(\lambda) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \\ u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.2)$$

where $q(x)$, $u(x)$ and $P_i(x)$, are analytic functions. It will always be assumed that (1.1) possesses a unique solution $u \in C^n(J)$. The values of λ for which the boundary value problem has nontrivial solution are called eigenvalues of (1.1). A nontrivial solution corresponding to an eigenvalue is called an eigenfunction.

Recently, Sturm-Liouville problems were treated in [El-Daou and Al-Matar (2010)] using Tau method, in [Chen and Shizgal (1996)] using spectral method, in [Çelik (2005); Yuan, De Meyer and VanDaele (2008)] using Chebyshev collocation method, in [Altıntan and Uğur (2008)] using variational iteration method, in [Ycel (2006)] using differential quadrature (DQ) method, in [Attili (2009)] using the Adomian decomposition method, in [Bujurkea and Salimatha (2008)] using Haar wavelets, in [?] using boundary value methods, in [Attili and Syam (2009)] using the homotopy perturbation method, in [Chanane (2007)] using Regularized sampling method, in [Chen and Ho (1996); Hassan (2002)] using the differential transformation, in [?] using finite-difference methods, in [Yano and De Meyer (1992)] using Milne-spline method, in [Pruess, De Hoog and Anderssen (1995)] using shooting methods and in [Eggert, Jarratt and Lund (1987); Jarratt, Lund and Bowers (1990); Lund and Riley (1984)] using the sinc-collocation method.

To our knowledge there is no study on the chebychev applications to Sturm-Liouville problems with parameter-dependent boundary conditions.

The paper is organized into five sections. Section II contains notation, definitions and some results of chebchev polynomial. Section III is devoted to the numerical computation of the eigenvalues problem of (1.1)-(1.2). Section IV is devoted to giving some examples exhibiting the technique. Finally, Section V provides conclusions of the study.

2 Some properties of chebyshev polynomials

In recent years, a lot of attention has been devoted to the study of the chebychev method to investigate various scientific models. The efficiency of the method has been formally proved by many researchers [Sezer and Kaynak (1996); ÇERDK-YASLAN and AKYZ-DACIOLU (2006)]. The goal of this section is to recall notation and definitions of the chebychev function, state some known results, and derive useful formulas that are important for this paper.

The well known Chebyshev polynomials are defined on the interval $[-1, 1]$ and the following definitions are necessary for this step [El-gamel and Sameeh (2012); Rivlin (1969)]

Definition 2.1. Chebyshev polynomial of degree n is defined as

$$T_n(x) = \cos(n \arccos(x)), \quad n = 0, 1, \dots, \quad x \in [-1, 1]$$

or, in a more instructive form,

$$T_n(x) = \cos n \theta, \quad x = \cos \theta, \quad \theta \in [0, \pi]$$

One sees at once that, on $[-1, 1]$,

1 T_n takes its maximal value with alternating signs $(n + 1)$ times:

$$\|T_n\| = 1, \quad T_n(x_k) = (-1)^k,$$

$$x_k = \cos\left(\frac{\pi k}{n}\right), \quad k = 0, \dots, n \quad (2.1)$$

2 T_n has n distinct zeros:

$$T_n(t_k) = 0, \quad t_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1, \dots, n$$

Lemma 2.1. Chebyshev polynomials T_n satisfy the recurrence relation

$$\begin{aligned} T_0(x) &= 1, & T_1(x) &= x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), & n &\geq 1. \end{aligned}$$

In particular, T_n is indeed an algebraic polynomial of degree n with the leading coefficient 2^{n-1} .
Also, orthogonality

$$\int_{-1}^1 T_m(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} = 0, \quad n \neq m$$

$$\int_{-1}^1 T_n^2(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \frac{\pi}{2}, & n > 0; \\ \pi, & n = 0. \end{cases} \quad (2.2)$$

In this paper we use orthonormal Chebyshev polynomials, noting property (2.2).

Theorem 2.2. On the interval $[-1, 1]$, among all polynomials of degree n with leading coefficient $a_n = 1$, the Chebyshev polynomial $\frac{1}{2^{n-1}} T_n$ deviates least from zero, i.e.,

$$\inf_{(a_i)} \|x^n + a_{n-1}x^{n-1} + \dots + a_0\| = \frac{1}{2^{n-1}} \|T_n\|.$$

We are concerned with the approximate solution by means of the Chebyshev polynomials in the form

$$u(x) = \sum_{r=0}^n a_r T_r(x), \quad -1 \leq x \leq 1 \quad (2.3)$$

where $T_r(x)$ denotes shifted Chebyshev polynomials of the first kind of degree r , a_r are unknown Chebyshev coefficients and n is chosen any positive integer such that $n \geq 2$. Let us assume that the function $u(x)$ and its derivatives have truncated Chebyshev series expansion of the form

$$u^{(k)}(x) = \sum_{r=0}^n a_r^{(k)} T_r(x), \quad k = 1, 2. \quad (2.4)$$

Then the solution expressed by (2.3) and its derivatives can be written in the matrix forms

$$\mathbf{A}^{(k)} = 2^k \mathbf{M}^k \mathbf{A}$$

$$\mathbf{U}^{(k)} = 2^k \mathbf{T} \mathbf{M}^k \mathbf{A}$$

where

$$\mathbf{T} = \begin{pmatrix} T_0(x_0) & T_1(x_0) & \dots & T_N(x_0) \\ T_0(x_1) & T_1(x_1) & \dots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & \dots & T_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_N) & T_1(x_N) & \dots & T_N(x_N) \end{pmatrix},$$

$$\mathbf{U}^{(k)} = \begin{pmatrix} u^{(k)}(x_0) \\ u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_N) \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix},$$

$$\mathbf{A} = \left[\frac{a_0}{2}, a_1, \dots, a_N \right]^T$$

and

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & m_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(N+1) \times (N+1)}. \quad (2.5)$$

where

$$\begin{aligned} m_1 &= \frac{N}{2}, & m_2 &= 0, & m_3 &= N & \text{if } N \text{ is odd} \\ m_1 &= 0, & m_2 &= N, & m_3 &= 0 & \text{if } N \text{ is even} \end{aligned}$$

The method can be developed for the problem defined in the domain $[a, b]$.

Definition 2.2. On the interval $[a, b]$, the Chebyshev polynomial is given by

$$T_n^*(x) = T_n(y), \quad y = \frac{2}{b-a} \left(x - \frac{a+b}{2} \right).$$

Notice that its leading coefficient is equal to $2^{n-1} \left(\frac{2}{b-a} \right)^n$.

To obtain the solution in terms of shifted Chebyshev polynomials $T_r^*(x)$ in the form

$$u(x) = \sum_{r=0}^n a_r^* T_r^*(x), \quad a \leq x \leq b,$$

where $T_r^*(x) = T_r \left(\frac{2}{b-a} \left(x - \frac{a+b}{2} \right) \right)$.

It is followed the previous procedure using the collocation points defined by

$$x_i = \frac{b-a}{2} \left[\left(\frac{a+b}{b-a} \right) + \cos \left(\frac{i\pi}{n} \right) \right], \quad i = 0, 1, \dots, n-1 \quad (2.6)$$

and the relation

$$A^{*(k)} = \left(\frac{4}{b-a} \right)^k \mathbf{M}^k \mathbf{A}^*, \quad k = 0, 1, 2.$$

where

$$\mathbf{A} = \left[\frac{a_0^*}{2}, a_1^*, \dots, a_n^* \right]^T$$

It is easily obtained $T = T^*$, because of the properties of Chebyshev polynomials.

3 The description of chebychev scheme

We assume that $u(x)$, the solution of (1.1), is approximated by the finite expansion of chebychev basis functions

$$u(x) = \sum_{r=0}^n a_r T_r^*(x), \quad 0 \leq x \leq 1 \quad (3.1)$$

where $T_r^*(x) = T_r(2x-1)$ presents the shifted Chebyshev polynomials of the first kind of degree r and a_r for $r = 0, 1, \dots, n$ are the undetermined Chebyshev coefficients and the Chebyshev collocation points in $[0, 1]$ are

$$x_i = \frac{1}{2} \left[1 + \cos \left(\frac{i\pi}{n} \right) \right], \quad i = 0, 1, \dots, n-1 \quad (3.2)$$

And its derivatives have truncated Chebyshev series expansion of the form

$$u^{(k)}(x) = \sum_{r=0}^n (a_r)^k T_r^*(x), \quad k = 0, 1, 2. \quad (3.3)$$

Using (3.1) and (3.3) and substituting $x = x_k$ in (3.2) and applying the collocation to it, we eventually obtain the following theorem

Theorem 3.1. If the assumed approximate solution of the problem (1.1) is (3.1), then the discrete Chebychev system is given by

$$\sum_{j=0}^2 P_j(x_i) u^{(j)}(x_i) = \lambda^2 q(x_i) u(x_i) \quad (3.4)$$

The fundamental matrix equation for (3.4) is

$$\mathbf{W}\mathbf{A} = 0 \quad (3.5)$$

where

$$\mathbf{W} = \sum_{j=0}^2 4^j \mathbf{P}_j \mathbf{T}^* \mathbf{M}^j - \mathbf{C} \mathbf{T}^*,$$

and

$$\mathbf{P}_j = \begin{bmatrix} p_j(x_0) & 0 & \dots & 0 \\ 0 & p_j(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_j(x_N) \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \lambda^2 q(x_0) & 0 & \dots & 0 \\ 0 & \lambda^2 q(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^2 q(x_N) \end{bmatrix}$$

The boundary condition is derived from equation (1.2) and matrices for conditions are

$$\alpha_{11}(\lambda) \mathbf{T}^*(0) \mathbf{A} + \alpha_{12}(\lambda) \mathbf{T}^*(0) \mathbf{M} \mathbf{A} + \beta_{11}(\lambda) \mathbf{T}^*(1) \mathbf{A} + 4 \beta_{12}(\lambda) \mathbf{T}^*(1) \mathbf{M} \mathbf{A} = 0 \quad (3.6)$$

$$\alpha_{21}(\lambda) \mathbf{T}^*(0) \mathbf{A} + \alpha_{22}(\lambda) \mathbf{T}^*(0) \mathbf{M} \mathbf{A} + \beta_{21}(\lambda) \mathbf{T}^*(1) \mathbf{A} + 4 \beta_{22}(\lambda) \mathbf{T}^*(1) \mathbf{M} \mathbf{A} = 0. \quad (3.7)$$

Consequently, replacing two rows of the augmented matrix by the equation (3.5), we have

$$\tilde{\mathbf{W}}\mathbf{A} = 0. \quad (3.8)$$

This set of equations has a non-trivial solution only if the determinant of the coefficients matrix vanishes. This gives a function of and the roots of this function are eigenvalues of the problem.

4 Examples and comparisons

In this section, we shall present a few examples to illustrate our method. In the following, λ_{exact} denotes the exact eigenvalue, λ_{cheby} denotes the approximate eigenvalue computed by the method of section 3. We use the absolute error which is defined as

$$E_{Cheby} = |\lambda_{exact} - \lambda_{cheby}|$$

Example 1: [Chanane (2005)] We considered the following example

$$-u'' = \lambda^2 u(x), \quad 0 < x < 1$$

and subject to the boundary conditions

$$u(0) + (\lambda^2 - 4\pi^2) u'(0) = 0$$

$$u(1) - \lambda^2 u'(1) = 0$$

The exact characteristic function is

$$\beta_{exact} = (1 + 4\pi^2 \lambda^4 - \lambda^6) \frac{\sin \lambda}{\lambda} - (2\lambda^2 - 4\pi^2) \cos \lambda$$

where zero is not an eigenvalue. The computed eigenvalues together with the "exact" ones are displayed in **Table 1**.

Table 1 The first three eigenvalues of λ_{exact} and λ_{cheby} for Example 1 at $n = 16$.

λ_k	λ_{exact}	λ_{cheby}
1	9.7308865782130820	9.730886578213059
2	88.763316252589763	88.76331847516812
3	157.88411043863472	157.8842052283192

Maximum absolute error are tabulated in **Table 2** for Chebychev method together with the analogous results of [Chanane (2005)].

Table 2 Maximum absolute errors for Example 1

λ_k	$\ E_{cheby}\ $	Results of [Chanane (2005)]
1	1.9539E-14	0.1554E-06
2	2.2225E-06	4.8758E-07
3	9.4789E-05	3.6142E-05

Example 2: [Chanane (2005)] Consider the Sturm-Liouville problem

$$-u(x)'' = \lambda^2 u(x), \quad 0 < x < 1$$

and subject to the boundary conditions

$$u(0) - 2u'(0) = 0$$

$$(1 + \lambda) u(1) + (1 - \lambda^2) u'(1) = 0$$

The exact characteristic function is

$$\beta_{exact} = \left(2 \cos \lambda + \frac{\sin \lambda}{\lambda} \right) + (1 - \lambda) [-2\lambda \sin \lambda + \cos \lambda]$$

1 is not an eigenvalue. The computed eigenvalues λ together with the "exact" ones are displayed in **Table 3**.

Table 3 The first three eigenvalues of λ_{exact} and λ_{cheby} for Example 2 at $n = 16$.

λ_k	λ_{exact}	λ_{cheby}
1	0.929679054283188	0.929679054283189
2	9.9387434140	9.938743414040601
3	11.2738742105212	11.27387421054431

Maximum absolute error are tabulated in **Table 4** for Chebychev method together with the analogous results of [Chanane (2005)].

Table 4 Maximum absolute errors for Example 2

λ_k	$\ E_{cheby}\ $	Results of [Chanane (2005)]
1	9.9920E-016	0.1554E-06
2	4.0602E-011	4.8758E-07
3	2.3119E-011	3.6142E-05

Example 3: [Chanane (2005)] Consider the Sturm-Liouville problem

$$-u''(x) + e^{i2x} u(x) = \lambda^2 u(x)$$

subject to boundary condition

$$\begin{aligned} u(0) + \lambda u(1) &= 0 \\ u'(0) &= 0 \end{aligned}$$

The exact characteristic function is

$$B_{exact} = \det \begin{pmatrix} J_{\sqrt{\lambda}}(1) + \sqrt{\lambda} J_{\sqrt{\lambda}}(e^i) & J_{-\sqrt{\lambda}}(1) + \sqrt{\lambda} J_{-\sqrt{\lambda}}(e^i) \\ \frac{1}{2} J_{-\sqrt{\lambda}-1}(1) - \frac{1}{2} J_{\sqrt{\lambda}+1}(1) & \frac{1}{2} J_{\sqrt{\lambda}-1}(1) - \frac{1}{2} J_{-\sqrt{\lambda}+1}(1) \end{pmatrix}$$

where J_a and J_{-a} are the Bessel functions of the first kind of order a . **Table 5** lists the first three eigenvalues of example 3 at $n = 10$.

Table 5 The first three eigenvalues in Example 3

λ_k	λ_{exact}	λ_{Cheby}
1	4.96854309+0.3906545 i	4.96854305+0.3906547 i
2	20.6027103+0.7502325 i	20.6027193+0.7502221 i
3	64.1403824+0.6842283 i	64.1382692+0.6839719 i

5 Conclusion

The Chebychev technique is applied to solve Sturm-Liouville problems with parameter dependent boundary condition. We have presented a few examples to illustrate the method and compared the computed eigenvalues with the exact ones obtained as the zeros of the exact characteristic functions. Numerical results are compared to those obtained by the regularized sampling method Chanane (2005) to illustrate the effectiveness of the proposed method. It is shown that the Chebychev technique is very promising in this problem. The results of example 3 clearly indicate that our method are accurate even when the coefficients p_i is a complex-valued function satisfying $p_i \in L^1_{loc}(0, 1)$.

References

- Aceto, I. Ghelardoni, P. and Magherini, C. (2009). Boundary value methods as an extension of Numerovs method for Sturm-Liouville eigenvalue estimates. J. Comput. Appl. Math. 59, 1644-1656.
- Altintan D. and Ugur, O. (2008). Variational iteration method for Sturm-Liouville differential equations. Computers and Math. Appl., 58, 322-328.
- Attili, B. (2005). The Adomian decomposition method for computing eigen-elements of Sturm-Liouville two-point boundary value problems. Appl. Math. Comput. 168, 1306-1316.
- Attili, B. and Syam, M. (2005). The homotopy perturbation method for eigenelements of a class of two-point boundary value problems. Adv. Stud. Contemp. Math. 14, 83-102.
- Bujurkea, N., Salimatha, C. and Shiralashettib, S. (2008). Computation of eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets. J. Comp. Appl. Math. 219, 90-101.

- Chanane, B. (2005). Computation of the eigenvalues of Sturm-Liouville problems with parameter dependent boundary conditions using the regularized sampling Method. Math. Comput. 74, 1793-1801.
- Chanane, B. (2007). Computing the spectrum of non-self-adjoint Sturm-Liouville problems with parameter-dependent boundary conditions. J. Comput. Appl. Math. 206, 229-237.
- Chen, C. and Ho S. (1996). Application of differential transformation to eigenvalue problems. Appl. Math. Comput. 79, 173-188.
- Chen, H. and Shizgal, B. (1996). A spectral solution of the Sturm-Liouville equation: comparison of classical and nonclassical basis sets. J. Comput. Appl. Math. 136, 17-35.
- Çelik, I. (2005). Approximate computation of eigenvalues with Chebyshev collocation method. Appl. Math. Comp. 168, 125-134.
- Eggert, N., Jarratt M. and Lund J. (1987). Sine function computation of the eigenvalues of Sturm-Liouville problems. J. Comput. Phys. 69, 209-229.
- El-Daou M. and Al-Matar N. (2010). Sine function computation of the eigenvalues of Sturm-Liouville problems. Appl. Math. Comput. 216, 1923-1937.
- El-gamel M. and Sameeh M. (2012). An efficient technique for finding the eigenvalues of fourth-order Sturm-Liouville problems. Applied Mathematics 3, 920-925.
- Gartland E. (1984). Accurate approximation of eigenvalues and zeros of selected eigenfunctions of regular Sturm-Liouville problems. Math. Comput. 42, 427-439.
- Ghelardoni P. (1997). Approximations of Sturm-Liouville eigenvalues using Boundary Value Methods. Appl. Numer. Math. 23, 311-325.
- Godart M. (1991). An iterative method for the solution of eigenvalue problems. Numer. Math. 59, 243-254.
- Hassan I. (2002). On solving some eigenvalue problems by using a differential transformation. Appl. Math. Comput. 127, 1-22.
- Jarratt I., Lund J. and Bowers K. (1990). Galerkin schemes and the sinc-Galerkin method for singular Sturm-Liouville problems. J. Comput. Phys. 89, 41-62.
- Lund J. and Riley B. (1984). A Sinc-collocation method for the computation of the eigenvalues of the radical Schrodinger equation. IMA J. Numer. Anal. 4, 83-98.
- Paine, J., De Hoog, F. and Anderssen R. (1981). On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems. Computing 26, 123-139.
- Pruess, S., Fulton, C. and Xie Y. (1995). An asymptotic numerical method for a class of singular Sturm-Liouville problems. SIAM J. Numer. Anal. 32, 1658-1676.
- Rivlin, T. (1969). An Introduction to the Approximation of Functions. Dover Publications, Inc., New York
- Sezer M. and Kaynak M. (1996). Chebyshev polynomial solutions of linear differential equations. Int. Math. Educ. Sci. Technol. 27, 607-618.

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- Vanden Berghe G. and De Meyer H. (1991). Accurate computation of higher Sturm-Liouville eigenvalues. Numer. Math. 59, 243-254.
- Vanden Berghe G., De Meyer H. and VanDaele M. (1992). A parallel approach to the modified Numerov-like eigenvalue determination for Sturm-Liouville problems. Comput. Math. Appl. 23, 69-74.
- ~~Yano T., Yokota T., Kawabata K., Matsushima, S., Ezawa, Y.~~ and Tomiyoshi S. (1992). Accurate computation of higher Sturm-Liouville eigenvalues. Comput. Phys. Comm. 75, 61-75.
- Yuan Q., He Z. and Leng H. (2008). An improvement for Chebyshev collocation method in solving certain Sturm-Liouville problems. Appl. Math. Comput. 195, 440-447.
- Yücel U. (2006). Approximations of Sturm-Liouville eigenvalues using differential quadrature (DQ) method. J. Comp. Appl. Math. 192, 310-319.
- Zhang X. (2010). Mapped barycentric Chebyshev differentiation matrix method for the solution of regular Sturm-Liouville problems. Appl. Math. Comput. 217, 2266-2276.
- Akyüz A. and Sezer M. (1999). A Chebyshev collocation method for the solution linear integro differential equations. J. Comput. Math. 72, 491-507.
- ÇERDK-YASLAN H. and AKYÜZ-DACIOĞLU A. (2006). Chebyshev polynomial solution of nonlinear Fredholm-Volterra integro-differential equations. J. Arts and Sciences 6, 89-101.
- Sezer M. and Kaynak M. (1996). Chebyshev polynomial solutions of linear differential equations. Int. Math. Educ. Sci. Technol. 27, 607-611.

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